Skill Practice Solve by using the Gauss-Jordan method.

11. $x + y + z = 2$
$x - y + z = 4$
$x + 4y + 2z = 1$

It is particularly easy to recognize a dependent or inconsistent system of equations from the reduced row echelon form of an augmented matrix. This is demonstrated in Examples 6 and 7.

Example 6 Solving a Dependent System of Equations by Using the Gauss-Jordan Method

Solve by using the Gauss-Jordan method.

1a

$$x - 3y = 4$$
$$\frac{1}{2}x - \frac{3}{2}y = 2$$

Solution:

$$\left[\begin{array}{cc|c} 1 & -3 & 4 \\ \frac{1}{2} & -\frac{3}{2} & 2 \end{array}\right]$$ Set up the augmented matrix.

$$\xrightarrow{-\frac{1}{2}R_1 + R_2 \Rightarrow R_2} \left[\begin{array}{cc|c} 1 & -3 & 4 \\ 0 & 0 & 0 \end{array}\right]$$ Multiply row 1 by $-\frac{1}{2}$ and add the result to row 2.

The second row of the augmented matrix represents the equation $0 = 0$; hence, the system is dependent. The solution is $\{(x, y) \mid x - 3y = 4\}$.

Skill Practice Solve by using the Gauss-Jordan method.

12. $4x - 6y = 16$
$6x - 9y = 24$

Example 7 Solving an Inconsistent System of Equations by Using the Gauss-Jordan Method

Solve by using the Gauss-Jordan method.

1b

$$x + 3y = 2$$
$$-3x - 9y = 1$$

Solution:

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ -3 & -9 & 1 \end{array}\right]$$ Set up the augmented matrix.

$$\xrightarrow{3R_1 + R_2 \Rightarrow R_2} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 0 & 7 \end{array}\right]$$ Multiply row 1 by 3 and add the result to row 2.

The second row of the augmented matrix represents the contradiction $0 = 7$; hence, the system is inconsistent. There is no solution.

Skill Practice Answers

11. $(1, -1, 2)$
12. Infinitely many solutions; $\{(x, y) \mid 4x - 6y = 16\}$; dependent system

8.3 No Solutions

1e

Linear systems sometimes have no solutions at all. For example, the system

$$\begin{aligned} 2x + 4y &= 10 \\ 3x + 6y &= 17 \end{aligned}$$

has no solution, since the corresponding lines are parallel in $\mathbb{R}^2$. Algebraically, if $2x + 4y = 10$, then $3x + 6y$ must be 15, not 17.

A system of equations with no solutions is sometimes said to be **inconsistent**.

Here are the first few steps of the row reduction for the above system:

$$\left[\begin{array}{cc|c} 2 & 4 & 10 \\ 3 & 6 & 17 \end{array}\right] \to \left[\begin{array}{cc|c} \mathbf{1} & 2 & 5 \\ 3 & 6 & 17 \end{array}\right] \to \left[\begin{array}{cc|c} \mathbf{1} & 2 & 5 \\ \mathbf{0} & 0 & 2 \end{array}\right]$$

At this point, the equation corresponding to the second row is

$$0x + 0y = 2$$

or more succinctly

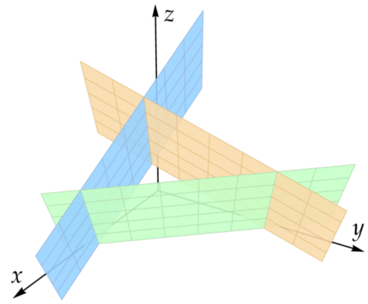
$$0 = 2$$

which is a contradiction.

In general, **a row whose coefficients are all zero but whose constant term is nonzero indicates a contradiction**. If such a row arises during a row reduction, it indicates that the original linear system had no solutions.

3×3 Systems with No Solutions

It is easy to see when a 2×2 system has no solutions, since the two lines must be parallel. For 3×3 systems, a contradiction can be much less obvious. For example, Figure 1 shows three planes that have no point in common, even though no two of the planes are parallel.



▲ **Figure 1:** Three planes that do not intersect at a common point.

An example of this phenomenon is the system

$$\begin{aligned} 2x + 4y + 4z &= 2 \\ 3x + 4y + 2z &= 5 \\ 5x + 8y + 6z &= 4 \end{aligned}$$

Even though no two of these planes are parallel, this 3×3 system has no solutions. The reason is that the *sum* of the first two equations is

$$5x + 8y + 6z = 7$$

which contradicts the third equation.

More generally, a 3×3 system will have no solutions if the third equation contradicts *any* equation that can be derived from the first two. For example, the linear system

$$\begin{aligned} 2x + 6y - 4z &= 2 \\ x - 5y + 5z &= 5 \\ 7x - 3y + 7z &= 6 \end{aligned}$$

has no solutions, and the reason is that *two* times the first equation plus *three* times the second equation is

$$7x - 3y + 7z = 19$$

which contradicts the third equation.

2a

Solution Using the Gauss–Jordan elimination method, we obtain the following sequence of equivalent augmented matrices:

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & -2 \\ 3 & -1 & -2 & 1 \\ 2 & 3 & -5 & -3 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \left[\begin{array}{rrr|r} 1 & 2 & -3 & -2 \\ 0 & -7 & 7 & 7 \\ 0 & -1 & 1 & 1 \end{array}\right] \xrightarrow{-\frac{1}{7}R_2}$$

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{array}\right] \xrightarrow[R_3 + R_2]{R_1 - 2R_2} \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last augmented matrix is in row-reduced form. Interpreting it as a system of linear equations gives

$$\begin{aligned} x - z &= 0 \\ y - z &= -1 \end{aligned}$$

a system of two equations in the three variables x, y, and z.

Let's now single out one variable—say, z—and solve for x and y in terms of it. We obtain

$$\begin{aligned} x &= z \\ y &= z - 1 \end{aligned}$$

If we assign a particular value to z—say, $z = 0$—we obtain $x = 0$ and $y = -1$, giving the solution $(0, -1, 0)$ to System (9). By setting $z = 1$, we obtain the solution $(1, 0, 1)$. In general, if we set $z = t$, where t represents some real number (called a parameter), we obtain a solution given by $(t, t - 1, t)$. Since the parameter t may be any real number, we see that System (9) has infinitely many solutions. Geometrically, the solutions of System (9) lie on the straight line in three-dimensional space given by the intersection of the three planes determined by the three equations in the system. ■

Note In Example 1 we chose the parameter to be z because it is more convenient to solve for x and y (both the x- and y-columns are in unit form) in terms of z. ■

The next example shows what happens in the elimination procedure when the system does not have a solution.

2b

EXAMPLE 2 A System of Equations That Has No Solution Solve the system of linear equations given by

$$\begin{aligned} x + y + z &= 1 \\ 3x - y - z &= 4 \\ x + 5y + 5z &= -1 \end{aligned} \tag{10}$$

Solution Using the Gauss–Jordan elimination method, we obtain the following sequence of equivalent augmented matrices:

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 3 & -1 & -1 & 4 \\ 1 & 5 & 5 & -1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 4 & 4 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

Observe that row 3 in the last matrix reads $0x + 0y + 0z = -1$—that is, $0 = -1$! We therefore conclude that System (10) is inconsistent and has no solution. Geometrically, we have a situation in which two of the planes intersect in a straight line but the third plane is parallel to this line of intersection of the two planes and does not intersect it. Consequently, there is no point of intersection of the three planes. ■

Example 21.6.17

Solve the system of equations using a matrix: $\begin{cases} 2x-5y+3z=8 \\ 3x-y+4z=7 \\ x+3y+2z=-3 \end{cases}$

Answer

$(6,-1,-3)$

Example 21.6.18

Solve the system of equations using a matrix: $\begin{cases} -3x+y+z=-4 \\ -x+2y-2z=1 \\ 2x-y-z=-1 \end{cases}$

Answer

$(5,7,4)$

So far our work with matrices has only been with systems that are consistent and independent, which means they have exactly one solution. Let's now look at what happens when we use a matrix for a dependent or inconsistent system.

Example 21.6.19

Solve the system of equations using a matrix: $\begin{cases} x+y+3z=0 \\ x+3y+5z=0 \\ 2x+4z=1 \end{cases}$

dc

Answer

	$\begin{cases} x+\ \ y+3z=0 \\ x+3y+5z=0 \\ 2x+\ \ \ \ \ \ \ \ 4z=1 \end{cases}$
Write the augmented matrix for the equations.	$\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 1 & 3 & 5 & 0 \\ 2 & 0 & 4 & 1 \end{array}\right]$
The entry in row 1, column 1 is 1.	
Using row operations, get zeros in column 1 below the 1.	$\begin{array}{c} \\ -1R_1+R_2 \\ \\ \end{array}\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 4 & 1 \end{array}\right]$
	$\begin{array}{c} \\ \\ -2R_1+R_3 \end{array}\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right]$
Continue the process until the matrix is in row-echelon form.	$\begin{array}{c} \\ \frac{1}{2}R_2 \\ \\ \end{array}\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right]$
Multiply row 2 by 2 and add it to row 3.	$\begin{array}{c} \\ \\ 2R_2+R_3 \end{array}\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$
At this point, we have all zeros on the left of row 3.	

Write the corresponding system of equations.	$\begin{cases} x+y+3z=0 \\ y+z=0 \\ 0 \neq 1 \end{cases}$
Since $0 \neq 1$ we have a false statement. Just as when we solved a system using other methods, this tells us we have an inconsistent system. There is no solution.	

Example 21.6.20

Solve the system of equations using a matrix: $\begin{cases} x-2y+2z=1 \\ -2x+y-z=2 \\ x-y+z=5 \end{cases}$

Answer

no solution

Example 21.6.21

Solve the system of equations using a matrix: $\begin{cases} 3x+4y-3z=-2 \\ -2x+3y-z=-1 \\ 2x+y-2z=6 \end{cases}$

Answer

no solution

The last system was inconsistent and so had no solutions. The next example is dependent and has infinitely many solutions.

2a

Example 21.6.22

Solve the system of equations using a matrix: $\begin{cases} x-2y+3z=1 \\ x+y-3z=7 \\ 3x-4y+5z=7 \end{cases}$

Answer

	$\begin{cases} x-2y+3z=1 \\ x+y-3z=7 \\ 3x-4y+5z=7 \end{cases}$
Write the augmented matrix for the equations.	$\left[\begin{array}{ccc\|c} 1 & -2 & 3 & 1 \\ 1 & 1 & -3 & 7 \\ 3 & -4 & 5 & 7 \end{array}\right]$
The entry in row 1, column 1 is 1.	
Using row operations, get zeros in column 1 below the 1.	$\begin{array}{c} \\ -1R_1+R_2 \\ \\ \end{array}\left[\begin{array}{ccc\|c} 1 & -2 & 3 & 1 \\ 0 & 3 & -6 & 6 \\ 3 & -4 & 5 & 7 \end{array}\right]$
	$\begin{array}{c} \\ \\ -3R_1+R_3 \end{array}\left[\begin{array}{ccc\|c} 1 & -2 & 3 & 1 \\ 0 & 3 & -6 & 6 \\ 0 & 2 & -4 & 4 \end{array}\right]$
Continue the process until the matrix is in row-echelon form.	

	$\begin{array}{r} \\ \frac{1}{3}R_2 \\ \\ \end{array}\left[\begin{array}{rrr\|r} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 2 & -4 & 4 \end{array}\right]$
Multiply row 2 by -2 and add it to row 3.	$\begin{array}{r} \\ \\ -2R_2+R_3 \end{array}\left[\begin{array}{rrr\|r} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$
At this point, we have all zeros in the bottom row.	
Write the corresponding system of equations.	$\begin{cases} x-2y+3z=1 \\ y-2z=2 \\ 0=0 \end{cases}$
Since $0=0$ we have a true statement. Just as when we solved by substitution, this tells us we have a dependent system. There are infinitely many solutions.	
Solve for y in terms of z in the second equation.	$y-2z=2$ $y=2z+2$
Solve the first equation for x in terms of z.	$x-2y+3z=1$
Substitute $y=2z+2$.	$x-2(2z+2)+3z=1$
Simplify.	$x-4z-4+3z=1$
Simplify.	$x-z-4=1$
Simplify.	$x=z+5$
The system has infinitely many solutions (x,y,z), where $x=z+5;\ y=2z+2;\ z$ is any real number.	

Example 21.6.23

Solve the system of equations using a matrix: $\begin{cases} x+y-z=0 \\ 2x+4y-2z=6 \\ 3x+6y-3z=9 \end{cases}$

Answer

infinitely many solutions (x,y,z), where $x=z-3;\ y=3;\ z$ is any real number.

Example 21.6.24

Solve the system of equations using a matrix: $\begin{cases} x-y-z=1 \\ -x+2y-3z=-4 \\ 3x-2y-7z=0 \end{cases}$

Answer

infinitely many solutions (x,y,z), where $x=5z-2;\ y=4z-3;\ z$ is any real number.

Access this online resource for additional instruction and practice with Gaussian Elimination.

- Gaussian Elimination

Key Concepts

- **Matrix:** A matrix is a rectangular array of numbers arranged in rows and columns. A matrix with m rows and n columns has *order* $m\times n$. The matrix on the left below has 2 rows and 3 columns and so it has order 2×3. We say it is a 2 by 3 matrix.

8.3 No Solutions

Linear systems sometimes have no solutions at all. For example, the system

$$\begin{aligned} 2x + 4y &= 10 \\ 3x + 6y &= 17 \end{aligned}$$

has no solution, since the corresponding lines are parallel in $\mathbb{R}^2$. Algebraically, if $2x + 4y = 10$, then $3x + 6y$ must be 15, not 17.

A system of equations with no solutions is sometimes said to be **inconsistent**.

Here are the first few steps of the row reduction for the above system:

$$\left[\begin{array}{cc|c} 2 & 4 & 10 \\ 3 & 6 & 17 \end{array}\right] \to \left[\begin{array}{cc|c} \mathbf{1} & 2 & 5 \\ 3 & 6 & 17 \end{array}\right] \to \left[\begin{array}{cc|c} \mathbf{1} & 2 & 5 \\ \mathbf{0} & 0 & 2 \end{array}\right]$$

At this point, the equation corresponding to the second row is

$$0x + 0y = 2$$

or more succinctly

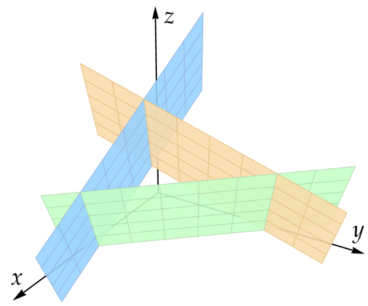
$$0 = 2$$

which is a contradiction.

In general, **a row whose coefficients are all zero but whose constant term is nonzero indicates a contradiction**. If such a row arises during a row reduction, it indicates that the original linear system had no solutions.

3×3 Systems with No Solutions

It is easy to see when a 2×2 system has no solutions, since the two lines must be parallel. For 3×3 systems, a contradiction can be much less obvious. For example, Figure 1 shows three planes that have no point in common, even though no two of the planes are parallel.



▲ **Figure 1:** Three planes that do not intersect at a common point.

An example of this phenomenon is the system

$$\begin{aligned} 2x + 4y + 4z &= 2 \\ 3x + 4y + 2z &= 5 \\ 5x + 8y + 6z &= 4 \end{aligned}$$

Even though no two of these planes are parallel, this 3×3 system has no solutions. The reason is that the *sum* of the first two equations is

$$5x + 8y + 6z = 7$$

which contradicts the third equation.

More generally, a 3×3 system will have no solutions if the third equation contradicts *any* equation that can be derived from the first two. For example, the linear system

$$\begin{aligned} 2x + 6y - 4z &= 2 \\ x - 5y + 5z &= 5 \\ 7x - 3y + 7z &= 6 \end{aligned}$$

has no solutions, and the reason is that *two* times the first equation plus *three* times the second equation is

$$7x - 3y + 7z = 19$$

which contradicts the third equation.

The following example illustrates how to use row reduction to detect a contradiction in a 3×3 system.

EXAMPLE 1

Solve the following linear system.

$$\begin{aligned} 2x + 4y + 4z &= 2 \\ 3x + 4y + 2z &= 5 \\ 5x + 8y + 6z &= 4 \end{aligned}$$

SOLUTION We row reduce the matrix in the usual way:

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 3 & 4 & 2 & 5 \\ 5 & 8 & 6 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 3 & 4 & 2 & 5 \\ 5 & 8 & 6 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & -2 & -4 & 2 \\ 5 & 8 & 6 & 4 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & -2 & -4 & 2 \\ 0 & -2 & -4 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -2 & -4 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

Arguably the contradiction was clear after the third row operation, since we had obtained the equations $-2y - 4z = 2$ and $-2y - 4z = -1$.

We can stop the row reduction at this point, since the last row is a contradiction ($0 = -3$). This means that the original linear system had no solutions.

Overdetermined Systems

As we have seen, a linear system with more equations than unknowns usually has no solutions. Again, the reason is always a contradiction in the original equations. For example, the system

$$\begin{aligned} x + 3y &= 2 \\ 2x + 3y &= 1 \\ 5x + 9y &= 3 \end{aligned}$$

has no solution, and the reason is that the first equation plus twice the second equation is

$$5x + 9y = 4$$

which contradicts the third equation. This contradiction can easily be detected using row reduction:

The first step of this row reduction is actually two row operations. Specifically, we add -2 times the first row to the second row, and we add -5 times the first row to the third row.

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 5 & 9 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -3 & -3 \\ 0 & -6 & -7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & -6 & -7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right]$$

The third row is now the equation $0 = -1$, which is a contradiction.

Of course, it's possible for an overdetermined system to have a solution. For example, the linear system

$$\begin{aligned} x + 3y &= 2 \\ 2x + 3y &= 1 \\ 5x + 9y &= 4 \end{aligned}$$

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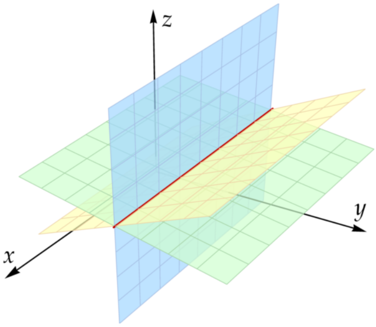

▲ **Figure 2:** It is possible for three planes to intersect along a line.

Redundant Equations

A linear system can have more free variables than expected if one of the equations is a consequence of the others. For example, consider the 3×3 system

$$\begin{aligned} x + 9y - z &= 27 \\ x - 8y + 16z &= 10 \\ 2x + y + 15z &= 37 \end{aligned}$$

Though a 3×3 system usually has a unique solution, in this system the third equation is a consequence of the first two. Specifically, the third equation here is simply the sum of the first two equations. As a result, any solution to the first two equations is also a solution to the third equation, so there is a whole line of solutions, as shown in Figure 2.

Redundant equations lead to rows of zeroes during row reduction. For example, here is what happens if we row reduce the matrix for the 3×3 system above:

$$\left[\begin{array}{ccc|c} 1 & 9 & -1 & 27 \\ 1 & -8 & 16 & 10 \\ 2 & 1 & 15 & 37 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 9 & -1 & 27 \\ 0 & -17 & 17 & -17 \\ 0 & -17 & 17 & -17 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 9 & -1 & 27 \\ 0 & 1 & -1 & 1 \\ 0 & -17 & 17 & -17 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 8 & 18 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The first and third steps here each consist of two row operations.

Because of the row of zeroes, only the first two columns have pivots, and therefore z is a free variable. In fact, we have the equations

$$x + 8z = 18, \qquad y - z = 1$$

and thus

$$x = 18 - 8t, \qquad y = 1 + t, \qquad z = t.$$

In general, a **redundant equation** in a linear system is an equation that is a consequence of the previous equations. A linear system with redundant equations behaves as though the extra equations weren't there. For example, the 3×3 system above has one redundant equation, so it behaves more like a 2×3 system, with one free variable and a line of solutions.

EXERCISES

1–8 ■ Use row reduction to find a parametric description for the solutions to the given linear system.

1. $$\begin{aligned} 2x + 6y - 2z &= 6 \\ -2x - 3y + 8z &= -15 \end{aligned}$$

2. $$\begin{aligned} x + 3y - 5z &= 2 \\ -3x - 7y + 8z &= -5 \end{aligned}$$

3. $$\begin{aligned} x + 3y + 6z &= 5 \\ 3x + 2y + 4z &= 8 \end{aligned}$$

4. $$\begin{aligned} -3x + 3y - 6z &= -6 \\ -x + 3y + 2z &= 4 \\ -3x + 7y + 2z &= 6 \end{aligned}$$

5. $$\begin{aligned} 3x_1 - 9x_2 + 12x_3 - 6x_4 &= 9 \\ -2x_1 + 5x_2 - 5x_3 + 4x_4 &= -8 \end{aligned}$$

6. $$\begin{aligned} x_1 + 2x_2 + 5x_3 + 3x_4 + 6x_5 &= 3 \\ 2x_1 + 3x_2 + 7x_3 + 4x_4 + 8x_5 &= 5 \end{aligned}$$

Step 3 Repeat step 1 with the **submatrix** formed by (mentally) deleting the row used in step 2 and all rows above this row.

Step 4 Repeat step 2 with the **entire matrix,** including the rows deleted mentally. Continue this process until the entire matrix is in reduced form.

Note: If at any point in this process we obtain a row with all zeros to the left of the vertical line and a nonzero number to the right, we can stop before we find the reduced form since we will have a contradiction: $0 = n$, $n \neq 0$. We can then conclude that the system has no solution.

Remarks

1. Even though each matrix has a unique reduced form, the sequence of steps presented here for transforming a matrix into a reduced form is not unique. For example, it is possible to use row operations in such a way that computations involving fractions are minimized. But we emphasize again that we are not interested in the most efficient hand methods for transforming small matrices into reduced forms. Our main interest is in giving you a little experience with a method that is suitable for solving large-scale systems on a graphing calculator or computer.
2. Most graphing calculators have the ability to find reduced forms. Figure 1 illustrates the solution of Example 2 on a TI-86 graphing calculator using the rref command (rref is an acronym for reduced row echelon form). Notice that in row 2 and column 4 of the reduced form the graphing calculator has displayed the very small number –3.5E – 13, instead of the exact value 0. This is a common occurrence on a graphing calculator and causes no problems. Just replace any very small numbers displayed in scientific notation with 0.

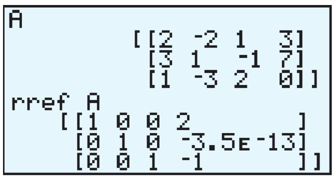

Figure 1 **Gauss–Jordan elimination on a graphing calculator**

EXAMPLE 3 Solving a System Using Gauss–Jordan Elimination Solve by Gauss–Jordan elimination:

$$\begin{aligned} 2x_1 - 4x_2 + x_3 &= -4 \\ 4x_1 - 8x_2 + 7x_3 &= 2 \\ -2x_1 + 4x_2 - 3x_3 &= 5 \end{aligned}$$

SOLUTION

$$\left[\begin{array}{rrr|r} 2 & -4 & 1 & -4 \\ 4 & -8 & 7 & 2 \\ -2 & 4 & -3 & 5 \end{array}\right] \quad 0.5R_1 \rightarrow R_1$$

$$\sim \left[\begin{array}{rrr|r} 1 & -2 & 0.5 & -2 \\ 4 & -8 & 7 & 2 \\ -2 & 4 & -3 & 5 \end{array}\right] \quad \begin{array}{l} (-4)R_1 + R_2 \rightarrow R_2 \\ 2R_1 + R_3 \rightarrow R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & -2 & 0.5 & -2 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & -2 & 1 \end{array}\right] \quad 0.2R_2 \rightarrow R_2$$

Note that column 3 is the leftmost nonzero column in this submatrix.

$$\sim \left[\begin{array}{rrr|r} 1 & -2 & 0.5 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{array}\right] \quad \begin{array}{l} (-0.5)R_2 + R_1 \rightarrow R_1 \\ 2R_2 + R_3 \rightarrow R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

We stop the Gauss–Jordan elimination, even though the matrix is not in reduced form, since the last row produces a contradiction.

The system has no solution.

Matched Problem 3 Solve by Gauss–Jordan elimination:

$$\begin{aligned} 2x_1 - 4x_2 - x_3 &= -8 \\ 4x_1 - 8x_2 + 3x_3 &= 4 \\ -2x_1 + 4x_2 + x_3 &= 11 \end{aligned}$$

CAUTION Figure 2 shows the solution to Example 3 on a graphing calculator with a built-in reduced-form routine. Notice that the graphing calculator does not stop when a contradiction first occurs, but continues on to find the reduced form. Nevertheless, the last row in the reduced form still produces a contradiction. Do not confuse this type of reduced form with one that represents a consistent system (see Fig. 1). ▲

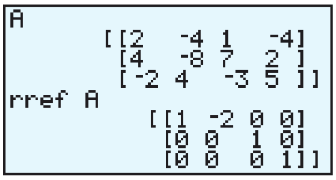

Figure 2 **Recognizing contradictions on a graphing calculator**

EXAMPLE 4 Solving a System Using Gauss–Jordan Elimination Solve by Gauss–Jordan elimination:

$$\begin{aligned} 3x_1 + 6x_2 - 9x_3 &= 15 \\ 2x_1 + 4x_2 - 6x_3 &= 10 \\ -2x_1 - 3x_2 + 4x_3 &= -6 \end{aligned}$$

SOLUTION

$$\left[\begin{array}{ccc|c} 3 & 6 & -9 & 15 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{array}\right] \quad \tfrac{1}{3}R_1 \to R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{array}\right] \quad \begin{array}{l} (-2)R_1 + R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

Note that we must interchange rows 2 and 3 to obtain a nonzero entry at the top of the second column of this submatrix.

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (-2)R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The matrix is now in reduced form. Write the corresponding reduced system and solve.

$$\begin{aligned} x_1 + x_3 &= -3 \\ x_2 - 2x_3 &= 4 \end{aligned}$$

We discard the equation corresponding to the third (all zero) row in the reduced form, since it is satisfied by all values of x_1, x_2, and x_3.

Note that the leftmost variable in each equation appears in one and only one equation. We solve for the leftmost variables x_1 and x_2 in terms of the remaining variable, x_3 :

$$x_1 = -x_3 - 3$$
$$x_2 = 2x_3 + 4$$

If we let $x_3 = t$, then for any real number t,

$$\begin{aligned} x_1 &= -t - 3 \\ x_2 &= 2t + 4 \\ x_3 &= t \end{aligned}$$

You should check that $(-t - 3, 2t + 4, t)$ is a solution of the original system for any real number t. Some particular solutions are

$t = 0$	$t = -2$	$t = 3.5$
$(-3, 4, 0)$	$(-1, 0, -2)$	$(-6.5, 11, 3.5)$

In general,

If the number of leftmost 1's in a reduced augmented coefficient matrix is less than the number of variables in the system and there are no contradictions, then the system is dependent and has infinitely many solutions.

Describing the solution set to this type of system is not difficult. In a reduced system, the **leftmost variables** correspond to the leftmost 1's in the corresponding reduced augmented matrix. The definition of reduced form for an augmented matrix ensures that each leftmost variable in the corresponding reduced system appears in one and only one equation of the system. Solving for each leftmost variable in terms of the remaining variables and writing a general solution to the system is usually easy. Example 5 illustrates a slightly more involved case.

Matched Problem 4 Solve by Gauss–Jordan elimination:

$$\begin{aligned} 2x_1 - 2x_2 - 4x_3 &= -2 \\ 3x_1 - 3x_2 - 6x_3 &= -3 \\ -2x_1 + 3x_2 + x_3 &= 7 \end{aligned}$$

Explore and Discuss 1 Explain why the definition of reduced form ensures that each leftmost variable in a reduced system appears in one and only one equation and no equation contains more than one leftmost variable. Discuss methods for determining whether a consistent system is independent or dependent by examining the reduced form.

EXAMPLE 5 Solving a System Using Gauss–Jordan Elimination Solve by Gauss–Jordan elimination:

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + x_4 - x_5 &= 1 \\ 2x_1 + 4x_2 + 8x_3 + 3x_4 - 4x_5 &= 2 \\ x_1 + 3x_2 + 7x_3 + 3x_5 &= -2 \end{aligned}$$

Thus, either there is no solution or there are infinitely many solutions corresponding to the points lying on a line of intersection of the two planes or on a single plane determined by the two equations. In the case where two planes intersect in a straight line, the solutions will involve one parameter, and in the case where the two planes are coincident, the solutions will involve two parameters.

Explore & Discuss

Give a geometric interpretation of Theorem 1 for a linear system composed of equations involving two variables. Specifically, illustrate what can happen if there are three linear equations in the system (the case involving two linear equations has already been discussed in Section 2.1). What if there are four linear equations? What if there is only one linear equation in the system?

3a

EXAMPLE 3 **A System with More Equations Than Variables** Solve the following system of linear equations:

$$\begin{aligned} x + 2y &= 4 \\ x - 2y &= 0 \\ 4x + 3y &= 12 \end{aligned}$$

Solution We obtain the following sequence of equivalent augmented matrices:

$$\left[\begin{array}{rr|r} 1 & 2 & 4 \\ 1 & -2 & 0 \\ 4 & 3 & 12 \end{array}\right] \xrightarrow[R_3 - 4R_1]{R_2 - R_1} \left[\begin{array}{rr|r} 1 & 2 & 4 \\ 0 & -4 & -4 \\ 0 & -5 & -4 \end{array}\right] \xrightarrow{-\frac{1}{4}R_2}$$

$$\left[\begin{array}{rr|r} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & -5 & -4 \end{array}\right] \xrightarrow[R_3 + 5R_2]{R_1 - 2R_2} \left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

The last row of the row-reduced augmented matrix implies that $0 = 1$, which is impossible, so we conclude that the given system has no solution. Geometrically, the three lines defined by the three equations in the system do not intersect at a point. (To see this for yourself, draw the graphs of these equations.)

3b

EXAMPLE 4 **A System with More Variables Than Equations** Solve the following system of linear equations:

$$\begin{aligned} x + 2y - 3z + w &= -2 \\ 3x - y - 2z - 4w &= 1 \\ 2x + 3y - 5z + w &= -3 \end{aligned}$$

Solution First, observe that the given system consists of three equations in four variables and so, by Theorem 1b, either the system has no solution or it has infinitely many solutions. To solve it we use the Gauss–Jordan method and obtain the following sequence of equivalent augmented matrices:

$$\left[\begin{array}{rrrr|r} 1 & 2 & -3 & 1 & -2 \\ 3 & -1 & -2 & -4 & 1 \\ 2 & 3 & -5 & 1 & -3 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \left[\begin{array}{rrrr|r} 1 & 2 & -3 & 1 & -2 \\ 0 & -7 & 7 & -7 & 7 \\ 0 & -1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{-\frac{1}{7}R_2}$$

$$\left[\begin{array}{rrrr|r} 1 & 2 & -3 & 1 & -2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & -1 & 1 & -1 & 1 \end{array}\right] \xrightarrow[R_3 + R_2]{R_1 - 2R_2} \left[\begin{array}{rrrr|r} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last augmented matrix is in row-reduced form. Observe that the given system is equivalent to the system

$$\begin{aligned} x - z - w &= 0 \\ y - z + w &= -1 \end{aligned}$$

of two equations in four variables. Thus, we may solve for two of the variables in terms of the other two. Letting $z = s$ and $w = t$ (where s and t are any real numbers), we find that

$$\begin{aligned} x &= s + t \\ y &= s - t - 1 \\ z &= s \\ w &= t \end{aligned}$$

The solutions may be written in the form $(s + t, s - t - 1, s, t)$. Geometrically, the three equations in the system represent three hyperplanes in four-dimensional space (since there are four variables) and their "points" of intersection lie in a two-dimensional subspace of four-space (since there are two parameters). ■

Note In Example 4, we assigned parameters to z and w rather than to x and y because x and y are readily solved in terms of z and w. ■

The following example illustrates a situation in which a system of linear equations has infinitely many solutions.

VIDEO

APPLIED EXAMPLE 5 Traffic Control Figure 7 shows the flow of downtown traffic in a certain city during the rush hours on a typical weekday. The arrows indicate the direction of traffic flow on each one-way road, and the average number of vehicles per hour entering and leaving each intersection appears beside each road. 5th Avenue and 6th Avenue can each handle up to 2000 vehicles per hour without causing congestion, whereas the maximum capacity of both 4th Street and 5th Street is 1000 vehicles per hour. The flow of traffic is controlled by traffic lights installed at each of the four intersections.

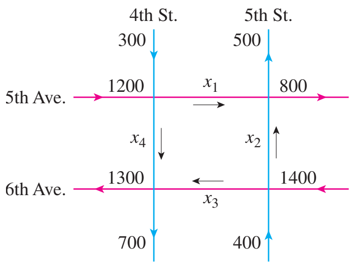

FIGURE **7**

a. Write a general expression involving the rates of flow—x_1, x_2, x_3, x_4—and suggest two possible flow patterns that will ensure no traffic congestion.

b. Suppose the part of 4th Street between 5th Avenue and 6th Avenue is to be resurfaced and that traffic flow between the two junctions must therefore be reduced to at most 300 vehicles per hour. Find two possible flow patterns that will result in a smooth flow of traffic.

Solution

a. To avoid congestion, all traffic entering an intersection must also leave that intersection. Applying this condition to each of the four intersections in a

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3. We obtain the following sequence of equivalent augmented matrices:

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ 2 & 3 & -1 & 4 \\ 1 & 5 & -4 & 2 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1}$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ 0 & 7 & -7 & -14 \\ 0 & 7 & -7 & -7 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ 0 & 7 & -7 & -14 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

Since the last row of the final augmented matrix is equivalent to the equation $0 = 7$, a contradiction, we conclude that the given system has no solution.

USING TECHNOLOGY

Systems of Linear Equations: Underdetermined and Overdetermined Systems

We can use the row operations of a graphing utility to solve a system of m linear equations in n unknowns by the Gauss–Jordan method, as we did in the previous technology section. We can also use the **rref** or equivalent operation to obtain the row-reduced form without going through all the steps of the Gauss–Jordan method. The SIMULT function, however, cannot be used to solve a system where the number of equations and the number of variables are not the same.

EXAMPLE 1 Solve the system

3e

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 2 \\ 2x_1 + x_2 - 2x_3 &= -1 \\ 3x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + 6x_2 - 12x_3 &= -6 \end{aligned}$$

Solution First, we enter the augmented matrix A into the calculator as

$$A = \left[\begin{array}{rrr|r} 1 & -2 & 4 & 2 \\ 2 & 1 & -2 & -1 \\ 3 & -1 & 2 & 1 \\ 2 & 6 & -12 & -6 \end{array}\right]$$

Then using the **rref** or equivalent operation, we obtain the equivalent matrix

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

in reduced form. Thus, the given system is equivalent to

$$\begin{aligned} x_1 \qquad\qquad &= 0 \\ x_2 - 2x_3 &= -1 \end{aligned}$$

If we let $x_3 = t$, where t is a parameter, then we find that the solutions are $(0, 2t - 1, t)$.

3d

Example 2.2. Solve the linear system

$$\left\{\begin{array}{rrrrcr} x_1 & -x_2 & +x_3 & -x_4 & = & 2 \\ x_1 & -x_2 & +x_3 & +x_4 & = & 0 \\ 4x_1 & -4x_2 & +4x_3 & & = & 4 \\ -2x_1 & +2x_2 & -2x_3 & +x_4 & = & -3 \end{array}\right.$$

Solution. Do the row operations:

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 0 \\ 4 & -4 & 4 & 0 & 4 \\ -2 & 2 & -2 & 1 & -3 \end{array}\right] \begin{array}{c} R_2 - R_1 \\ R_3 - 4R_1 \\ \sim \\ R_4 + 2R_1 \end{array} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right] \begin{array}{c} (1/2)R_2 \\ R_3 - 2R_2 \\ \sim \\ R_4 + (1/2)R_2 \end{array}$$

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 + R_2 \\ \sim \end{array} \left[\begin{array}{cccc|r} (1) & [-1] & [1] & 0 & 1 \\ 0 & 0 & 0 & (1) & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The linear system is equivalent to

$$\left\{\begin{array}{lcl} x_1 & = & 1 + x_2 - x_3 \\ x_4 & = & -1 \end{array}\right.$$

We see that the variables x_2, x_3 can take arbitrary numbers; they are called **free variables**. Let $x_2 = c_1$, $x_3 = c_2$, where $c_1, c_2 \in \mathbb{R}$. Then $x_1 = 1 + c_1 - c_2$, $x_4 = -1$. All solutions of the system are given by

$$\left\{\begin{array}{lcl} x_1 & = & 1 + c_1 - c_2 \\ x_2 & = & c_1 \\ x_3 & = & c_2 \\ x_4 & = & -1 \end{array}\right.$$

The general solutions may be written as

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{where } c_1, c_2 \in \mathbb{R}.$$

Set $c_1 = c_2 = 0$, i.e., set $x_2 = x_3 = 0$, we have a **particular solution**

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

For the **corresponding homogeneous linear system** $A\boldsymbol{x} = \boldsymbol{0}$, i.e.,

$$\left\{\begin{array}{rrrrcr} x_1 & -x_2 & +x_3 & -x_4 & = & 0 \\ x_1 & -x_2 & +x_3 & +x_4 & = & 0 \\ 4x_1 & -4x_2 & +4x_3 & & = & 0 \\ -2x_1 & +2x_2 & -2x_3 & +x_4 & = & 0 \end{array}\right.$$

we have

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 1 & 0 \\ 4 & -4 & 4 & 0 & 0 \\ -2 & 2 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} (1) & [-1] & [1] & -1 & 0 \\ 0 & 0 & 0 & (1) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

3. Solve the following systems of equations:

(a)
$$\begin{aligned} x + 2y &= 4 \\ x - 2y &= 0 \\ 4x + 3y &= 12 \end{aligned}$$

(b)
$$\begin{aligned} w + x + 2y - 3z &= -2 \\ -4w + 3x - y - 2z &= 1 \\ w + 2x + 3y - 5z &= -3 \end{aligned}$$

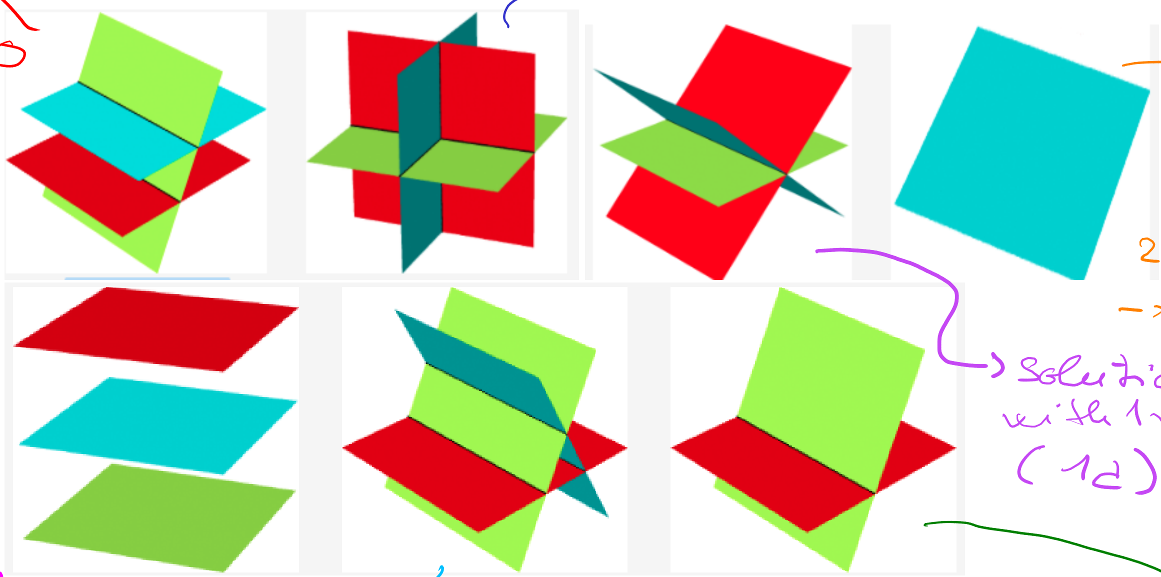
(c)
$$\begin{aligned} x - 2y + 4z &= 2 \\ 2x + y - 2z &= -1 \\ 3x - y + 2z &= 1 \\ 2x + 6y - 12z &= -6 \end{aligned}$$

(d)
$$\begin{aligned} -w + x - y + z &= 2 \\ w + x - y + z &= 0 \\ 4x - 4y + 4z &= 4 \\ w - 2x + 2y - 2z &= -3 \end{aligned}$$

2 parallels
$x - y - z = 1$
$x - y - z = 2$
$x + y + 2z = 0$

4. Choose an image and find a system of 3 equations, which can be represented by your image. Repeat twice more.

anything with 1 solution (last lesson)



1 equation
$x + y + z = 1$
$2x + 2y + 2z = 2$
$-x - y - z = -1$

solution with 1 parameter (1d)

Source 1: https://www.chegg.com/homework-help/questions-and-answers/graph-graph-system-three-linear-equations-three-unknowns-form-ax-b-ax-b--determine-whether-q34440776

3 parallels
$x + y + z = 1$
$x + y + z = 2$
$x + y + z = 3$

∄ solution, for ex. 1c

5. The network shows a plan for the traffic flow around a new park. The plan calls for a computerized traffic light at the north exit. The diagram indicates the average number of vehicles per hour that are expected to flow in and out of the streets around the park. All streets are one-way.

   (a) How many vehicles per hour should the traffic light let trough? The average number of vehicles flowing into is the same as the average number of vehicles flowing out.

the same 2 equations
$2x - y + z = 2$
$4x - 2y + 2z = 4$
$3x + y + 5z = -2$

A common problem in network analysis is to use known flow rates in certain branches to find the flow rates in all of the branches. Here is an example.

EXAMPLE 1 | Network Analysis Using Linear Systems

Figure 1.10.1 shows a network with four nodes in which the flow rate and direction of flow in certain branches are known. Find the flow rates and directions of flow in the remaining branches.

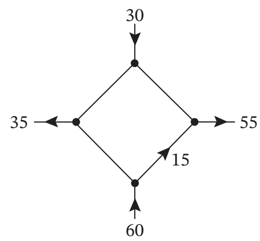


FIGURE 1.10.1

Solution As illustrated in **Figure 1.10.2**, we have assigned arbitrary directions to the unknown flow rates x_1, x_2, and x_3. We need not be concerned if some of the directions are incorrect, since an incorrect direction will be signaled by a negative value for the flow rate when we solve for the unknowns.

It follows from the conservation of flow at node A that

$$x_1 + x_2 = 30$$

Similarly, at the other nodes we have

$$\begin{aligned} x_2 + x_3 &= 35 \quad \text{(node } B\text{)} \\ x_3 + 15 &= 60 \quad \text{(node } C\text{)} \\ x_1 + 15 &= 55 \quad \text{(node } D\text{)} \end{aligned}$$

These four conditions produce the linear system

$$\begin{aligned} x_1 + x_2 &= 30 \\ x_2 + x_3 &= 35 \\ x_3 &= 45 \\ x_1 &= 40 \end{aligned}$$

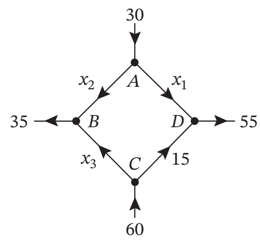


FIGURE 1.10.2

which we can now try to solve for the unknown flow rates. In this particular case the system is sufficiently simple that it can be solved by inspection (work from the bottom up). We leave it for you to confirm that the solution is

$$x_1 = 40, \quad x_2 = -10, \quad x_3 = 45$$

The fact that x_2 is negative tells us that the direction assigned to that flow in Figure 1.10.2 is incorrect; that is, the flow in that branch is *into* node A.

5

EXAMPLE 2 | Design of Traffic Patterns

The network in **Figure 1.10.3*a*** shows a proposed plan for the traffic flow around a new park that will house the Liberty Bell in Philadelphia, Pennsylvania. The plan calls for a computerized traffic light at the north exit on Fifth Street, and the diagram indicates the average number of vehicles per hour that are expected to flow in and out of the streets that border the complex. All streets are one-way.

(*a*) How many vehicles per hour should the traffic light let through to ensure that the average number of vehicles per hour flowing into the complex is the same as the average number of vehicles flowing out?

(*b*) Assuming that the traffic light has been set to balance the total flow in and out of the complex, what can you say about the average number of vehicles per hour that will flow along the streets that border the complex?

5

Solution (*a*) If, as indicated in **Figure 1.10.3*b***, we let x denote the number of vehicles per hour that the traffic light must let through, then the total number of vehicles per hour that flow in and out of the complex will be

Flowing in: $500 + 400 + 600 + 200 = 1700$

Flowing out: $x + 700 + 400$

Equating the flows in and out shows that the traffic light should let $x = 600$ vehicles per hour pass through.

Solution (*b*) To avoid traffic congestion, the flow in must equal the flow out at each intersection. For this to happen, the following conditions must be satisfied:

Intersection	Flow In		Flow Out
A	$400 + 600$	$=$	$x_1 + x_2$
B	$x_2 + x_3$	$=$	$400 + x$
C	$500 + 200$	$=$	$x_3 + x_4$
D	$x_1 + x_4$	$=$	700

Thus, with $x = 600$, as computed in part (a), we obtain the following linear system:

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & & & & & = & 1000 \\ & & x_2 & + & x_3 & & & = & 1000 \\ & & & & x_3 & + & x_4 & = & 700 \\ x_1 & & & & & + & x_4 & = & 700 \end{array}$$

We leave it for you to show that the system has infinitely many solutions and that these are given by the parametric equations

$$x_1 = 700 - t, \quad x_2 = 300 + t, \quad x_3 = 700 - t, \quad x_4 = t \qquad (1)$$

However, the parameter t is not completely arbitrary here, since there are physical constraints to be considered. For example, the average flow rates must be nonnegative since we have assumed the streets to be one-way, and a negative flow rate would indicate a flow in the wrong direction. This being the case, we see from (1) that t can be any real number that satisfies $0 \le t \le 700$, which implies that the average flow rates along the streets will fall in the ranges

$$0 \le x_1 \le 700, \quad 300 \le x_2 \le 1000, \quad 0 \le x_3 \le 700, \quad 0 \le x_4 \le 700$$

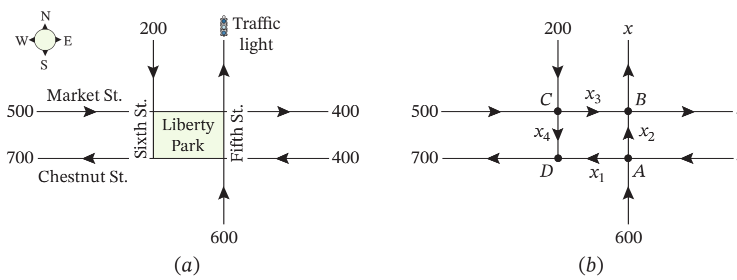


FIGURE 1.10.3

Electrical Circuits

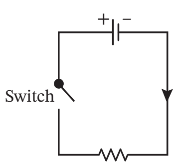


FIGURE 1.10.4

Next we will show how network analysis can be used to analyze electrical circuits consisting of batteries and resistors. A ***battery*** is a source of electric energy, and a ***resistor***, such as a lightbulb, is an element that dissipates electric energy. **Figure 1.10.4** shows a schematic diagram of a circuit with one battery (represented by the symbol), one resistor (represented by the symbol), and a switch. The battery has a ***positive pole*** (+) and a ***negative pole*** (−). When the switch is closed, electrical current is considered to