1. $x = -4,\ y = 22$

2. $x = 13, y = 12$

3. $2x + 1 = 5,\ 3x = 6,\ 3y - 5 = 4$

$x = 2,\ y = 3$

4. $x + 2 = 2x + 6$ $2y = 18$

$-4 = x$ $y = 9$

$2x = -8$ $y + 2 = 11$

$x = -4$ $y = 9$

5. (a) $A + B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1+2 & -1-1 \\ 2-1 & -1+8 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 7 \end{bmatrix}$

(b) $A - B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1-2 & -1+1 \\ 2+1 & -1-8 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 3 & -9 \end{bmatrix}$

(c) $3A = 3\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(-1) \\ 3(2) & 3(-1) \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 6 & -3 \end{bmatrix}$

(d) $3A - 2B = \begin{bmatrix} 3 & -3 \\ 6 & -3 \end{bmatrix} - 2\begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 6 & -3 \end{bmatrix} + \begin{bmatrix} -4 & 2 \\ 2 & -16 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 8 & -19 \end{bmatrix}$

6. (a) $A + B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1-3 & 2-2 \\ 2+4 & 1+2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 6 & 3 \end{bmatrix}$

(b) $A - B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1+3 & 2+2 \\ 2-4 & 1-2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ -2 & -1 \end{bmatrix}$

(c) $3A = 3\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) \\ 3(2) & 3(1) \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 3 \end{bmatrix}$

(d) $3A - 2B = \begin{bmatrix} 3 & 6 \\ 6 & 3 \end{bmatrix} - 2\begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 3+6 & 6+4 \\ 6-8 & 3-4 \end{bmatrix} = \begin{bmatrix} 9 & 10 \\ -2 & -1 \end{bmatrix}$

7 **7.** $A = \begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix},\ B = \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix}$

(a) $A + B = \begin{bmatrix} 7 & 3 \\ 1 & 9 \\ -2 & 15 \end{bmatrix}$ (b) $A - B = \begin{bmatrix} 5 & -5 \\ 3 & -1 \\ -4 & -5 \end{bmatrix}$ (c) $3A = \begin{bmatrix} 18 & -3 \\ 6 & 12 \\ -9 & 15 \end{bmatrix}$

(d) $3A - 2B = \begin{bmatrix} 18 & -3 \\ 6 & 12 \\ -9 & 15 \end{bmatrix} - \begin{bmatrix} 2 & 8 \\ -2 & 10 \\ 2 & 20 \end{bmatrix} = \begin{bmatrix} 16 & -11 \\ 8 & 2 \\ -11 & -5 \end{bmatrix}$

8. (a) $A + B = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 2+2 & 1-3 & 1+4 \\ -1-3 & -1+1 & 4-2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 5 \\ -4 & 0 & 2 \end{bmatrix}$

(b) $A - B = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 2-2 & 1-(-3) & 1-4 \\ -1-(-3) & -1-1 & 4-(-2) \end{bmatrix} = \begin{bmatrix} 0 & 4 & -3 \\ 2 & -2 & 6 \end{bmatrix}$

(c) $3A = 3\begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 3(2) & 3(1) & 3(1) \\ 3(-1) & 3(-1) & 3(4) \end{bmatrix} = \begin{bmatrix} 6 & 3 & 3 \\ -3 & -3 & 12 \end{bmatrix}$

(d) $3A - 2B = \begin{bmatrix} 6 & 3 & 3 \\ -3 & -3 & 12 \end{bmatrix} - 2\begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 3 \\ -3 & -3 & 12 \end{bmatrix} + \begin{bmatrix} -4 & 6 & -8 \\ 6 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 9 & -5 \\ 3 & -5 & 16 \end{bmatrix}$

20. $55\left(\begin{bmatrix}14&-11\\-22&19\end{bmatrix}+\begin{bmatrix}-22&20\\13&6\end{bmatrix}\right)=\begin{bmatrix}-440&495\\-495&1375\end{bmatrix}$

21. $-\begin{bmatrix}3.211&6.829\\-1.004&4.914\\0.055&-3.889\end{bmatrix}-\begin{bmatrix}-1.630&-3.090\\5.256&8.335\\-9.768&4.251\end{bmatrix}=\begin{bmatrix}-1.581&-3.739\\-4.252&-13.249\\9.713&-0.362\end{bmatrix}$

22. $-12\left(\begin{bmatrix}6&20\\1&-9\\-2&5\end{bmatrix}+\begin{bmatrix}14&-15\\-8&-6\\7&0\end{bmatrix}+\begin{bmatrix}-31&-19\\16&10\\24&-10\end{bmatrix}\right)=\begin{bmatrix}132&168\\-108&60\\-348&60\end{bmatrix}$

23. $X=3\begin{bmatrix}-2&-1\\1&0\\3&-4\end{bmatrix}-2\begin{bmatrix}0&3\\2&0\\-4&-1\end{bmatrix}=\begin{bmatrix}-6&-3\\3&0\\9&-12\end{bmatrix}-\begin{bmatrix}0&6\\4&0\\-8&-2\end{bmatrix}=\begin{bmatrix}-6&-9\\-1&0\\17&-10\end{bmatrix}$

24. $2X=2A-B$

2

$$X=A-\tfrac{1}{2}B=\begin{bmatrix}-2&-1\\1&0\\3&-4\end{bmatrix}-\tfrac{1}{2}\begin{bmatrix}0&3\\2&0\\-4&-1\end{bmatrix}=\begin{bmatrix}-2&-1\\1&0\\3&-4\end{bmatrix}-\begin{bmatrix}0&\frac{3}{2}\\1&0\\-2&-\frac{1}{2}\end{bmatrix}=\begin{bmatrix}-2&-\frac{5}{2}\\0&0\\5&-\frac{7}{2}\end{bmatrix}$$

25. $X=-\tfrac{3}{2}A+\tfrac{1}{2}B=-\tfrac{3}{2}\begin{bmatrix}-2&-1\\1&0\\3&-4\end{bmatrix}+\tfrac{1}{2}\begin{bmatrix}0&3\\2&0\\-4&-1\end{bmatrix}=\begin{bmatrix}3&\frac{3}{2}\\-\frac{3}{2}&0\\-\frac{9}{2}&6\end{bmatrix}+\begin{bmatrix}0&\frac{3}{2}\\1&0\\-2&-\frac{1}{2}\end{bmatrix}=\begin{bmatrix}3&3\\-\frac{1}{2}&0\\-\frac{13}{2}&\frac{11}{2}\end{bmatrix}$

26. $2A+4B=-2X$

$$X=-A-2B=-1\begin{bmatrix}-2&-1\\1&0\\3&-4\end{bmatrix}-2\begin{bmatrix}0&3\\2&0\\-4&-1\end{bmatrix}=\begin{bmatrix}2&1\\-1&0\\-3&4\end{bmatrix}+\begin{bmatrix}0&-6\\-4&0\\8&2\end{bmatrix}=\begin{bmatrix}2&-5\\-5&0\\5&6\end{bmatrix}$$

27. A is 3×2 and B is 3×3. AB is not possible.

28. A is 2×4, B is 2×2. AB is not possible.

29. A is 3×3, B is $3\times 2 \implies AB$ is 3×2.

$$\begin{bmatrix}0&-1&0\\4&0&2\\8&-1&7\end{bmatrix}\begin{bmatrix}2&1\\-3&4\\1&6\end{bmatrix}=\begin{bmatrix}(0)(2)+(-1)(-3)+(0)(1)&(0)(1)+(-1)(4)+(0)(6)\\(4)(2)+(0)(-3)+(2)(1)&(4)(1)+(0)(4)+(2)(6)\\(8)(2)+(-1)(-3)+(7)(1)&(8)(1)+(-1)(4)+(7)(6)\end{bmatrix}=\begin{bmatrix}3&-4\\10&16\\26&46\end{bmatrix}$$

30. A is 3×2, B is $2\times 2 \implies AB$ is 3×2.

$$AB=\begin{bmatrix}-1&3\\4&-5\\0&2\end{bmatrix}\begin{bmatrix}1&2\\0&7\end{bmatrix}=\begin{bmatrix}-1&19\\4&-27\\0&14\end{bmatrix}$$

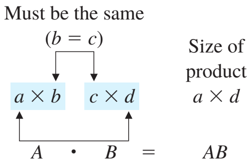

Figure 3

It is important to check sizes before starting the multiplication process. If A is an $a \times b$ matrix and B is a $c \times d$ matrix, then if $b = c$, the product AB will exist and will be an $a \times d$ matrix (see Fig. 3). If $b \neq c$, the product AB does not exist. The definition is not as complicated as it might first seem. An example should help clarify the process.

For

$$A = \begin{bmatrix} 2 & 3 & -1 \\ -2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$$

A is 2×3 and B is 3×2, so AB is 2×2. To find the first row of AB, we take the product of the first row of A with every column of B and write each result as a real number, not as a 1×1 matrix. The second row of AB is computed in the same manner. The four products of row and column matrices used to produce the four elements in AB are shown in the following dashed box. These products are usually calculated mentally or with the aid of a calculator, and need not be written out. The shaded portions highlight the steps involved in computing the element in the first row and second column of AB.

$$\overset{2 \times 3}{\begin{bmatrix} 2 & 3 & -1 \\ -2 & 1 & 2 \end{bmatrix}} \overset{3 \times 2}{\begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}} = \begin{bmatrix} [2 \quad 3 \quad -1]\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} & [2 \quad 3 \quad -1]\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \\ [-2 \quad 1 \quad 2]\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} & [-2 \quad 1 \quad 2]\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (2)(1) + (3)(2) + (-1)(-1) & (2)(3) + (3)(0) + (-1)(2) \\ (-2)(1) + (1)(2) + (2)(-1) & (-2)(3) + (1)(0) + (2)(2) \end{bmatrix} = \overset{2 \times 2}{\begin{bmatrix} 9 & 4 \\ -2 & -2 \end{bmatrix}}$$

EXAMPLE 8 Matrix Multiplication Find the indicated matrix product, if it exists, where:

3

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad E = [2 \quad -3 \quad 0] \quad F = \begin{bmatrix} -5 \\ 2 \\ -2 \end{bmatrix}$$

(A) $$AB = \overset{3 \times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}} \overset{2 \times 4}{\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix}}$$

$$= \begin{bmatrix} (2)(1)+(1)(2) & (2)(-1)+(1)(1) & (2)(0)+(1)(2) & (2)(1)+(1)(0) \\ (1)(1)+(0)(2) & (1)(-1)+(0)(1) & (1)(0)+(0)(2) & (1)(1)+(0)(0) \\ (-1)(1)+(2)(2) & (-1)(-1)+(2)(1) & (-1)(0)+(2)(2) & (-1)(1)+(2)(0) \end{bmatrix}$$

$$= \overset{3 \times 4}{\begin{bmatrix} 4 & -1 & 2 & 2 \\ 1 & -1 & 0 & 1 \\ 3 & 3 & 4 & -1 \end{bmatrix}}$$

3b (B) $$BA = \underset{2\times 4}{\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}} \quad \text{Not defined}$$

c (C) $$CD = \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} (2)(1)+(6)(3) & (2)(2)+(6)(6) \\ (-1)(1)+(-3)(3) & (-1)(2)+(-3)(6) \end{bmatrix} = \begin{bmatrix} 20 & 40 \\ -10 & -20 \end{bmatrix}$$

d (D) $$DC = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} (1)(2)+(2)(-1) & (1)(6)+(2)(-3) \\ (3)(2)+(6)(-1) & (3)(6)+(6)(-3) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

e (E) $$EF = \begin{bmatrix} 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} -5 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} (2)(-5)+(-3)(2)+(0)(-2) \end{bmatrix} = [-16]$$

f (F) $$FE = \begin{bmatrix} -5 \\ 2 \\ -2 \end{bmatrix}\begin{bmatrix} 2 & -3 & 0 \end{bmatrix} = \begin{bmatrix} (-5)(2) & (-5)(-3) & (-5)(0) \\ (2)(2) & (2)(-3) & (2)(0) \\ (-2)(2) & (-2)(-3) & (-2)(0) \end{bmatrix} = \begin{bmatrix} -10 & 15 & 0 \\ 4 & -6 & 0 \\ -4 & 6 & 0 \end{bmatrix}$$

g (G) $$A^{2*} = AA = \underset{3\times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}} \quad \text{Not defined}$$

h (H) $$C^2 = CC = \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} (2)(2)+(6)(-1) & (2)(6)+(6)(-3) \\ (-1)(2)+(-3)(-1) & (-1)(6)+(-3)(-3) \end{bmatrix} = \begin{bmatrix} -2 & -6 \\ 1 & 3 \end{bmatrix}$$

Matched Problem 8 Find each product, if it is defined:

(A) $$\begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$$

(B) $$\begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$$

(C) $$\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$$

(D) $$\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

(E) $$\begin{bmatrix} 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

(F) $$\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 3 & -2 & 1 \end{bmatrix}$$

*Following standard algebraic notation, we write $A^2 = AA$, $A^3 = AAA$, and so on.

$1A = A$	*Scalar unit*
$0A = O$	*Scalar zero*
$A(BC) = (AB)C$	*Multiplicative associative law*
$c(AB) = (cA)B$	*Multiplicative associative law*
$c(dA) = (cd)A$	*Multiplicative associative law*
$AI = IA = A$	*Multiplicative identity law*
$A(B + C) = AB + AC$	*Distributive law*
$(A + B)C = AC + BC$	*Distributive law*
$OA = AO = O$	*Multiplication by zero matrix*

Note that we have not included a multiplicative commutative law for matrices, because the equation $AB = BA$ does not hold in general. In other words, matrix multiplication is *not* exactly like multiplication of numbers. (You have to be a little careful because it is easy to apply the commutative law without realizing it.)

We should also say a bit more about transposition. Transposition and multiplication have an interesting relationship. We write down the properties of transposition again, adding one new one.

Properties of Transposition

$$(A + B)^T = A^T + B^T$$
$$(cA)^T = c(A^T)$$
$$(AB)^T = B^T A^T$$

Notice the change in order in the last one. The order is crucial.

quick **Examples**

4

1. $\left(\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 5 & -1 \end{bmatrix}\right)^T = \begin{bmatrix} -2 & 1 \\ 10 & -2 \end{bmatrix}^T = \begin{bmatrix} -2 & 10 \\ 1 & -2 \end{bmatrix}$ $\quad (AB)^T$

2. $\begin{bmatrix} 3 & 0 \\ 5 & -1 \end{bmatrix}^T\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 10 \\ 1 & -2 \end{bmatrix}$ $\quad B^T A^T$

3. $\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}^T\begin{bmatrix} 3 & 0 \\ 5 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -3 & -7 \end{bmatrix}$ $\quad A^T B^T$

These properties give you a glimpse of the field of mathematics known as **abstract algebra.** Algebraists study operations like these that resemble the operations on numbers but differ in some way, such as the lack of commutativity for multiplication seen here.

We end this section with more on the relationship between linear equations and matrix equations, which is one of the important applications of matrix multiplication.

Example 7 Matrix Form of a System of Linear Equations

a. If

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -1 \\ -3 & 1 & 1 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ and } B = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

rewrite the matrix equation $AX = B$ as a system of linear equations.

application like this, not only do the dimensions have to match, but also the *labels* have to match for the result to be meaningful. The labels on the three columns of P are the parts that were sold, and these are also the labels on the three rows of Q. Therefore, we can "cancel labels" at the same time that we cancel the dimensions in the product. However, the labels on the two columns of Q do not match the labels on the two rows of P, and there is no useful interpretation of the product QP in this situation. ■

There are very special square matrices of every size: $1 \times 1, 2 \times 2, 3 \times 3$, and so on, called the **identity** matrices.

Identity Matrix

The $n \times n$ identity matrix I is the matrix with 1s down the **main diagonal** (the diagonal starting at the top left) and 0s everywhere else. In symbols,

$$I_{ii} = 1, \quad \text{and}$$
$$I_{ij} = 0 \quad \text{if } i \neq j$$

quick Examples

1. 1×1 identity matrix $\quad I = [1]$

2. 2×2 identity matrix $\quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3. 3×3 identity matrix $\quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. 4×4 identity matrix $\quad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Note Identity matrices are always square matrices, meaning that they have the same number of rows as columns. There is no such thing, for example, as the "2×4 identity matrix."

The next example shows why I is interesting. ■

5

Example 6 Identity Matrix

Evaluate the products AI and IA, where $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and I is the 3×3 identity matrix.

Solution

First notice that A is arbitrary; it could be any 3×3 matrix.

$$AI = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

5

using *Technology*

See the Technology Guides at the end of the chapter to see how to get an identity matrix using a TI-83/84 or Excel.
In the online Matrix Algebra Tool at
Chapter 3
→ Tools
→ Matrix Algebra Tool
use `I` in a formula to refer to the identity matrix of any dimension. The program will choose the correct dimension in the context of the formula. For example, if A is a 3 x 3 matrix, then the expression `I-A` uses the 3 x 3 identity matrix for `I`.

and

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

In both cases, the answer is the matrix A we started with. In symbols,

$$AI = A$$

and

$$IA = A$$

no matter which 3×3 matrix A you start with. Now this should remind you of a familiar fact from arithmetic:

$$a \cdot 1 = a$$

and

$$1 \cdot a = a$$

That is why we call the matrix I the 3×3 *identity* matrix, because it appears to play the same role for 3×3 matrices that the identity 1 does for numbers.

Before we go on... Try a similar calculation using 2×2 matrices: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, let I be the 2×2 identity matrix, and check that $AI = IA = A$. In fact, the equation

$$AI = IA = A$$

works for square matrices of every dimension. It is also interesting to notice that $AI = A$ if I is the 2×2 identity matrix and A is any 3×2 matrix (try one). In fact, if I is any identity matrix, then $AI = A$ whenever the product is defined, and $IA = A$ whenever this product is defined. ■

We can now add to the list of properties we gave for matrix arithmetic at the end of Section 3.1 by writing down properties of matrix multiplication. In stating these properties, we shall assume that all matrix products we write are defined—that is, that the matrices have correctly matching dimensions. The first eight properties are the ones we've already seen; the rest are new.

Properties of Matrix Addition and Multiplication

If A, B and C are matrices, if O is a zero matrix, and if I is an identity matrix, then the following hold:

$A + (B + C) = (A + B) + C$	*Additive associative law*
$A + B = B + A$	*Additive commutative law*
$A + O = O + A = A$	*Additive identity law*
$A + (-A) = O = (-A) + A$	*Additive inverse law*
$c(A + B) = cA + cB$	*Distributive law*
$(c + d)A = cA + dA$	*Distributive law*

6

42. (a) $AB = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 2(0) + (-1)3 & 2(0) + (-1)(-3) \\ 1(0) + 4(3) & 1(0) + 4(-3) \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 12 & -12 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 0 & 0 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 0(2) + (0)1 & 0(-1) + (0)(4) \\ 3(2) + (-3)(1) & 3(-1) + (-3)4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 3 & -15 \end{bmatrix}$

(c) $A^2 = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2(2) + (-1)(1) & 2(-1) + (-1)4 \\ 1(2) + 4(1) & 1(-1) + 4(4) \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 6 & 15 \end{bmatrix}$

7

43. (a) $AB = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} (3)(1) + (-1)(3) & (3)(-3) + (-1)(1) \\ (1)(1) + (3)(3) & (1)(-3) + (3)(1) \end{bmatrix} = \begin{bmatrix} 0 & -10 \\ 10 & 0 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} (1)(3) + (-3)(1) & (1)(-1) + (-3)(3) \\ (3)(3) + (1)(1) & (3)(-1) + (1)(3) \end{bmatrix} = \begin{bmatrix} 0 & -10 \\ 10 & 0 \end{bmatrix}$

(c) $A^2 = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} (3)(3) + (-1)(1) & (3)(-1) + (-1)(3) \\ (1)(3) + (3)(1) & (1)(-1) + (3)(3) \end{bmatrix} = \begin{bmatrix} 8 & -6 \\ 6 & 8 \end{bmatrix}$

44. (a) $AB = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1(1) + (-1)(-3) & 1(3) + (-1)(1) \\ 1(1) + 1(-3) & 1(3) + 1(1) \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1(1) + (3)1 & 1(-1) + 3(1) \\ -3(1) + (1)(1) & -3(-1) + 1(1) \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix}$

(c) $A^2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1(1) + (-1)(1) & 1(-1) + (-1)(1) \\ 1(1) + (1)(1) & 1(-1) + 1(1) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

45. (a) $AB = \begin{bmatrix} 7 \\ 8 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 7(1) & 7(1) & 7(2) \\ 8(1) & 8(1) & 8(2) \\ -1(1) & -1(1) & -1(2) \end{bmatrix} = \begin{bmatrix} 7 & 7 & 14 \\ 8 & 8 & 16 \\ -1 & -1 & -2 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 8 \\ -1 \end{bmatrix} = [(1)(7) + (1)(8) + (2)(-1)] = [13]$

(c) A^2 is not possible.

46. (a) $AB = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = [3(2) + 2(3) + 1(0)] = [12]$

(b) $BA = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2(3) & 2(2) & 2(1) \\ 3(3) & 3(2) & 3(1) \\ 0(3) & 0(2) & 0(1) \end{bmatrix} = \begin{bmatrix} 6 & 4 & 2 \\ 9 & 6 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

(c) The number of columns of A does not equal the number of rows of A; the multiplication is not possible.

47. $\begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ -4 & -16 \end{bmatrix}$

48. $3\left(\begin{bmatrix} 6 & 5 & -1 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -1 & -3 \\ 4 & 1 \end{bmatrix}\right) = -3\left(\begin{bmatrix} 6(0) + 5(-1) + (-1)(4) & 6(3) + 5(-3) + (-1)(1) \\ 1(0) + (-2)(-1) + (0)(4) & 1(3) + (-2)(-3) + (0)(1) \end{bmatrix}\right)$

$= -3\begin{bmatrix} -9 & 2 \\ 2 & 9 \end{bmatrix}$

$= \begin{bmatrix} 27 & -6 \\ -6 & -27 \end{bmatrix}$

5. Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Find

(a) AI (b) IA

6. Let

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 3 & -3 \end{pmatrix}$$

Find

(a) AB (b) BA

7. Let

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$$

Find

(a) AB (b) BA

8. True or false? (Assume that all operations are defined.)

(TRUE – FALSE) $A + B = B + A$

(TRUE – FALSE) $AB = BA$ see 6 vs. 7

(TRUE – FALSE) $(AB)^T = A^T B^T$

(TRUE – FALSE) $(AB)^T = B^T A^T$ see 4

(TRUE – FALSE) $IA = AI = A$

see 5

$$
\begin{pmatrix} 225 & 110 & 50 \\ 75 & 180 & 225 \\ 280 & 85 & 110 \\ 0 & 250 & 225 \end{pmatrix} \cdot \begin{pmatrix} 40 \\ 60 \\ 80 \end{pmatrix} = \begin{array}{cl} \begin{pmatrix} 19\,600 \\ 31\,800 \\ 25\,100 \\ 33\,000 \end{pmatrix} & \begin{array}{l} \text{Ozo} \\ \text{Atlas} \\ \text{Aero} \\ \text{Mat} \end{array} \end{array}
$$

$$
\overline{\quad 109\,500 \quad}
$$