10. $AB = \frac{1}{3}\begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3 & 1 & 1 & -3 \\ -3 & -1 & 2 & -3 \\ 0 & 1 & 1 & 0 \\ -3 & -2 & 1 & 0 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 3-3+3 & -1-1+2 & -1+2-1 & 3-3 \\ -3+3 & 1+1+1 & 1-2+1 & -3+3 \\ 3-3 & -1-1+2 & -1+2+2 & 3-3 \\ 3-3 & 1+1-2 & -2+1+1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$BA = \frac{1}{3}\begin{bmatrix} -3 & 1 & 1 & -3 \\ -3 & -1 & 2 & -3 \\ 0 & 1 & 1 & 0 \\ -3 & -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 3+1-1 & -3-1+1+3 & 1+2-3 & 3-3 \\ 3-1-2 & -3+1+2+3 & -1+4-3 & 3-3 \\ 1-1 & -1+1 & 1+2 & 0 \\ 3-2-1 & -3+2+1 & -2+2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

11. $[A \;\vdots\; I] = \left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right]$

$\begin{array}{l} \frac{1}{2}R_1 \rightarrow \\ \frac{1}{3}R_2 \rightarrow \end{array} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{3} \end{array}\right] = [I \;\vdots\; A^{-1}]$

$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

12. $[A \;\vdots\; I] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 7 & 0 & 1 \end{array}\right]$

1a

$-3R_1 + R_2 \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{array}\right]$

$-2R_2 + R_1 \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 7 & -2 \\ 0 & 1 & -3 & 1 \end{array}\right] = [I \;\vdots\; A^{-1}]$

$A^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$

1b

13. $[A \;\vdots\; I] = \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right]$

$-2R_1 + R_2 \rightarrow \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$

$2R_2 + R_1 \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right] = [I \;\vdots\; A^{-1}]$

$A^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$

14. $[A \;\vdots\; I] = \left[\begin{array}{cc|cc} -7 & 33 & 1 & 0 \\ 4 & -19 & 0 & 1 \end{array}\right]$

$2R_2 + R_1 \rightarrow \left[\begin{array}{cc|cc} 1 & -5 & 1 & 2 \\ 4 & -19 & 0 & 1 \end{array}\right]$

$-4R_1 + R_2 \rightarrow \left[\begin{array}{cc|cc} 1 & -5 & 1 & 2 \\ 0 & 1 & -4 & -7 \end{array}\right]$

$5R_2 + R_1 \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -19 & -33 \\ 0 & 1 & -4 & -7 \end{array}\right] = [I \;\vdots\; A^{-1}]$

$A^{-1} = \begin{bmatrix} -19 & -33 \\ -4 & -7 \end{bmatrix}$

15. $[A \;\vdots\; I] = \begin{bmatrix} -1 & 1 & \vdots & 1 & 0 \\ -2 & 1 & \vdots & 0 & 1 \end{bmatrix}$

$-R_2 + R_1 \longrightarrow \begin{bmatrix} 1 & 0 & \vdots & 1 & -1 \\ -2 & 1 & \vdots & 0 & 1 \end{bmatrix}$

$2R_1 + R_2 \longrightarrow \begin{bmatrix} 1 & 0 & \vdots & 1 & -1 \\ 0 & 1 & \vdots & 2 & -1 \end{bmatrix} = [I \;\vdots\; A^{-1}]$

$A^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

16. $[A \;\vdots\; I] = \begin{bmatrix} 11 & 1 & \vdots & 1 & 0 \\ -1 & 0 & \vdots & 0 & 1 \end{bmatrix}$

$10R_2 + R_1 \longrightarrow \begin{bmatrix} 1 & 1 & \vdots & 1 & 10 \\ -1 & 0 & \vdots & 0 & 1 \end{bmatrix}$

$R_1 + R_2 \longrightarrow \begin{bmatrix} 1 & 1 & \vdots & 1 & 10 \\ 0 & 1 & \vdots & 1 & 11 \end{bmatrix}$

$-R_2 + R_1 \longrightarrow \begin{bmatrix} 1 & 0 & \vdots & 0 & -1 \\ 0 & 1 & \vdots & 1 & 11 \end{bmatrix} = [I \;\vdots\; A^{-1}]$

$A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 11 \end{bmatrix}$

1d

17. $[A \;\vdots\; I] = \begin{bmatrix} 2 & 4 & \vdots & 1 & 0 \\ 4 & 8 & \vdots & 0 & 1 \end{bmatrix}$

$-2R_1 + R_2 \longrightarrow \begin{bmatrix} 2 & 4 & \vdots & 1 & 0 \\ 0 & 0 & \vdots & -2 & 1 \end{bmatrix}$

The two zeros in the second row imply that the inverse does not exist.

1e

18. $[A \;\vdots\; I] = \begin{bmatrix} 2 & 3 & \vdots & 1 & 0 \\ 1 & 4 & \vdots & 0 & 1 \end{bmatrix}$

$\begin{matrix} R_2 \\ R_1 \end{matrix} \begin{bmatrix} 1 & 4 & \vdots & 0 & 1 \\ 2 & 3 & \vdots & 1 & 0 \end{bmatrix}$

$-2R_1 + R_2 \longrightarrow \begin{bmatrix} 1 & 4 & \vdots & 0 & 1 \\ 0 & -5 & \vdots & 1 & -2 \end{bmatrix}$

$-\frac{1}{5}R_2 \longrightarrow \begin{bmatrix} 1 & 4 & \vdots & 0 & 1 \\ 0 & 1 & \vdots & -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$

$-4R_2 + R_1 \longrightarrow \begin{bmatrix} 1 & 0 & \vdots & \frac{4}{5} & -\frac{3}{5} \\ 0 & 1 & \vdots & -\frac{1}{5} & \frac{2}{5} \end{bmatrix} = [I \;\vdots\; A^{-1}]$

$A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

1e

19. $A = \begin{bmatrix} 2 & 7 & 1 \\ -3 & -9 & 2 \end{bmatrix}$ A has no inverse because it is not square.

20. $A = \begin{bmatrix} -2 & 5 \\ 6 & -15 \\ 0 & 1 \end{bmatrix}$ A has no inverse because it is not square.

2a

21. $[A \;\vdots\; I] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 & 0 & 0 \\ 3 & 5 & 4 & \vdots & 0 & 1 & 0 \\ 3 & 6 & 5 & \vdots & 0 & 0 & 1 \end{bmatrix}$

$\begin{matrix} \\ -3R_1 + R_2 \longrightarrow \\ -3R_1 + R_3 \longrightarrow \end{matrix} \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 2 & 1 & \vdots & -3 & 1 & 0 \\ 0 & 3 & 2 & \vdots & -3 & 0 & 1 \end{bmatrix}$

$\frac{1}{2}R_2 \longrightarrow \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \vdots & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 3 & 2 & \vdots & -3 & 0 & 1 \end{bmatrix}$

$\begin{matrix} -R_2 + R_1 \longrightarrow \\ \\ -3R_2 + R_3 \longrightarrow \end{matrix} \begin{bmatrix} 1 & 0 & \frac{1}{2} & \vdots & \frac{5}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \vdots & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \vdots & \frac{3}{2} & -\frac{3}{2} & 1 \end{bmatrix}$

$\begin{matrix} -R_3 + R_1 \longrightarrow \\ -R_3 + R_2 \longrightarrow \\ \\ \end{matrix} \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 & 1 & -1 \\ 0 & 1 & 0 & \vdots & -3 & 2 & -1 \\ 0 & 0 & \frac{1}{2} & \vdots & \frac{3}{2} & -\frac{3}{2} & 1 \end{bmatrix}$

$\begin{matrix} \\ \\ 2R_3 \longrightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 & 1 & -1 \\ 0 & 1 & 0 & \vdots & -3 & 2 & -1 \\ 0 & 0 & 1 & \vdots & 3 & -3 & 2 \end{bmatrix} = [I \;\vdots\; A^{-1}]$

$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{bmatrix}$

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22. $$[A \;\vdots\; I] = \left[\begin{array}{rrr|rrr} 1 & 2 & 2 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 1 & 0 \\ -1 & -4 & -7 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -3R_1 + R_2 \rightarrow \\ R_1 + R_3 \rightarrow \end{array} \left[\begin{array}{rrr|rrr} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -3 & 1 & 0 \\ 0 & -2 & -5 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} -2R_2 + R_1 \rightarrow \\ \\ 2R_2 + R_3 \rightarrow \end{array} \left[\begin{array}{rrr|rrr} 1 & 0 & -4 & 7 & -2 & 0 \\ 0 & 1 & 3 & -3 & 1 & 0 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{array}\right]$$

$$\begin{array}{r} 4R_3 + R_1 \rightarrow \\ -3R_3 + R_2 \rightarrow \\ \\ \end{array} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -13 & 6 & 4 \\ 0 & 1 & 0 & 12 & -5 & -3 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{array}\right] = [I \;\vdots\; A^{-1}]$$

$$A^{-1} = \begin{bmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{bmatrix}$$

23. $$[A \;\vdots\; I] = \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 2 & 5 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -3R_1 + R_2 \rightarrow \\ -2R_1 + R_3 \rightarrow \end{array} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & -3 & 1 & 0 \\ 0 & 5 & 5 & -2 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -\frac{5}{4}R_2 + R_3 \rightarrow \end{array} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & -3 & 1 & 0 \\ 0 & 0 & 5 & \frac{7}{4} & -\frac{5}{4} & 1 \end{array}\right]$$

$$\begin{array}{r} \\ \frac{1}{4}R_2 \rightarrow \\ \frac{1}{5}R_3 \rightarrow \end{array} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{7}{20} & -\frac{1}{4} & \frac{1}{5} \end{array}\right] = [I \;\vdots\; A^{-1}]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & \frac{1}{4} & 0 \\ \frac{7}{20} & -\frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

24. $$[A \;\vdots\; I] = \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 5 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{r} \\ -3R_1 + R_2 \rightarrow \\ -2R_1 + R_3 \rightarrow \end{array} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 & 0 \\ 0 & 5 & 5 & -2 & 0 & 1 \end{array}\right]$$

Since the first three entries of row 2 are all zeros, the inverse of A does not exist.

25. $$[A \;\vdots\; I] = \left[\begin{array}{rrrr|rrrr} -8 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -5 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} -\frac{1}{8}R_1 \rightarrow \\ \\ \frac{1}{4}R_3 \rightarrow \\ -\frac{1}{5}R_4 \rightarrow \end{array} \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & -\frac{1}{8} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{5} \end{array}\right] = [I \;\vdots\; A^{-1}]$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{8} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & -\frac{1}{5} \end{bmatrix}$$

Although there is a general formula for the inverse of a matrix, it is not a simple one. In fact, using the formula for anything larger than a 3×3 matrix is so inefficient that the row-reduction procedure is the method of choice even for computers. However, the general formula is very simple for the special case of 2×2 matrices:

Formula for the Inverse of a 2 × 2 Matrix

The inverse of a 2×2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \qquad \text{provided } ad - bc \neq 0$$

If the quantity $ad - bc$ is zero, then the matrix is singular (noninvertible). The quantity $ad - bc$ is called the **determinant** of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

quick Examples

1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \dfrac{1}{(1)(4) - (2)(3)}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = -\dfrac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$

2. $\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$ has determinant $ad - bc = (1)(-2) - (-1)(2) = 0$ and so is singular.

The formula for the inverse of a 2×2 matrix can be obtained using the technique of row reduction. (See the Communication and Reasoning Exercises at the end of the section.)

As we have mentioned above, not every square matrix has an inverse, as we see in the next example.

2e

Example 3 Singular 3 × 3 Matrix

Find the inverse of the matrix $S = \begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & 4 \\ 3 & 1 & -2 \end{bmatrix}$, if it exists.

Solution We proceed as before.

$$\begin{array}{c} \qquad S \qquad\qquad\qquad I \\ \left[\begin{array}{rrr|rrr} 1 & 1 & 2 & 1 & 0 & 0 \\ -2 & 0 & 4 & 0 & 1 & 0 \\ 3 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 - 3R_1 \end{array} \rightarrow \left[\begin{array}{rrrrrr} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 8 & 2 & 1 & 0 \\ 0 & -2 & -8 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} 2R_1 - R_2 \\ \\ R_3 + R_2 \end{array}$$

$$\rightarrow \left[\begin{array}{rrrrrr} 2 & 0 & -4 & 0 & -1 & 0 \\ 0 & 2 & 8 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

We stopped here, even though the reduction is incomplete, because there is *no hope* of getting the identity on the left-hand side. Completing the row reduction will not change the three zeros in the bottom row. So what is wrong? Nothing. As in Example 1, we have here a singular matrix. Any square matrix that, after row reduction, winds up with a row of zeros is singular (see Exercise 77).

Finding the Inverse of a Matrix

Given the $n \times n$ matrix A:

1. Adjoin the $n \times n$ identity matrix I to obtain the augmented matrix

$$[A \mid I]$$

2. Use a sequence of row operations to reduce $[A \mid I]$ to the form

$$[I \mid B]$$

if possible.

Then the matrix B is the inverse of A.

Note Although matrix multiplication is not generally commutative, it is possible to prove that if A has an inverse and $AB = I$, then $BA = I$ also. Hence, to verify that B is the inverse of A, it suffices to show that $AB = I$.

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EXAMPLE 1 Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

Solution We form the augmented matrix

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

and use the Gauss–Jordan elimination method to reduce it to the form $[I \mid B]$:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} -1 & -1 & 0 & 1 & -1 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 + 3R_1 \\ R_3 + 2R_1}]{-R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 3 & -2 & 0 \\ 0 & -1 & 2 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{-R_2 \\ R_3 - R_2}]{R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 + R_3]{R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

The inverse of A is the matrix

$$A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

We leave it to you to verify these results.

Example 2 illustrates what happens to the reduction process when a matrix A does *not* have an inverse.

A Method for Inverting Matrices

As a first application of Theorem 1.5.3, we will develop a procedure (or algorithm) that can be used to tell whether a given matrix is invertible, and if so, produce its inverse. To derive this algorithm, assume for the moment, that A is an invertible $n \times n$ matrix. In Equation (3), the elementary matrices execute a sequence of row operations that reduce A to I_n. If we multiply both sides of this equation on the right by A^{-1} and simplify, we obtain

$$A^{-1} = E_k \cdots E_2 E_1 I_n$$

But this equation tells us that *the same sequence of row operations that reduces A to I_n will transform I_n to A^{-1}*. Thus, we have established the following result.

Inversion Algorithm To find the inverse of an invertible matrix A, find a sequence of elementary row operations that reduces A to the identity and then perform that same sequence of operations on I_n to obtain A^{-1}.

A simple method for carrying out this procedure is given in the following example.

EXAMPLE 4 | Using Row Operations to Find A^{-1}

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Solution We want to reduce A to the identity matrix by row operations and simultaneously apply these operations to I to produce A^{-1}. To accomplish this we will adjoin the identity matrix to the right side of A, thereby producing a partitioned matrix of the form

$$[A \mid I]$$

Then we will apply row operations to this matrix until the left side is reduced to I; these operations will convert the right side to A^{-1}, so the final matrix will have the form

$$[I \mid A^{-1}]$$

The computations are as follows:

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

We added -2 times the first row to the second and -1 times the first row to the third.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

We added 2 times the second row to the third.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We multiplied the third row by -1.

2e

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We added 3 times the third row to the second and -3 times the third row to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We added -2 times the second row to the first.

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Often it will not be known in advance if a given $n \times n$ matrix A is invertible. However, if it is not, then by parts (a) and (c) of Theorem 1.5.3 it will be impossible to reduce A to I_n by elementary row operations. This will be signaled by a row of zeros appearing on the *left side* of the partition at some stage of the inversion algorithm. If this occurs, then you can stop the computations and conclude that A is not invertible.

EXAMPLE 5 | Showing That a Matrix Is Not Invertible

Consider the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Applying the procedure of Example 4 yields

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

We added -2 times the first row to the second and added the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

We added the second row to the third.

Since we have obtained a row of zeros on the left side, A is not invertible.

EXAMPLE 6 | Analyzing Homogeneous Systems

Use Theorem 1.5.3 to determine whether the given homogeneous system has nontrivial solutions.

(a)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 \qquad\;\; + 8x_3 &= 0 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 6x_2 + 4x_3 &= 0 \\ 2x_1 + 4x_2 - x_3 &= 0 \\ -x_1 + 2x_2 + 5x_3 &= 0 \end{aligned}$$

Solution From parts (a) and (b) of Theorem 1.5.3 a homogeneous linear system has only the trivial solution if and only if its coefficient matrix is invertible. From Examples 4 and 5 the coefficient matrix of system (a) is invertible and that of system (b) is not. Thus, system (a) has only the trivial solution while system (b) has nontrivial solutions.

3

If a product of matrices is singular, then at least one of the factors must be singular. Why?

We leave it for you to show that

$$AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}, \quad (AB)^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

and also that

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}, \quad B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Thus, $(AB)^{-1} = B^{-1}A^{-1}$ as guaranteed by Theorem 1.4.6.

Powers of a Matrix

If A is a *square* matrix, then we define the nonnegative integer powers of A to be

$$\boxed{A^0 = I \quad \text{and} \quad A^n = AA\cdots A \quad [n \text{ factors}]}$$

and if A is invertible, then we define the negative integer powers of A to be

$$\boxed{A^{-n} = (A^{-1})^n = A^{-1}A^{-1}\cdots A^{-1} \quad [n \text{ factors}]}$$

Because these definitions parallel those for real numbers, the usual laws of nonnegative exponents hold; for example,

$$\boxed{A^rA^s = A^{r+s} \quad \text{and} \quad (A^r)^s = A^{rs}}$$

In addition, we have the following properties of negative exponents.

Theorem 1.4.7

If A is invertible and n is a nonnegative integer, then:

(*a*) A^{-1} is invertible and $(A^{-1})^{-1} = A$.

(*b*) A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.

(*c*) kA is invertible for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$.

We will prove part (*c*) and leave the proofs of parts (*a*) and (*b*) as exercises.

***Proof* (c)** Properties (*m*) and (*l*) of Theorem 1.4.1 imply that

$$(kA)(k^{-1}A^{-1}) = k^{-1}(kA)A^{-1} = (k^{-1}k)AA^{-1} = (1)I = I$$

and similarly, $(k^{-1}A^{-1})(kA) = I$. Thus, kA is invertible and $(kA)^{-1} = k^{-1}A^{-1}$. ■

EXAMPLE 10 | Properties of Exponents

Let A and A^{-1} be the matrices in Example 9; that is,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Then

$$A^{-3} = (A^{-1})^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

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The transpose of the identity matrix is still the identity matrix $I^T = I$. Knowing this and using our above result it's quick to get the transpose of an inverse

$$AA^{-1} = I = I^T = (AA^{-1})^T = (A^{-1})^T A^T$$

So, the inverse of A^T is $(A^{-1})^T$. Stated otherwise $(A^T)^{-1} = (A^{-1})^T$. In words, the inverse of the transpose is the transpose of the inverse.

Example - Find A^T and A^{-1} and $(A^{-1})^T$ and $(A^T)^{-1}$ for

$$A = \begin{pmatrix} 1 & 0 \\ 9 & 3 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 9 \\ 0 & 3 \end{pmatrix} \qquad (A^T)^{-1} = \begin{pmatrix} 1 & -3 \\ 0 & \frac{1}{3} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & \frac{1}{3} \end{pmatrix} \qquad (A^{-1})^T = \begin{pmatrix} 1 & -3 \\ 0 & \frac{1}{3} \end{pmatrix}$$

2 Symmetric Matrices

A symmetric matrix is a matrix that is its own transpose. Stated slightly more mathematically, a matrix A is symmetric if $A = A^T$. Note that, obviously, all symmetric matrices are square matrices.

For example, the matrix

5. True or false? (Assume that all operations are defined.)

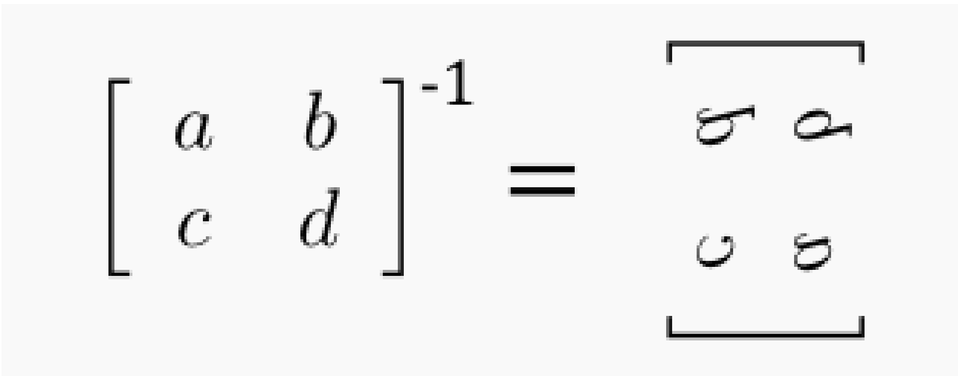
(TRUE – FALSE) $A + B = B + A$

(TRUE – FALSE) $AB = BA$

(TRUE – FALSE) $(AB)^T = A^T B^T$

(TRUE – FALSE) $(AB)^T = B^T A^T$

(TRUE – FALSE) $IA = AI = A$



Source 1: http://mathfail.com/movabletype/mt/mt-search.cgi?blog_id=4&tag=matrices&limit=20