Section 8.4 The Determinant of a Square Matrix

- You should be able to determine the determinant of a matrix of order 2×2 by using the difference of the products of the diagonals.
- You should be able to use expansion by cofactors to find the determinant of a matrix of order 3×3 or greater.
- The determinant of a triangular matrix equals the product of the entries on the main diagonal.

Vocabulary Check

1. determinant **2.** minor **3.** cofactor **4.** expanding by cofactors

1. 5

2. -8

3. $\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 2(4) - 1(3) = 8 - 3 = 5$

1a

4. $\begin{vmatrix} -3 & 1 \\ 5 & 2 \end{vmatrix} = (-3)(2) - (5)(1) = -11$

1d

5. $\begin{vmatrix} 5 & 2 \\ -6 & 3 \end{vmatrix} = 5(3) - 2(-6) = 15 + 12 = 27$

1b

6. $\begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = (2)(3) - (4)(-2) = 14$

7. $\begin{vmatrix} -7 & 0 \\ 3 & 0 \end{vmatrix} = -7(0) - 0(3) = 0$

8. $\begin{vmatrix} 4 & -3 \\ 0 & 0 \end{vmatrix} = (4)(0) - (0)(-3) = 0$

9. $\begin{vmatrix} 2 & 6 \\ 0 & 3 \end{vmatrix} = 2(3) - 6(0) = 6$

1c

10. $\begin{vmatrix} 2 & -3 \\ -6 & 9 \end{vmatrix} = (2)(9) - (-6)(-3) = 0$

11. $\begin{vmatrix} -3 & -2 \\ -6 & -1 \end{vmatrix} = (-3)(-1) - (-2)(-6) = 3 - 12 = -9$

12. $\begin{vmatrix} 4 & 7 \\ -2 & 5 \end{vmatrix} = (4)(5) - (-2)(7) = 34$

13. $\begin{vmatrix} 9 & 0 \\ 7 & 8 \end{vmatrix} = 9(8) - 0(7) = 72 - 0 = 72$

14. $\begin{vmatrix} 0 & 6 \\ -3 & 2 \end{vmatrix} = (0)(2) - (-3)(6) = 18$

15. $\begin{vmatrix} -\frac{1}{2} & \frac{1}{3} \\ -6 & \frac{1}{3} \end{vmatrix} = -\frac{1}{2}\left(\frac{1}{3}\right) - \frac{1}{3}(-6) = -\frac{1}{6} + 2 = \frac{11}{6}$

16. $\begin{vmatrix} \frac{2}{3} & \frac{4}{3} \\ -1 & -\frac{1}{3} \end{vmatrix} = \left(\frac{2}{3}\right)\left(-\frac{1}{3}\right) - (-1)\left(\frac{4}{3}\right) = \frac{10}{9}$

17. $\begin{vmatrix} 0.3 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 \\ -0.4 & 0.4 & 0.3 \end{vmatrix} = -0.002$

18. $\begin{vmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{vmatrix} = -0.022$

19. $\begin{vmatrix} 0.9 & 0.7 & 0 \\ -0.1 & 0.3 & 1.3 \\ -2.2 & 4.2 & 6.1 \end{vmatrix} = -4.842$

20. $\begin{vmatrix} 0.1 & 0.1 & -4.3 \\ 7.5 & 6.2 & 0.7 \\ 0.3 & 0.6 & -1.2 \end{vmatrix} = -11.217$

2a

21. $\begin{vmatrix} 1 & 4 & -2 \\ 3 & 6 & -6 \\ -2 & 1 & 4 \end{vmatrix} = 0$

2b

22. $\begin{vmatrix} 2 & 3 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & -2 \end{vmatrix} = -20$

Finding the Determinant of a 3×3 Matrix

The determinant of a matrix is a value associated with a matrix. You can only find the determinant of a square matrix (2 rows and 2 columns, 3 rows and 3 columns, etc.). The determinant of the matrix can be used to solve systems of equations, but first we need to discuss how to find the determinant of a matrix. Here we will be learning how to find the determinant of a 3×3 matrix.

There are a few different ways to find the determinant of a 3×3 matrix. I have chosen to show this technique because it seems a lot easier for me to remember. I do not have to worry about things like minors and cofactors. If you know another way to find the determinant of a 3×3 matrix consider giving this technique a try.

First let's take care of the notation used for determinants. If we were using matrix A, it would be denoted as [A]. To express the determinant of matrix A, we use the notation |A|.

Here are the steps we follow to find the determinant of a 3×3 matrix:

Step 1:	Rewrite the first two columns of the matrix.
Step 2:	Multiply diagonally downward and diagonally upward.
Step 3:	Add the downward numbers together.
Step 4:	Add the upward numbers together.
Step 5:	Subtract the upward sum from the downward sum to get the determinant.

The steps are a lot easier to understand when you look at some examples, so here we go.

2e

Example 1: Find the determinant of $\begin{vmatrix} -3 & 3 & 2 \\ 5 & 4 & -1 \\ 2 & 1 & 4 \end{vmatrix}$.

Step 1: Rewrite the first two columns of the matrix.

$$\begin{vmatrix} -3 & 3 & 2 \\ 5 & 4 & -1 \\ 2 & 1 & 4 \end{vmatrix} \rightarrow \left|\begin{array}{ccc|cc} -3 & 3 & 2 & -3 & 3 \\ 5 & 4 & -1 & 5 & 4 \\ 2 & 1 & 4 & 2 & 1 \end{array}\right.$$

Step 2: Multiply diagonally downward and diagonally upward.

$$\begin{array}{ccccc} & & 16 & 3 & 60 \\ -3 & 3 & 2 & -3 & 3 \\ 5 & 4 & -1 & 5 & 4 \\ 2 & 1 & 4 & 2 & 1 \\ & & -48 & -6 & 10 \end{array}$$

Step 3: Add the downward numbers together.

–48 + (–6) + 10 = –44

Step 4: Add the upward numbers together.

16 + 3 + 60 = 79

Step 5: Subtract the upward sum from the downward sum to get the determinant.

–44 – 79 = –123

The determinant of $\begin{vmatrix} -3 & 3 & 2 \\ 5 & 4 & -1 \\ 2 & 1 & 4 \end{vmatrix}$ is –123.

2d

Example 2: Find the determinant of $\begin{vmatrix} 2 & -3 & 4 \\ -1 & 4 & 5 \\ -3 & -2 & 1 \end{vmatrix}$.

Step 1: Rewrite the first two columns of the matrix.

$$\begin{vmatrix} 2 & -3 & 4 \\ -1 & 4 & 5 \\ -3 & -2 & 1 \end{vmatrix} \rightarrow \begin{vmatrix} 2 & -3 & 4 \\ -1 & 4 & 5 \\ -3 & -2 & 1 \end{vmatrix} \begin{matrix} 2 & -3 \\ -1 & 4 \\ -3 & -2 \end{matrix}$$

Step 2: Multiply diagonally downward and diagonally upward.

$$\begin{array}{ccccc} & & -48 & -20 & 3 \\ \left|\begin{matrix} 2 \\ -1 \\ -3 \end{matrix}\right. & \begin{matrix} -3 \\ 4 \\ -2 \end{matrix} & \left.\begin{matrix} 4 \\ 5 \\ 1 \end{matrix}\right| & \begin{matrix} 2 \\ -1 \\ -3 \end{matrix} & \begin{matrix} -3 \\ 4 \\ -2 \end{matrix} \\ & & 8 & 45 & 8 \end{array}$$

Step 3: Add the downward numbers together.

8 + 45 + 8 = 61

Step 4: Add the upward numbers together.

–48 + (–20) + 3 = –65

Step 5: Subtract the upward sum from the downward sum to get the determinant.

61 – (–65) = 126

The determinant of $\begin{vmatrix} 2 & -3 & 4 \\ -1 & 4 & 5 \\ -3 & -2 & 1 \end{vmatrix}$ is 126.

33. (a) $\begin{vmatrix} 5 & 0 & -3 \\ 0 & 12 & 4 \\ 1 & 6 & 3 \end{vmatrix} = 0\begin{vmatrix} 0 & -3 \\ 6 & 3 \end{vmatrix} + 12\begin{vmatrix} 5 & -3 \\ 1 & 3 \end{vmatrix} - 4\begin{vmatrix} 5 & 0 \\ 1 & 6 \end{vmatrix} = 0(18) + 12(18) - 4(30) = 96$

(b) $\begin{vmatrix} 5 & 0 & -3 \\ 0 & 12 & 4 \\ 1 & 6 & 3 \end{vmatrix} = 0\begin{vmatrix} 0 & 4 \\ 1 & 3 \end{vmatrix} + 12\begin{vmatrix} 5 & -3 \\ 1 & 3 \end{vmatrix} - 6\begin{vmatrix} 5 & -3 \\ 0 & 4 \end{vmatrix} = 0(-4) + 12(18) - 6(20) = 96$

34. (a) $\begin{vmatrix} 10 & -5 & 5 \\ 30 & 0 & 10 \\ 0 & 10 & 1 \end{vmatrix} = 0\begin{vmatrix} -5 & 5 \\ 0 & 10 \end{vmatrix} - 10\begin{vmatrix} 10 & 5 \\ 30 & 10 \end{vmatrix} + \begin{vmatrix} 10 & -5 \\ 30 & 0 \end{vmatrix} = 0(-50) - 10(-50) + 150 = 650$

(b) $\begin{vmatrix} 10 & -5 & 5 \\ 30 & 0 & 10 \\ 0 & 10 & 1 \end{vmatrix} = 10\begin{vmatrix} 0 & 10 \\ 10 & 1 \end{vmatrix} - 30\begin{vmatrix} -5 & 5 \\ 10 & 1 \end{vmatrix} + 0\begin{vmatrix} -5 & 5 \\ 0 & 10 \end{vmatrix} = 10(-100) - 30(-55) + 0(-50) = 650$

35. (a) $\begin{vmatrix} 6 & 0 & -3 & 5 \\ 4 & 13 & 6 & -8 \\ -1 & 0 & 7 & 4 \\ 8 & 6 & 0 & 2 \end{vmatrix} = -4\begin{vmatrix} 0 & -3 & 5 \\ 0 & 7 & 4 \\ 6 & 0 & 2 \end{vmatrix} + 13\begin{vmatrix} 6 & -3 & 5 \\ -1 & 7 & 4 \\ 8 & 0 & 2 \end{vmatrix} - 6\begin{vmatrix} 6 & 0 & 5 \\ -1 & 0 & 4 \\ 8 & 6 & 2 \end{vmatrix} - 8\begin{vmatrix} 6 & 0 & -3 \\ -1 & 0 & 7 \\ 8 & 6 & 0 \end{vmatrix}$

3a

$= -4(-282) + 13(-298) - 6(-174) - 8(-234) = 170$

(b) $\begin{vmatrix} 6 & 0 & -3 & 5 \\ 4 & 13 & 6 & -8 \\ -1 & 0 & 7 & 4 \\ 8 & 6 & 0 & 2 \end{vmatrix} = 0\begin{vmatrix} 4 & 6 & -8 \\ -1 & 7 & 4 \\ 8 & 0 & 2 \end{vmatrix} + 13\begin{vmatrix} 6 & -3 & 5 \\ -1 & 7 & 4 \\ 8 & 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 6 & -3 & 5 \\ 4 & 6 & -8 \\ 8 & 0 & 2 \end{vmatrix} + 6\begin{vmatrix} 6 & -3 & 5 \\ 4 & 6 & -8 \\ -1 & 7 & 4 \end{vmatrix}$

$= 0 + 13(-298) + 0 + 6(674) = 170$

36. (a) $\begin{vmatrix} 10 & 8 & 3 & -7 \\ 4 & 0 & 5 & -6 \\ 0 & 3 & 2 & 7 \\ 1 & 0 & -3 & 2 \end{vmatrix} = 0\begin{vmatrix} 8 & 3 & -7 \\ 0 & 5 & -6 \\ 0 & -3 & 2 \end{vmatrix} - 3\begin{vmatrix} 10 & 3 & -7 \\ 4 & 5 & -6 \\ 1 & -3 & 2 \end{vmatrix} + 2\begin{vmatrix} 10 & 8 & -7 \\ 4 & 0 & -6 \\ 1 & 0 & 2 \end{vmatrix} - 7\begin{vmatrix} 10 & 8 & 3 \\ 4 & 0 & 5 \\ 1 & 0 & -3 \end{vmatrix}$

$= 0(-64) - 3(-3) + 2(-112) - 7(136) = -1167$

(b) $\begin{vmatrix} 10 & 8 & 3 & -7 \\ 4 & 0 & 5 & -6 \\ 0 & 3 & 2 & 7 \\ 1 & 0 & -3 & 2 \end{vmatrix} = 10\begin{vmatrix} 0 & 5 & -6 \\ 3 & 2 & 7 \\ 0 & -3 & 2 \end{vmatrix} - 4\begin{vmatrix} 8 & 3 & -7 \\ 3 & 2 & 7 \\ 0 & -3 & 2 \end{vmatrix} + 0\begin{vmatrix} 8 & 3 & -7 \\ 0 & 5 & -6 \\ 0 & -3 & 2 \end{vmatrix} - 1\begin{vmatrix} 8 & 3 & -7 \\ 0 & 5 & -6 \\ 3 & 2 & 7 \end{vmatrix}$

$= 10(24) - 4(245) + 0(-64) - 1(427) = -1167$

37. Expand along Column 1.

$\begin{vmatrix} 2 & -1 & 0 \\ 4 & 2 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} - 4\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} + 4\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = 2(0) - 4(-1) + 4(-1) = 0$

38. Expand along Row 3.

$\begin{vmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{vmatrix} = 0\begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 1 & 0 \end{vmatrix} + 4\begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix}$

$= 0(3) - 1(-3) + 4(0) = 3$

39. Expand along Row 2.

$$\begin{vmatrix} 6 & 3 & -7 \\ 0 & 0 & 0 \\ 4 & -6 & 3 \end{vmatrix} = 0\begin{vmatrix} 3 & -7 \\ -6 & 3 \end{vmatrix} - 0\begin{vmatrix} 6 & -7 \\ 4 & 3 \end{vmatrix} + 0\begin{vmatrix} 6 & 3 \\ 4 & -6 \end{vmatrix} = 0$$

40. Expand along Column 3.

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ -2 & 0 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= 2(2) - 0(2) + 3(-2) = -2$$

41. $\begin{vmatrix} -1 & 2 & -5 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{vmatrix} = (-1)(3)(3) = -9$ (Upper triangular)

42. Expand along Row 1.

$$\begin{vmatrix} 1 & 0 & 0 \\ -4 & -1 & 0 \\ 5 & 1 & 5 \end{vmatrix} = 1\begin{vmatrix} -1 & 0 \\ 1 & 5 \end{vmatrix} - 0\begin{vmatrix} -4 & 0 \\ 5 & 5 \end{vmatrix} + 0\begin{vmatrix} -4 & -1 \\ 5 & 1 \end{vmatrix}$$

$$= 1(-5) - 0(-20) + 0(1) = -5$$

43. Expand along Column 3.

$$\begin{vmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{vmatrix} = -2\begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix}$$

$$= -2(14) + 3(-10) = -58$$

44. Expand along Row 3.

$$\begin{vmatrix} 2 & -1 & 3 \\ 1 & 4 & 4 \\ 1 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & 3 \\ 4 & 4 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} = 1(-16) - 0(5) + 2(9) = 2$$

45. $\begin{vmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{vmatrix} = (2)(3)(-5) = -30$ (Upper triangular)

46. Expand along Row 1.

$$\begin{vmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{vmatrix} = -3\begin{vmatrix} 11 & 0 \\ 2 & 2 \end{vmatrix} - 0\begin{vmatrix} 7 & 0 \\ 1 & 2 \end{vmatrix} + 0\begin{vmatrix} 7 & 11 \\ 1 & 2 \end{vmatrix}$$

$$= -3(22) - 0(14) + 0(3) = -66$$

47. Expand along Column 3.

$$\begin{vmatrix} 2 & 6 & 6 & 2 \\ 2 & 7 & 3 & 6 \\ 1 & 5 & 0 & 1 \\ 3 & 7 & 0 & 7 \end{vmatrix} = 6\begin{vmatrix} 2 & 7 & 6 \\ 1 & 5 & 1 \\ 3 & 7 & 7 \end{vmatrix} - 3\begin{vmatrix} 2 & 6 & 2 \\ 1 & 5 & 1 \\ 3 & 7 & 7 \end{vmatrix} = 6(-20) - 3(16) = -168$$

48. Expand along Row 2.

3b

$$\begin{vmatrix} 3 & 6 & -5 & 4 \\ -2 & 0 & 6 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 3 & -1 & -1 \end{vmatrix} = -(-2)\begin{vmatrix} 6 & -5 & 4 \\ 1 & 2 & 2 \\ 3 & -1 & -1 \end{vmatrix} - 6\begin{vmatrix} 3 & 6 & 4 \\ 1 & 1 & 2 \\ 0 & 3 & -1 \end{vmatrix} = 2(-63) - 6(-3) = -108$$

49. Expand along Column 1.

$$\begin{vmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{vmatrix} = 5\begin{vmatrix} 6 & 4 & 12 \\ 2 & -3 & 4 \\ 1 & -2 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & 0 & 6 \\ 2 & -3 & 4 \\ 1 & -2 & 2 \end{vmatrix} = 5(0) - 4(0) = 0$$

$$\det(A) = (-1)^{1+1}(2)\det\begin{bmatrix} 6 & 7 \\ 9 & 1 \end{bmatrix} + (-1)^{1+2}(3)\det\begin{bmatrix} 5 & 7 \\ 8 & 1 \end{bmatrix} + (-1)^{1+3}(4)\det\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$$
$$= (1)(2)(-57) + (-1)(3)(-51) + (1)(4)(-3) = -114 + 153 - 12 = 27.$$

Cofactor If A is a square matrix, the **ij^{th} cofactor of A** is defined to be $(-1)^{i+j}\det(A_{ij})$. The notation C_{ij} will sometimes be used to denote the ij^{th} cofactor of A.

Example Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Then $C_{11} = (-1)^{1+1}\det\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} = (1)(45 - 48) = -3$,

$C_{12} = (-1)^{1+2}\det\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} = (-1)(36-42) = 6$ and $C_{23} = (-1)^{2+3}\det\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} = (-1)(8-14) = 6$.

In the definition of the determinant, part (2) consists of multiplying each first row entry of A by its cofactor and then summing these products. For this reason it is called a first row cofactor expansion.

3c

Example Let $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 \\ 5 & 3 & 4 & 1 \\ 4 & 2 & 3 & 5 \end{bmatrix}$. Use a first row cofactor expansion to evaluate det(A).

Solution det(A) =

$$(-1)^{1+1}(2)\det\begin{bmatrix} 3 & 2 & 4 \\ 3 & 4 & 1 \\ 2 & 3 & 5 \end{bmatrix} + (-1)^{1+2}(3)\det\begin{bmatrix} 1 & 2 & 4 \\ 5 & 4 & 1 \\ 4 & 3 & 5 \end{bmatrix} + (-1)^{1+3}(4)\det\begin{bmatrix} 1 & 3 & 4 \\ 5 & 3 & 1 \\ 4 & 2 & 5 \end{bmatrix} + (-1)^{1+4}(5)\det\begin{bmatrix} 1 & 3 & 2 \\ 5 & 3 & 4 \\ 4 & 2 & 3 \end{bmatrix}$$

$$=(1)(2)\left\{(-1)^{1+1}(3)\det\begin{bmatrix} 4 & 1 \\ 3 & 5 \end{bmatrix} + (-1)^{1+2}(2)\det\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} + (-1)^{1+3}(4)\det\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\right\}$$

$$+(-1)(3)\left\{(-1)^{1+1}(1)\det\begin{bmatrix} 4 & 1 \\ 3 & 5 \end{bmatrix} + (-1)^{1+2}(2)\det\begin{bmatrix} 5 & 1 \\ 4 & 5 \end{bmatrix} + (-1)^{1+3}(4)\det\begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix}\right\}$$

$$+(1)(4)\left\{(-1)^{1+1}(1)\det\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} + (-1)^{1+2}(3)\det\begin{bmatrix} 5 & 1 \\ 4 & 5 \end{bmatrix} + (-1)^{1+3}(4)\det\begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix}\right\}$$

$$+(-1)(5)\left\{(-1)^{1+1}(1)\det\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} + (-1)^{1+2}(3)\det\begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix} + (-1)^{1+3}(2)\det\begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix}\right\}$$

$=(1)(2)\{(1)(3)(20-3) + (-1)(2)(15-2) + (1)(4)(9-8)\}$
$\quad +(-1)(3)\{(1)(1)(20-3) + (-1)(2)(25-4) + (1)(4)(15-16)\}$
$\quad +(1)(4)\{(1)(1)(15-2) + (-1)(3)(25-4) + (1)(4)(10-12)\}$
$\quad +(-1)(5)\{(1)(1)(9-8) + (-1)(3)(15-16) + (1)(2)(10-12)\}$
$=(1)(2)\{51-26+4\} + (-1)(3)\{17-42-4\} + (1)(4)\{13-63-8\} + (-1)(5)\{1+3-4\}$
$=58 + 87 - 232 + 0 = -87$

50. Expand along Row 3.

$$\begin{vmatrix} 1 & 4 & 3 & 2 \\ -5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \end{vmatrix} = 0$$

51. Expand along Column 2, then along Column 4.

21

$$\begin{vmatrix} 3 & 2 & 4 & -1 & 5 \\ -2 & 0 & 1 & 3 & 2 \\ 1 & 0 & 0 & 4 & 0 \\ 6 & 0 & 2 & -1 & 0 \\ 3 & 0 & 5 & 1 & 0 \end{vmatrix} = -2\begin{vmatrix} -2 & 1 & 3 & 2 \\ 1 & 0 & 4 & 0 \\ 6 & 2 & -1 & 0 \\ 3 & 5 & 1 & 0 \end{vmatrix} = (-2)(-2)\begin{vmatrix} 1 & 0 & 4 \\ 6 & 2 & -1 \\ 3 & 5 & 1 \end{vmatrix} = 4(103) = 412$$

52. Expand along Column 1.

$$\begin{vmatrix} 5 & 2 & 0 & 0 & -2 \\ 0 & 1 & 4 & 3 & 2 \\ 0 & 0 & 2 & 6 & 3 \\ 0 & 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} = 5\begin{vmatrix} 1 & 4 & 3 & 2 \\ 0 & 2 & 6 & 3 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 5 \cdot 1\begin{vmatrix} 2 & 6 & 3 \\ 3 & 4 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 5(-20) = -100$$

53. $\begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 8 & 1 & 6 \end{vmatrix} = -126$

54. $\begin{vmatrix} 5 & -8 & 0 \\ 9 & 7 & 4 \\ -8 & 7 & 1 \end{vmatrix} = 223$

55. $\begin{vmatrix} 7 & 0 & -14 \\ -2 & 5 & 4 \\ -6 & 2 & 12 \end{vmatrix} = 0$

56. $\begin{vmatrix} 3 & 0 & 0 \\ -2 & 5 & 0 \\ 12 & 5 & 7 \end{vmatrix} = 105$

57. $\begin{vmatrix} 1 & -1 & 8 & 4 \\ 2 & 6 & 0 & -4 \\ 2 & 0 & 2 & 6 \\ 0 & 2 & 8 & 0 \end{vmatrix} = -336$

58. $\begin{vmatrix} 0 & -3 & 8 & 2 \\ 8 & 1 & -1 & 6 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{vmatrix} = 7441$

59. $\begin{vmatrix} 3 & -2 & 4 & 3 & 1 \\ -1 & 0 & 2 & 1 & 0 \\ 5 & -1 & 0 & 3 & 2 \\ 4 & 7 & -8 & 0 & 0 \\ 1 & 2 & 3 & 0 & 2 \end{vmatrix} = 410$

60. $\begin{vmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{vmatrix} = -48$

61. (a) $\begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} = -3$

(b) $\begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} = -2$

(c) $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$

(d) $\begin{vmatrix} -2 & 0 \\ 0 & -3 \end{vmatrix} = 6$

62. (a) $|A| = \begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix} = 0$

(b) $|B| = \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = -1$

(c) $AB = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 4 & 10 \end{bmatrix}$

(d) $|AB| = \begin{vmatrix} -2 & -5 \\ 4 & 10 \end{vmatrix} = 0$

Exercises

1. Find the determinant

   (a) $\begin{vmatrix} -3 & 1 \\ 5 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix}$ (c) $\begin{vmatrix} 2 & -3 \\ -6 & 9 \end{vmatrix}$ (d) $\begin{vmatrix} 5 & 2 \\ -6 & 3 \end{vmatrix}$

2. Find the determinant

   (a) $\begin{vmatrix} 1 & 4 & -2 \\ 3 & 6 & -6 \\ -2 & 1 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 3 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & -2 \end{vmatrix}$ (c) $\begin{vmatrix} -3 & 3 & 2 \\ 5 & 4 & -1 \\ 2 & 1 & 4 \end{vmatrix}$ (d) $\begin{vmatrix} 2 & -3 & 4 \\ -1 & 4 & 5 \\ -3 & -2 & 1 \end{vmatrix}$

3. Find the determinant

   (a) $\begin{vmatrix} 6 & 0 & -3 & 5 \\ 4 & 13 & 6 & -8 \\ -1 & 0 & 7 & 4 \\ 8 & 6 & 0 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 3 & 6 & -5 & 4 \\ -2 & 0 & 6 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 3 & -1 & -1 \end{vmatrix}$ (c) $\begin{vmatrix} 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 \\ 5 & 3 & 4 & 1 \\ 4 & 2 & 3 & 5 \end{vmatrix}$

4. Find the determinant $\begin{vmatrix} 3 & 2 & 4 & -1 & 5 \\ -2 & 0 & 1 & 3 & 2 \\ 1 & 0 & 0 & 4 & 0 \\ 6 & 0 & 2 & -1 & 0 \\ 3 & 0 & 5 & 1 & 0 \end{vmatrix}$

5. Explore the following properties (Choose suitable matrices (2x2 is enough), give some hypotesis and verify.

   (a) $\det 3A = ?$ $3^n \cdot \det A$ (n is size (n×n))

   (b) $\det AB = ?$ $\det A \cdot \det B$

   (c) $\det A^{-1} = ?$ $1/\det A$

   (d) $\det A^T = ?$ $\det A$

   (e) $\det(A + B) = ?$ nothing :(

   (f) det of a matrix with two interchanged rows? $= -\det A$

   (g) det of a matrix with two interchanged columns? $= -\det A$

Lecture 4f

Calculating the Determinant Using Row Operations

(pages 268-9)

So far, we've seen that determinant calculations get easier when a matrix has zero entries. And it is particularly easy to calculate the determinant of triangular matrices (either upper- or lower-). Most recently we've seen that it is easy to keep track of the changes that occur to a determinant when we apply elementary row operations to a matrix. So, what we'll do now is use elementary row operations to find a row equivalent matrix whose determinant is easy to calculate, and then compensate for the changes to the determinant that took place.

Summarizing the results of the previous lecture, we have the following:

Summary: If A is an $n \times n$ matrix, then

(1) if B is obtained from A by multiplying one row of A by the non-zero scalar r, $\det B = r \det A$
(2) if B is obtained from A by interchanging two rows, then $\det B = -\det A$
(3) if B is obtained from A by adding a scalar multiple of one row to another, then $\det B = \det A$.

Notice that with the third operation, there is no change to the determinant:

6a

Example: Let $A = \begin{bmatrix} 3 & 7 & -9 \\ 6 & 1 & 4 \\ -9 & 5 & 2 \end{bmatrix}$. Then we can get a row echelon form matrix B that is row equivalent to A as follows:

$$A = \begin{bmatrix} 3 & 7 & -9 \\ 6 & 1 & 4 \\ -9 & 5 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + 3R_1 \end{matrix} \sim \begin{bmatrix} 3 & 7 & -9 \\ 0 & -13 & 22 \\ 0 & 26 & -25 \end{bmatrix} \; R_3 + 2R_2$$
$$\sim \begin{bmatrix} 3 & 7 & -9 \\ 0 & -13 & 22 \\ 0 & 0 & 19 \end{bmatrix} = B.$$

B is an upper-triangular matrix, so $\det B$ is the product of its diagonal entries: $(3)(-13)(19) = -741$. And since the only row operations used to get from A to B where operations of type (3), we have that $\det A = \det B$, and so $\det A = -741$.

In theory, you can get a matrix into row echelon form only using operations of type (3), but the reality of such a process can get quite complicated. So let's look at an example using all the types of operations.

65

Example: Let $A = \begin{bmatrix} 0 & 5 & -2 & -4 \\ 2 & 4 & -2 & 8 \\ -3 & 4 & -1 & 1 \\ 5 & 5 & -8 & 9 \end{bmatrix}$. Then we can get a row echelon form matrix B that is row equivalent to A as follows:

$$A = \begin{bmatrix} 0 & 5 & -2 & -4 \\ 2 & 4 & -2 & 8 \\ -3 & 4 & -1 & 1 \\ 5 & 5 & -8 & 9 \end{bmatrix} \begin{array}{l} R_1 \updownarrow R_2 \\ \\ \\ \\ \end{array} \sim B_1 = \begin{bmatrix} 2 & 4 & -2 & 8 \\ 0 & 5 & -2 & -4 \\ -3 & 4 & -1 & 1 \\ 5 & 5 & -8 & 9 \end{bmatrix} \begin{array}{l} (1/2)R_1 \\ \\ \\ \\ \end{array}$$

$$\sim B_2 = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & -2 & -4 \\ -3 & 4 & -1 & 1 \\ 5 & 5 & -8 & 9 \end{bmatrix} \begin{array}{l} \\ \\ R_3 + 3R_1 \\ R_4 - 5R_1 \end{array} \sim B_3 = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & -2 & -4 \\ 0 & 10 & -4 & 13 \\ 0 & -5 & -3 & -11 \end{bmatrix} \begin{array}{l} \\ \\ R_3 - 2R_2 \\ R_4 + R_2 \end{array}$$

$$\sim B_4 = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & -2 & -4 \\ 0 & 0 & 0 & 21 \\ 0 & 0 & -5 & -15 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_3 \updownarrow R_4 \end{array} \sim B = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & -2 & -4 \\ 0 & 0 & -5 & -15 \\ 0 & 0 & 0 & 21 \end{bmatrix}.$$

Since B is an upper-triangular matrix, we know that $\det B = (1)(5)(-5)(21) = -525$. Now we need to get $\det A$ from $\det B$. To do this, we will look over the row operations we used. The first row operation we used was a row swap, which means we need to multiply the determinant by (-1), giving us $\det B_1 = -\det A$. The next row operation was to multiply row 1 by 1/2, so we have that $\det B_2 = (1/2)\det B_1 = (1/2)(-1)\det A$. The next matrix was obtained from B_2 by adding multiples of row 1 to rows 3 and 4. Since these row operations do not change the determinant, we have $\det B_3 = \det B_2 = (1/2)(-1)\det A$. B_4 was obtained from B_3 by adding a multiples of row 2 to rows 3 and 4. Again, these row operations do not change the determinant, so we have $\det B_4 = \det B_3 = (1/2)(-1)\det A$. Our final matrix, B, was obtained from B_4 by swapping two rows. This means that $\det B = -\det B_4 = (-1)(1/2)(-1)\det A$.

But we already know $\det B$; what we want to know is $\det A$. So, we solve for $\det A$, and get $\det A = (-1)(2)(-1)\det B = 2\det B = -1050$.

There are some important lessons to be learned from this example. The first is that we can ignore any row operations of type (3), since they do not affect the determinant. The second is that two row swaps cancel each other out, in terms of calculating the determinant, so we only care whether an odd or even number of row swaps take place. The third thing to notice is the most important, and it is the fact that if you multiply a row by r in your row reduction from A to B, then you will need to multiply $\det B$ by (1/r) to get to $\det A$. If the original theorem had been stated as $\det A = (1/r)\det B$ instead of $\det B = r\det A$, then this would have been more clear. And so, we have developed the following technique for calculating the determinant of A:

EXAMPLE 2 Using Cramer's Rule to Solve a Linear System

Use Cramer's rule to solve the system:

$$\begin{cases} 5x - 4y = 2 \\ 6x - 5y = 1. \end{cases}$$

Solution Because

$$x = \frac{D_x}{D} \quad \text{and} \quad y = \frac{D_y}{D},$$

we will set up and evaluate the three determinants D, D_x, and D_y.

1. D, the determinant in both denominators, consists of the x- and y-coefficients.

$$D = \begin{vmatrix} 5 & -4 \\ 6 & -5 \end{vmatrix} = (5)(-5) - (6)(-4) = -25 + 24 = -1$$

Because this determinant is not zero, we continue to use Cramer's rule to solve the system.

2. D_x, the determinant in the numerator for x, is obtained by replacing the x-coefficients in D, 5 and 6, by the constants on the right sides of the equations, 2 and 1.

$$D_x = \begin{vmatrix} 2 & -4 \\ 1 & -5 \end{vmatrix} = (2)(-5) - (1)(-4) = -10 + 4 = -6$$

3. D_y, the determinant in the numerator for y, is obtained by replacing the y-coefficients in D, -4 and -5, by the constants on the right sides of the equations, 2 and 1.

$$D_y = \begin{vmatrix} 5 & 2 \\ 6 & 1 \end{vmatrix} = (5)(1) - (6)(2) = 5 - 12 = -7$$

4. Thus,

$$x = \frac{D_x}{D} = \frac{-6}{-1} = 6 \quad \text{and} \quad y = \frac{D_y}{D} = \frac{-7}{-1} = 7.$$

As always, the solution (6, 7) can be checked by substituting these values into the original equations. The solution set is $\{(6, 7)\}$.

Check Point 2 Use Cramer's rule to solve the system:

$$\begin{cases} 5x + 4y = 12 \\ 3x - 6y = 24. \end{cases}$$

3 Evaluate a third-order determinant.

The Determinant of a 3 × 3 Matrix

Associated with every square matrix is a real number called its determinant. The determinant for a 3 × 3 matrix is defined as follows:

Definition of a Third-Order Determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

These four third-order determinants are given by

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ These are the coefficients of the variables x, y, and z.

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$ Replace x-coefficients in D with the constants on the right of the three equations.

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$ Replace y-coefficients in D with the constants on the right of the three equations.

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$ Replace z-coefficients in D with the constants on the right of the three equations.

EXAMPLE 5 Using Cramer's Rule to Solve a Linear System in Three Variables

7b

Use Cramer's rule to solve:

$$\begin{cases} x + 2y - z = -4 \\ x + 4y - 2z = -6 \\ 2x + 3y + z = 3. \end{cases}$$

Solution Because

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad \text{and} \quad z = \frac{D_z}{D},$$

we need to set up and evaluate four determinants.

Step 1 Set up the determinants.

1. D, the determinant in all three denominators, consists of the x-, y-, and z-coefficients.

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 4 & -2 \\ 2 & 3 & 1 \end{vmatrix}$$

2. D_x, the determinant in the numerator for x, is obtained by replacing the x-coefficients in D, 1, 1, and 2, with the constants on the right sides of the equations, -4, -6, and 3.

$$D_x = \begin{vmatrix} -4 & 2 & -1 \\ -6 & 4 & -2 \\ 3 & 3 & 1 \end{vmatrix}$$

3. D_y, the determinant in the numerator for y, is obtained by replacing the y-coefficients in D, 2, 4, and 3, with the constants on the right sides of the equations, -4, -6, and 3.

$$D_y = \begin{vmatrix} 1 & -4 & -1 \\ 1 & -6 & -2 \\ 2 & 3 & 1 \end{vmatrix}$$

4. D_z, the determinant in the numerator for z, is obtained by replacing the z-coefficients in D, -1, -2, and 1, with the constants on the right sides of the equations, -4, -6, and 3.

$$D_z = \begin{vmatrix} 1 & 2 & -4 \\ 1 & 4 & -6 \\ 2 & 3 & 3 \end{vmatrix}$$