6th lesson

https://www2.karlin.mff.cuni.cz/~kuncova/en/teaching.php
kuncova@karlin.mff.cuni.cz

Theory

Definition 1. Let $k, n \in \mathbb{N}$ and $\vec{v}^1, \dots, \vec{v}^k \in \mathbb{R}^n$. We say that a vector $\vec{u} \in \mathbb{R}^n$ is a *linear combination of the vectors* $\vec{v}^1, \dots, \vec{v}^k$ *with coefficients* $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ if

$$\vec{u} = \lambda_1 \vec{v}^1 + \cdots + \lambda_k \vec{v}^k.$$

By a *trivial linear combination* of vectors $\vec{v}^1, \dots, \vec{v}^k$ we mean the linear combination $0 \cdot \vec{v}^1 + \cdots + 0 \cdot \vec{v}^k$. Linear combination which is not trivial is called *non-trivial.*

Definition 2. We say that vectors $\vec{v}^1, \dots, \vec{v}^k \in \mathbb{R}^n$ are *linearly dependent* if there exists their non-trivial linear combination which is equal to the zero vector. We say that vectors $\vec{v}^1, \dots, \vec{v}^k \in \mathbb{R}^n$ are *linearly independent* if they are not linearly dependent, i.e. if whenever $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ satisfy $\lambda_1 \vec{v}^1 + \cdots + \lambda_k \vec{v}^k = \vec{o}$, then $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$.

Remarks 3. Vectors $\vec{v}^1, \dots, \vec{v}^k$ are linearly dependent if and only if one of them can be expressed as a linear combination of the others.

Definition 4. Let $\boldsymbol{A} \in M(m \times n)$. The *rank* of the matrix $\boldsymbol{A}$ is the maximal number of linearly independent row vectors of $\boldsymbol{A}$, i.e. the rank is equal to $k \in \mathbb{N}$ if

1. there is k linearly independent row vectors of $\boldsymbol{A}$ and

2. each l-tuple of row vectors of $\boldsymbol{A}$, where $l > k$, is linearly dependent.

The rank of the zero matrix is zero. Rank of $\boldsymbol{A}$ is denoted by $\operatorname{rank}(\boldsymbol{A})$.

Remarks 5. It can be shown that $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}^T)$ for any $\boldsymbol{A} \in M(m \times n)$.

Theorem 6. Let $\boldsymbol{A} \in M(n \times n)$. Then $\boldsymbol{A}$ is invertible if and only if $\operatorname{rank}(\boldsymbol{A}) = n$.

Theorem 7 (Rouché-Fontené)**.** The system $(A|b)$ has a solution if and only if its coefficient matrix has the same rank as its augmented matrix.

Exercises

1. Find the rank of the matrices

   (a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 2 & 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 5 & 3 \\ 2 & 6 & 2 \\ 3 & 7 & 1 \\ 4 & 8 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

2. Decide whether the vectors are linearly dependent or independent.

   (a) $(-1, 3)$, $(5, 6)$, $(1, 4)$.

   (b) $(1, 2, 3)$, $(-2, 1, 0)$, $(1, 0, 1)$.

   (c) $(1, 2, 3, 2)$, $(2, 5, 5, 5)$, $(2, 6, 4, 6)$.

   (d) $(1, 3, -1, 0)$, $(4, 9, -2, 1)$, $(2, 3, 0, 1)$.

   (e) $(3, 1, 6)$, $(2, 0, 4)$, $(2, 1, 4)$ (with determinant).

3. Express the vector

   (a) $(6, 6)$ as the linear combination of $(0, 3)$ and $(2, 1)$.

   (b) $(9, 6)$ as the linear combination of $(1, 2)$ and $(1, -4)$.

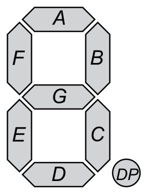
   (c) $(1, 5, 4)$ as the linear combination of $(1, 0, 3)$, $(0, 2, 0)$ and $(0, 3, 1)$.

4. Write the polynomial $x + 1$ as a linear combination of $2x^2 - x + 1$ and $-x^2 + x$.

5. What can you say about solutions of this systems? About rank of matrix and augmented matrix?

   (a) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right)$ (b) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{array}\right)$ (c) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

6. Let us imagine the 7-segments display



Source 1: `https://en.wikipedia.org/wiki/File:7_Segment_Display_with_Labeled_Segments.svg`

At this display we can put on the light various characters. List of all 128 of them can be found here: `https://en.wikipedia.org/wiki/File:7-segment.svg`.

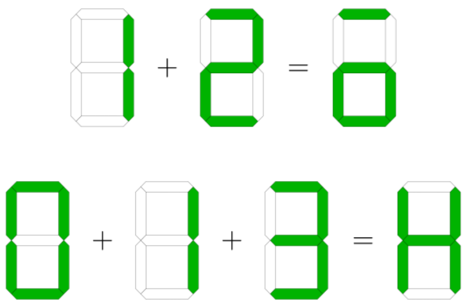
Especially we can express letters:

Latin alphabet

Case	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Upper	A		C		E	F	G	H	I	J		L			O	P			S		U					
Lower		b	c	d				h	i					n	o		q	r		t	u				y	

Source 2: `https://en.wikipedia.org/wiki/Seven-segment_display`

Furthermore, we can define 'addition' of these characters (it is XOR):

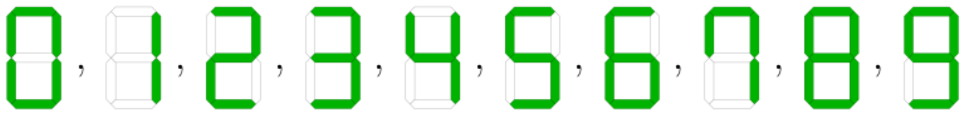


Source 3: http://isoptera.lcsc.edu/hamoon/files/Math340S19/ApplInspiredLACh7.pdf

Your tasks:

(a) Choose a short word (5 letters max) and send it to your classmates wihtout spelling it. Pretend, you have only dumbphone. Of course, you should settle some code with your classmates and you should use our new 7-segments system.

(b) Choose a character (Character means any combinations of 7 LCD panels, not only letters or numbers). Then try to express it with combination of this 10 digits:



Source 4: http://isoptera.lcsc.edu/hamoon/files/Math340S19/ApplInspiredLACh8.pdf

(c) Is it possible to omit some of this 10 digits and still have all possible combinations of characters?