The proof is simple and left to you as an exercise.

Example 4. The ranks of the following matrices are 2 and 3, respectively.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\square$

Lemmas 1 and 3 suggest the following approach to compute the rank of a matrix $\boldsymbol{A}$. First, convert $\boldsymbol{A}$ to a matrix $\boldsymbol{A'}$ of row echelon form, and then, count the number of non-zero rows of $\boldsymbol{A'}$.

Example 5. Next, we use the approach to calculate the rank of the matrix in Example 2 (in the derivation below, $\Rightarrow$ indicates applying row elementary operations):

1a

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

rank = 2

$\square$

Example 6. Compute the rank of the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

Solution.

1b

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 2 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -4 & -8 & -12 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the original matrix has rank 2. $\square$

Lemma 3. *Suppose that* $\boldsymbol{r_1}$, $\boldsymbol{r_2}$, ..., $\boldsymbol{r_k}$ *are linearly independent, but* $\boldsymbol{r_1}$, $\boldsymbol{r_2}$, ..., $\boldsymbol{r_{k+1}}$ *are linearly dependent. Then,* $\boldsymbol{r_{k+1}}$ *must be a linear combination of* $\boldsymbol{r_1}$, $\boldsymbol{r_2}$, ..., $\boldsymbol{r_k}$.

Proof. Since $\boldsymbol{r_1}$, $\boldsymbol{r_2}$, ..., $\boldsymbol{r_{k+1}}$ are linearly dependent, there exist $c_1, ..., c_{k+1}$ such that (i) they are not all zero, and (ii)

$$c_1\boldsymbol{r_1} + c_2\boldsymbol{r_2} + ... + c_k\boldsymbol{r_k} + c_{k+1}\boldsymbol{r_{k+1}} \quad = \quad 0.$$

Note that c_{k+1} cannot be 0. Otherwise, it will follow that $c_1\boldsymbol{r_1} + c_2\boldsymbol{r_2} + ... + c_k\boldsymbol{r_k} = 0$. Since $c_1, ..., c_k$ cannot be all zero, this means that $\boldsymbol{r_1}, ..., \boldsymbol{r_k}$ were linearly dependent, which is a contradiction.

Now that $c_{k+1} \neq 0$, we have:

$$\boldsymbol{r_{k+1}} \quad = \quad \frac{c_1}{c_{k+1}}\boldsymbol{r_1} + \frac{c_2}{c_{k+1}}\boldsymbol{r_2} + ... + \frac{c_k}{c_{k+1}}\boldsymbol{r_k}.$$

Therefore, $\boldsymbol{r_{k+1}}$ is a linear combination of $\boldsymbol{r_1}$, $\boldsymbol{r_2}$, ..., $\boldsymbol{r_k}$. $\square$

This implies that, if a matrix has rank k, then there are only k "effective" rows, in the sense that every other row can be derived as a linear combination of those k rows. For instance, consider the matrix in Example 6; we know that its rank is 2, and that the first two rows are linearly independent. Thus, we must be able to represent the 3rd row as a linear combination of the first two. Indeed, this is true:

$$[3, 2, 1, 0] \quad = \quad (-2) \cdot [1, 2, 3, 4] + [5, 6, 7, 8].$$

3 An Important Property of Ranks

In this section, we will prove a non-trivial lemma about ranks.

Lemma 4. *The rank of a matrix $\boldsymbol{A}$ is the same as the rank of $\boldsymbol{A}^T$.*

Proof. (Sketch) Define the *column-rank* of $\boldsymbol{A}$ to be the maximum number of independent column vectors of $\boldsymbol{A}$. Note that the column-rank of $\boldsymbol{A}$ is exactly the same as the rank of $\boldsymbol{A}^T$. Hence, to prove the lemma, it suffices to show that the rank of $\boldsymbol{A}$ is the same as the column-rank of $\boldsymbol{A}$.

We first show:

- Fact 1: If $\boldsymbol{A}$ is in row echelon form, then the rank of $\boldsymbol{A}$ cannot be less than its column-rank.
- Fact 2: Elementary row operations on $\boldsymbol{A}$ do not change its column rank.

The proofs of the above facts are left to you as an exercise. But here are the hints:

- For Fact 1: assume that $\boldsymbol{A}$ has rank k; take k columns appropriately and prove that they must be linearly independent.
- For Fact 2: it can proved in the same "style" as the proof of Lemma 1.

Since elementary row operations on $\boldsymbol{A}$ do not change its rank, combining both facts shows that the rank of $\boldsymbol{A}$ is at most its column-rank.

On the other hand, reversing the above argument shows that the column-rank of $\boldsymbol{A}$ is at most its rank. With this, we complete the lemma. □

Example 7. Let

1e

$$A \quad = \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

Compute the rank of A^T.

Solution. From Example 6, we know that the rank of A is 2. Lemma 4 tells us that the rank of A^T must also be 2. □

3. no solution.

Based on the above possibilities, we have the following definition.

Definition 2.5.1 (Consistent, Inconsistent) A linear system is called CONSISTENT if it admits a solution and is called INCONSISTENT if it admits no solution.

The question arises, as to whether there are conditions under which the linear system $A\mathbf{x} = \mathbf{b}$ is consistent. The answer to this question is in the affirmative. To proceed further, we need a few definitions and remarks.

Recall that the row reduced echelon form of a matrix is unique and therefore, the number of non-zero rows is a unique number. Also, note that the number of non-zero rows in either the row reduced form or the row reduced echelon form of a matrix are same.

Definition 2.5.2 (Row rank of a Matrix) The number of non-zero rows in the row reduced form of a matrix is called the row-rank of the matrix.

By the very definition, it is clear that row-equivalent matrices have the same row-rank. For a matrix A, we write 'row-rank (A)' to denote the row-rank of A.

Example 2.5.3 1. Determine the row-rank of $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

Solution: To determine the row-rank of A, we proceed as follows.

1b

(a) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{R_{21}(-2), R_{31}(-1)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2(-1), R_{32}(1)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3(1/2), R_{12}(-2)} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_{23}(-1), R_{13}(1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The last matrix in Step 1d is the row reduced form of A which has 3 non-zero rows. Thus, row-rank$(A) = 3$. This result can also be easily deduced from the last matrix in Step 1b.

2. Determine the row-rank of $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

1e

Solution: Here we have

(a) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_{21}(-2), R_{31}(-1)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{R_2(-1), R_{32}(1)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

From the last matrix in Step 2b, we deduce row-rank$(A) = 2$.

Remark 2.5.4 *Let $A\mathbf{x} = \mathbf{b}$ be a linear system with m equations and n unknowns. Then the row-reduced echelon form of A agrees with the first n columns of $[A \;\; \mathbf{b}]$, and hence*

$$\text{row-rank}(A) \leq \text{row-rank}([A \;\; \mathbf{b}]).$$

The reader is advised to supply a proof.

Remark 2.5.5 *Consider a matrix A. After application of a finite number of elementary column operations (see Definition 2.4.16) to the matrix A, we can have a matrix, say B, which has the following properties:*

1. *The first nonzero entry in each column is 1.*
2. *A column containing only 0's comes after all columns with at least one non-zero entry.*
3. *The first non-zero entry (the leading term) in each non-zero column moves down in successive columns.*

Therefore, we can define **column-rank** of A as the number of non-zero columns in B. It will be proved later that

$$\text{row-rank}(A) = \text{column-rank}(A).$$

Thus we are led to the following definition.

Definition 2.5.6 The number of non-zero rows in the row reduced form of a matrix A is called the **rank** of A, denoted rank (A).

Theorem 2.5.7 Let A be a matrix of rank r. Then there exist elementary matrices $E_1, E_2, \ldots, E_s$ and $F_1, F_2, \ldots, F_\ell$ such that

$$E_1 E_2 \ldots E_s \; A \; F_1 F_2 \ldots F_\ell = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof. Let C be the row reduced echelon matrix obtained by applying elementary row operations to the given matrix A. As rank$(A) = r$, the matrix C will have the first r rows as the non-zero rows. So by Remark 2.4.5, C will have r leading columns, say $i_1, i_2, \ldots, i_r$. Note that, for $1 \leq s \leq r$, the i_s^{th} column will have 1 in the s^{th} row and zero elsewhere.

We now apply column operations to the matrix C. Let D be the matrix obtained from C by successively interchanging the s^{th} and i_s^{th} column of C for $1 \leq s \leq r$. Then the matrix D can be written in the form $\begin{bmatrix} I_r & B \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, where B is a matrix of appropriate size. As the $(1,1)$ block of D is an identity matrix, the block $(1,2)$ can be made the zero matrix by application of column operations to D. This gives the required result. $\square$

Exercise 2.5.8

1. Determine the ranks of the coefficient and the augmented matrices that appear in Part 1 and Part 2 of Exercise 2.4.12.
2. For any matrix A, prove that rank$(A) = $ rank(A^t).
3. Let A be an $n \times n$ matrix with rank$(A) = n$. Then prove that A is row-equivalent to I_n.

2a

5.2.1 Example Determine whether the following vectors in $\mathbf{R}^2$ are linearly dependent or linearly independent:

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Solution Suppose we have a linear combination of the vectors equal to $\mathbf{0}$:

$$\begin{aligned} \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 &= \mathbf{0} \\ \alpha_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 5 \\ 6 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -\alpha_1 + 5\alpha_2 + \alpha_3 \\ 3\alpha_1 + 6\alpha_2 + 4\alpha_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Equating components we get a system with augmented matrix

$$\left[\begin{array}{ccc|c} -1 & 5 & 1 & 0 \\ 3 & 6 & 4 & 0 \end{array}\right]\begin{array}{c} 3 \\ \end{array} \sim \left[\begin{array}{ccc|c} -1 & 5 & 1 & 0 \\ 0 & 21 & 7 & 0 \end{array}\right].$$

Since α_3 is free, we can choose it to be anything. In particular, we can choose it to be nonzero. Therefore, the vectors are linearly dependent. □

2b

5.2.2 Example Determine whether the following vectors in $\mathbf{R}^3$ are linearly dependent or linearly independent:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Solution Suppose we have a linear combination of the vectors equal to $\mathbf{0}$:

$$\begin{aligned} \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 &= \mathbf{0} \\ \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \alpha_1 - 2\alpha_2 + \alpha_3 \\ 2\alpha_1 + \alpha_2 \\ 3\alpha_1 + \alpha_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Equating components we get a system with augmented matrix

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right]\begin{array}{cc} -2 & -3 \\ & \\ & \end{array} &\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & 6 & -2 & 0 \end{array}\right]\begin{array}{c} \\ -6 \\ 5 \end{array} \\ &\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]. \end{aligned}$$

Since there is a pivot in every column except for the augmented column, there is a unique solution, namely, $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_3 = 0$.

The computation shows that the only way to get a linear combination of the vectors to equal $\mathbf{0}$ is by making all of the scalar factors 0. Therefore, the vectors are linearly independent. $\square$

The solutions to these last two examples show that the question of whether some given vectors are linearly independent can be answered just by looking at a row-reduced form of the matrix obtained by writing the vectors side by side. The following theorem uses a new term: A matrix has **full rank** if a row-reduced form of the matrix has a pivot in every column.

> THEOREM. *Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s$ be vectors in $\mathbf{R}^n$ and let $\mathbf{A}$ be the matrix formed by writing these vectors side by side:*
>
> $$\mathbf{A} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_s \end{bmatrix}.$$
>
> *The vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s$ are linearly independent if and only if $\mathbf{A}$ has full rank.*

5.2.3 Example Use the last theorem to determine whether the vectors $[1, 3, -1, 0]^T$, $[4, 9, -2, 1]^T$, and $[2, 3, 0, 1]^T$ are linearly independent.

Solution We have

$$\begin{bmatrix} 1 & 4 & 2 \\ 3 & 9 & 3 \\ -1 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{array}{cc} -3 & 1 \\ & \\ & \\ & \end{array} \sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{array}{cc} & \\ 2 & 1 \\ 3 & \\ & 3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the matrix does not have full rank, the vectors are *not* linearly independent. $\square$

5.3 Facts about linear dependence/independence

The next theorem says that if a vector is written as a linear combination of linearly independent vectors, then the scaling factors are uniquely determined.

EXAMPLE 1 Determine whether the three vectors $\mathbf{u} = (1,2,3,2)$, $\mathbf{v} = (2,5,5,5)$, and $\mathbf{w} = (2,6,4,6)$ are linearly dependent.

2c

SOLUTION We begin by constructing a matrix whose rows are $\mathbf{u}^T$, $\mathbf{v}^T$, and $\mathbf{w}^T$:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 5 & 5 & 5 \\ 2 & 6 & 4 & 6 \end{bmatrix}.$$

Reducing to echelon form gives

$$\mathbf{E} = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $\mathbf{E}$ has a row of zeroes, the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent. ■

The next theorem discusses how the columns of a matrix are affected by elementary row operations. For the following theorem, we will use the notation

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_k \end{bmatrix}$$

to mean that $\mathbf{A}$ is a matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$.

Theorem 2 Row Operations and Dependence of Columns

Consider two $n \times k$ matrices

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_k \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix},$$

where $\mathbf{B}$ is obtained from $\mathbf{A}$ using one or more elementary row operations. If the columns of $\mathbf{A}$ satisfy an equation of the form

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_k\mathbf{a}_k = \mathbf{0},$$

where $c_1, c_2, \ldots, c_k$ are scalars, then the columns of $\mathbf{B}$ satisfy the same equation:

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_k\mathbf{b}_k = \mathbf{0}.$$

This theorem says that the relationship between the columns of a matrix does not change when we perform elementary row operations. The following example illustrates this phenomenon.

Since there is a pivot in every column except for the augmented column, there is a unique solution, namely, $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_3 = 0$.

The computation shows that the only way to get a linear combination of the vectors to equal $\mathbf{0}$ is by making all of the scalar factors 0. Therefore, the vectors are linearly independent. □

The solutions to these last two examples show that the question of whether some given vectors are linearly independent can be answered just by looking at a row-reduced form of the matrix obtained by writing the vectors side by side. The following theorem uses a new term: A matrix has **full rank** if a row-reduced form of the matrix has a pivot in every column.

THEOREM. *Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s$ be vectors in $\mathbf{R}^n$ and let $\mathbf{A}$ be the matrix formed by writing these vectors side by side:*

$$\mathbf{A} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_s \end{bmatrix}.$$

The vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s$ are linearly independent if and only if $\mathbf{A}$ has full rank.

5.2.3 Example Use the last theorem to determine whether the vectors $[1, 3, -1, 0]^T$, $[4, 9, -2, 1]^T$, and $[2, 3, 0, 1]^T$ are linearly independent.

2d

Solution We have

$$\begin{array}{l} \begin{bmatrix} 1 & 4 & 2 \\ 3 & 9 & 3 \\ -1 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{array}{cc} -3 & 1 \\ & \\ & \\ & \end{array} \sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{array}{cc} & \\ 2 & 1 \\ 3 & \\ & 3 \end{array} \\ \sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{array}$$

Since the matrix does not have full rank, the vectors are *not* linearly independent. □

5.3 Facts about linear dependence/independence

The next theorem says that if a vector is written as a linear combination of linearly independent vectors, then the scaling factors are uniquely determined.

2e

EXAMPLE 5 Determine whether the vectors $(3, 1, 6)$, $(2, 0, 4)$, and $(2, 1, 4)$ are linearly dependent.

SOLUTION We compute the determinant of the matrix whose rows are the given vectors:

$$\begin{vmatrix} 3 & 1 & 6 \\ 2 & 0 & 4 \\ 2 & 1 & 4 \end{vmatrix} = 3(-4) - 1(0) + 6(2) = 0.$$

Since the determinant is zero, the given vectors are linearly dependent. ■

In the last example, it would work just as well to make the given vectors the columns of a matrix. Also, note that this method only works if the matrix that you get is square, since you can't take the determinant of a non-square matrix.

Theorem 4 can also be useful for recognizing when a determinant is zero:

EXAMPLE 6 Evaluate the following determinant:

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{vmatrix}.$$

SOLUTION Observe that the third row of this matrix is equal to the sum of the first two rows. Since the rows of this matrix are linearly dependent, the determinant must be zero. ■

CS131 Mathematics for Computer Scientists II *Note 5*

LINEAR COMBINATIONS AND SUBSPACES

Linear combinations. In $\mathbb{R}^2$ the vector $(5,3)$ can be written in the form $(5,3) = 5(1,0) + 3(0,1)$ and also in the form $(5,3) = 1(2,0) + 3(1,1)$. In each case we say that $(5,3)$ is a *linear combination* of the two vectors on the right hand side.

If $\underline{u}, \underline{v} \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$, then a vector of the form $\alpha\underline{u} + \beta\underline{v}$ is a linear combination of $\underline{u}$ and $\underline{v}$.

Problem. Express the vector $(6,6)$ as a linear combination of $(0,3)$ and $(2,1)$.

Solution. We want to find numbers α and β with

3α

$$(6,6) = \alpha(0,3) + \beta(2,1).$$

Now from the properties of vectors:

$$\begin{aligned}(6,6) = \alpha(0,3) + \beta(2,1) &\Leftrightarrow (6,6) = (2\beta, 3\alpha + \beta)\\ &\Leftrightarrow 2\beta = 6 \quad \text{and} \quad 3\alpha + \beta = 6\\ &\Leftrightarrow \beta = 3 \quad \text{and} \quad \alpha = 1\end{aligned}$$

so we have $\underline{(6,6) = 1(0,3) + 3(2,1)}$.

We can similarly define linear combinations of more than two vectors and of vectors in $\mathbb{R}^n$.

> If $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_m$ are vectors in $\mathbb{R}^n$ and $\alpha_1, \alpha_2, \ldots, \alpha_m$ are real numbers, then any vector of the form
>
> $$\alpha_1\underline{u}_1 + \alpha_2\underline{u}_2 + \cdots + \alpha_m\underline{u}_m$$
>
> is called a *linear combination* of $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_m$.

Problem. Express the vector $(3,0)$ as a linear combination of the vectors $(1,1), (1,0)$ and $(1,-1)$ in two different ways.

Linear Combinations and Span

Given two vectors $\mathbf{v}$ and $\mathbf{w}$, a **linear combination** of $\mathbf{v}$ and $\mathbf{w}$ is any vector of the form

$$a\mathbf{v} + b\mathbf{w}$$

where a and b are scalars. For example, the vector $(6, 8, 10)$ is a linear combination of the vectors $(1, 1, 1)$ and $(1, 2, 3)$, since

$$\begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

More generally, a **linear combination** of n vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is any vector of the form

For example, a linear combination of three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ would have the form $a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$, where a, b, and c are scalars.

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

where $a_1, a_2, \ldots, a_n$ are scalars. For $n = 2$, this reduces to the definition for two vectors given above.

It is all right if some of the scalars in a linear combination are either zero or negative. For example, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors, then

$$2\mathbf{u} - 3\mathbf{v} + 4\mathbf{w}, \qquad 3\mathbf{u} + 5\mathbf{w}, \qquad \mathbf{v} + \mathbf{w}, \qquad \mathbf{w} - \mathbf{u}, \qquad \text{and} \qquad 5\mathbf{v}$$

are some possible linear combinations of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

We will sometimes want to discuss linear combinations of a single vector. If $\mathbf{v}$ is a vector, a linear combination of just $\mathbf{v}$ is the same thing as a scalar multiple of $\mathbf{v}$:

$$a\mathbf{v}.$$

Thus $(3, 12, 6)$ is a linear combination of $(1, 4, 2)$, since $(3, 12, 6) = 3(1, 4, 2)$.

Expressing a Vector as a Linear Combination

Sometimes you want to express one vector as a linear combination of others. For example, can we express the vector $(8, 3, 3)$ as a linear combination of $(1, 1, 1)$ and $(1, 0, 0)$? A moment's thought reveals the answer:

$$\begin{bmatrix} 8 \\ 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For more complicated examples, you can express one vector as a linear combination of others by solving a system of linear equations.

EXAMPLE 1 Express the vector $(9, 6)$ as a linear combination of the vectors $(1, 2)$ and $(1, -4)$.

30

SOLUTION We are looking for scalars x_1 and x_2 so that

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}.$$

We can write this equation as a system of linear equations:

$$\begin{aligned} x_1 + x_2 &= 9 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Solving gives $x_1 = 7$ and $x_2 = 2$. Thus

$$\begin{bmatrix} 9 \\ 6 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

■

Solution. (i) $2\underline{u} + 3\underline{v} + 4\underline{w} = (2,0,6) + (0,6,0) + (0,12,4) = (2,18,10)$.
(ii) We have

$$\begin{aligned}
(1,5,4) = \alpha\underline{u} + \beta\underline{v} + \gamma\underline{w} \;&\Leftrightarrow\; (1,5,4) = \alpha(1,0,3) + \beta(0,2,0) + \gamma(0,3,1)\\
&\Leftrightarrow\; (1,5,4) = (\alpha, 2\beta + 3\gamma, 3\alpha + \gamma)\\
&\Leftrightarrow\; \alpha = 1,\ 2\beta + 3\gamma = 5,\ 3\alpha + \gamma = 4\\
&\Leftrightarrow\; \alpha = 1,\ \gamma = 1,\ \beta = 1
\end{aligned}$$

so $(1,5,4) = (1,0,3) + (0,2,0) + (0,3,1)$.
(iii) We have

$$\begin{aligned}
(1,5,4) = \alpha\underline{u} + \beta\underline{v} \;&\Leftrightarrow\; (1,5,4) = \alpha(1,0,3) + \beta(0,2,0) = (\alpha, 2\beta, 3\alpha)\\
&\Leftrightarrow\; \alpha = 1,\ 2\beta = 5, 3\alpha = 4
\end{aligned}$$

But we cannot have both $\alpha = 1$ and $3\alpha = 4$ so these equations cannot be solved for α and β. Hence $(1,5,4)$ *cannot* be written as a linear combination of $\underline{u}$ and $\underline{v}$.

Span. If $U = \{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_m\}$ is a finite set of vectors in $\mathbb{R}^n$, then the *span* of U is the set of all linear combinations of $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_m$ and is denoted by span U. Hence

$$\text{span } U = \{\alpha_1\underline{u}_1 + \alpha_2\underline{u}_2 + \cdots + \alpha_m\underline{u}_m \mid \alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{R}\}$$

Examples.

(1) If $U = \{\underline{u}\}$ contains just a single vector, then span $\{\underline{u}\} = \{\alpha\underline{u} \mid \alpha \in \mathbb{R}\}$ is the set of all multiples of $\underline{u}$.

(2) In $\mathbb{R}^2$ if $U = \{(1,0),(0,1)\}$ then the span of U is $\mathbb{R}^2$. To see this note that we can write an arbitrary vector (x,y) in $\mathbb{R}^2$ as a linear combination of $(1,0)$ and $(0,1)$ as follows: $(x,y) = x(1,0) + y(0,1)$.

(3) In $\mathbb{R}^3$ the span of the set $\{(1,0,0),(0,1,0),(0,0,1)\}$ is $\mathbb{R}^3$.

(4) In $\mathbb{R}^3$ let $\underline{u} = (1,0,1)$ and $\underline{v} = (2,0,3)$. Then

$$\alpha\underline{u} + \beta\underline{v} = \alpha(1,0,1) + \beta(2,0,3) = (\alpha + 2\beta, 0, \alpha + 3\beta)$$

so any linear combination of $\underline{u}$ and $\underline{v}$ has 0 for its middle component. In fact any vector with middle component 0 is a linear combination of $\underline{u}$ and $\underline{v}$. To see this note that for any $x, z \in \mathbb{R}$

$$\begin{aligned}
(x,0,z) = \alpha\underline{u} + \beta\underline{v} \;&\Leftrightarrow\; (x,0,z) = \alpha(1,0,1) + \beta(2,0,3)\\
&\Leftrightarrow\; (x,0,z)) = (\alpha + 2\beta, 0, \alpha + 3\beta)\\
&\Leftrightarrow\; \alpha + 2\beta = x,\ \alpha + 3\beta = z\\
&\Leftrightarrow\; \beta = z - x,\ \alpha = 3x - 2z
\end{aligned}$$

LINEAR ALGEBRA

HOMEWORK 2

4

(1) Write the polynomial $x+1$ as a linear combination of the polynomials $2x^2 - x + 1$ and $-x^2 + x$.

Comparing coefficients in the equation

$$x + 1 = a(2x^2 - x + 1) + b(-x^2 + x)$$

gives $2a - b = 0$, $-a + b = 1$, and $a = 1$. The only solution is $a = 1$, $b = 2$, hence

$$x + 1 = 1(2x^2 - x + 1) + 2(-x^2 + x).$$

(2) Show that the vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ are linearly independent. From $a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$ we get the system of equations

$$\begin{aligned} a + c &= 0 \\ b + 2c &= 0 \\ b - c &= 0 \end{aligned}$$

and this gives $c = 0$, $b = 0$, $a = 0$ as the only solution. Thus there is no nontrivial relation between the given vectors, and therefore they are linearly independent.

(3) For which values of $c \in \mathbb{R}$ are the vectors $x + 3$ and $2x + c + 2$ in the vector space of polynomials of degree ≤ 2 linearly dependent?

Note: of course the given polynomials are also contained in the vector space o polynomials of degree ≤ 1 (or in those of degree ≤ 7). For solving the problem, however, this is irrelevant. If they are linearly dependent in P_2, then they are also linearly dependent in P_1 or P_7, because the relation is valid in any of these spaces.

Checking for linear independence means solving the equation $a(x+3) + b(2x+c+2) = 0$. This gives $a + 2b = 0$ and $3a + b(c+2) = 0$. Eliminating a shows that $b(c+2) - 6b = 0$, i.e., $b(c-4) = 0$. We need a nontrivial relation, hence we must have $b \neq 0$. But then we must have $c - 4 = 0$, that is, $c = 4$, and in this case we actually have the nontrivial relation $2(x+3) = 2x + 6$.

Answer: The polynomials $x + 3$ and $2x + c + 2$ are linearly dependent if and only if $c = 4$.

(4) Find a basis for all vectors of the form $(a + c, a - b, b + c, -a + b)$ for $a, b, c \in \mathbb{R}$. What is the dimension of this vector space V? Does the vector $(3, 1, 2, -1)$ lie in this vector space? If yes, write it as a linear combination of your basis.

Exercises

1. Find the rank of the matrices

(a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 2 & 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 5 & 3 \\ 2 & 6 & 2 \\ 3 & 7 & 1 \\ 4 & 8 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

2. Decide whether the vectors are linearly dependent or independent.

   (a) $(1, 2, 3, 2)$, $(2, 5, 5, 5)$, $(2, 6, 4, 6)$.

   (b) $(-1, 3)$, $(5, 6)$, $(1, 4)$.

   (c) $(1, 2, 3)$, $(-2, 1, 0)$, $(1, 0, 1)$.

   (d) $(1, 3, -1, 0)$, $(4, 9, -2, 1)$, $(2, 3, 0, 1)$.

   (e) $(3, 1, 6)$, $(2, 0, 4)$, $(2, 1, 4)$ (with determinant).

3. Express the vector

   (a) $(6, 6)$ as the linear combination of $(0, 3)$ and $(2, 1)$.

   (b) $(9, 6)$ as the linear combination of $(1, 2)$ and $(1, -4)$.

   (c) $(1, 5, 4)$ as the linear combination of $(1, 0, 3)$ and $(0, 2, 0)$.

4. Write the polynomial $x + 1$ as a linear combination of $2x^2 - x + 1$ and $-x^2 + x$.

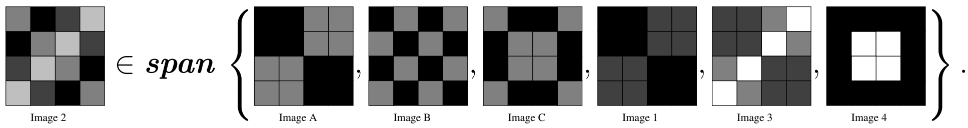
5. What can you say about solutions of this systems? About rank of matrix and augmented matrix?

(a) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right)$ (b) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{array}\right)$ (c) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

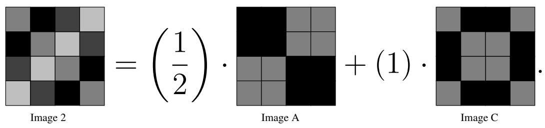
	(a)	(b)	(c)
	1 sol.	1∅ sol.	∞ sol.
hodn A	3	2	2
	=	< !	=
hodn A|b	3	3	2

Suppose that $W = \{v_1, v_2, \cdots, v_n\}$ is a subset of a vector space $(V, +, \cdot)$. If $v_k \in \mathbf{span}\,\{v_1, v_2, \cdots, v_{k-1}, v_{k+1}, \cdots, v_n\}$ for some k, then W is linearly dependent. We can see this in a more concrete way by considering the set when $n = 5$ for example. If $W = \{v_1, v_2, v_3, v_4, v_5\}$ and $v_2 \in \mathbf{span}\,\{v_1, v_3, v_4, v_5\}$ then we know that W is linearly dependent.

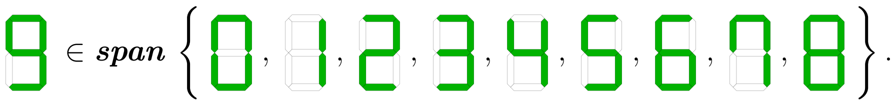
Example 8.1.1. *Consider the 4×4 image example from the beginning of the chapter. We can say that the set of seven images is linearly dependent because, for example,*



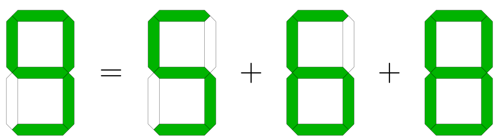
We know this is true because Image 1 can be written as a linear combination of the other images:



Example 8.1.2. *Consider the vector space $\mathcal{D}(Z_2)$. We can say that the set of ten LCD character images is linearly dependent because, for example,*



We know that this is true because image d_9 can be written as a linear combination of d_5, d_6 and d_8:



for example 9 can be omitted