7th lesson

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Theory

Remarks 1. Let **A** be a square matrix. We say that a number $\lambda \in \mathbb{R}$ is its *eigenvalue*, if

$$\det(A - \lambda I) = 0.$$

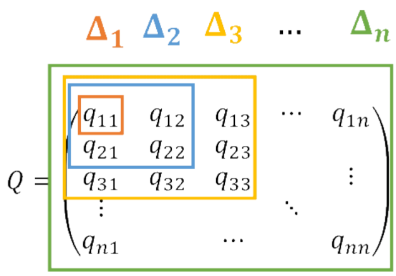
Theorem 2 (Eigenvalue test)**.** Let **A** be a **symmetric** matrix $n \times n$.

1. **A** is positive definite if and only if all its eigenvalues are positive;
2. **A** is negative definite if and only if all its eigenvalues are negative;
3. **A** is positive semi-definite if and only if all its eigenvalues are non-negative;
4. **A** is negative semi-definite if and only if all its eigenvalues are non-positive;
5. **A** is indefinite if and only if it has both positive and negative eigenvalues.

Theorem 3 (Sylvester's criterion)**.** Let **A** be a **symmetric** matrix $n \times n$. Let D_k denotes the determinant of $k \times k$ submatrix taken from the top left corner of **A**.

1. **A** is positive definite if and only if $D_k > 0$ for all k;
2. **A** negative definite if and only if $D_1 < 0$, $D_2 > 0$, $D_3 < 0 \ldots$;
3. **A** is indefinite if $D_k \neq 0$ for all k and neither of the previous two conditions is satisfied.

(If some of $D_k = 0$ we have no information about the definiteness - matrix can be positive semi-definite, negative semi-definite or indefinite.)



Source 1: http://www.princeton.edu/~aaa/Public/Teaching/ORF363_COS323/F15/ORF363_COS323_F15_Lec2.pdf

Exercises

1. Decide about the definiteness (use the eigenvalue test):

(a) $\begin{pmatrix} 4 & -3 \\ -3 & 12 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 4 \\ 4 & 7 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$

(d) $\begin{pmatrix} -3 & 2 \\ 2 & -6 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

2. Decide about the definiteness (use the Sylvester's criterion):

(a) $\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$

(b) $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & -4 \\ -4 & 5 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 3 & -2 & 8 \end{pmatrix}$

(e) $\begin{pmatrix} -4 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

(f) $\begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix}$

(g) $\begin{pmatrix} -3 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix}$