1a

$$\begin{vmatrix} 4-\lambda & -3 \\ -3 & 12-\lambda \end{vmatrix} = (4-\lambda)(12-\lambda) - 9 = \lambda^2 - 16\lambda + 39$$
$$= (\lambda-3)(\lambda-13)$$

$\lambda_1 = 3 \qquad \lambda_2 = 13 \qquad \rightarrow$ Pos. def.

1b

$$\begin{vmatrix} 1-\lambda & 4 \\ 4 & 7-\lambda \end{vmatrix} = (1-\lambda)(7-\lambda) - 16 = \lambda^2 - 8\lambda - 9$$
$$= (\lambda+1)(\lambda-9)$$

$\lambda_1 = -1. \quad \lambda_2 = 9 \qquad \rightarrow$ Indef

1c

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 9-\lambda \end{vmatrix} = (1-\lambda)(9-\lambda) - 9 = \lambda^2 - 10\lambda = \lambda(\lambda-10)$$

$\lambda_1 = 0 \qquad \lambda_2 = 10 \qquad \rightarrow$ Pos. semi-def

1d

$$\begin{vmatrix} -3-\lambda & 2 \\ 2 & -6-\lambda \end{vmatrix} = (-3-\lambda)(-6-\lambda) - 4 = \lambda^2 + 9\lambda + 14$$
$$= (\lambda+2)(\lambda+7)$$

$\lambda_1 = -2 \qquad \lambda_2 = -7$ Negative def.

A^2 and A^{-1} and $A + 4I$ keep the *same eigenvectors as* A. Their eigenvalues are λ^2 and λ^{-1} and $\lambda + 4$:

$$A^2 \text{ has eigenvalues } 1^2 = 1 \text{ and } 3^2 = 9 \qquad A^{-1} \text{ has } \frac{1}{1} \text{ and } \frac{1}{3} \qquad A + 4I \text{ has } \begin{matrix} 1+4=5 \\ 3+4=7 \end{matrix}$$

The trace of A^2 is $5 + 5$ which agrees with $1 + 9$. The determinant is $25 - 16 = 9$.

Notes for later sections: A has *orthogonal eigenvectors* (Section 6.4 on symmetric matrices). A can be *diagonalized* since $\lambda_1 \neq \lambda_2$ (Section 6.2). A is *similar* to any 2 by 2 matrix with eigenvalues 1 and 3 (Section 6.6). A is a *positive definite matrix* (Section 6.5) since $A = A^{\mathrm{T}}$ and the λ's are positive.

6.1 B Find the eigenvalues and eigenvectors of this 3 by 3 matrix A:

Symmetric matrix
Singular matrix
Trace $1 + 2 + 1 = 4$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution Since all rows of A add to zero, the vector $\boldsymbol{x} = (1, 1, 1)$ gives $A\boldsymbol{x} = \mathbf{0}$. This is an eigenvector for the eigenvalue $\lambda = 0$. To find λ_2 and λ_3 I will compute the 3 by 3 determinant:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} \begin{aligned} &= (1-\lambda)(2-\lambda)(1-\lambda) - 2(1-\lambda) \\ &= (1-\lambda)[(2-\lambda)(1-\lambda) - 2] \\ &= (1-\lambda)(-\lambda)(3-\lambda). \end{aligned}$$

That factor $-\lambda$ confirms that $\lambda = 0$ is a root, and an eigenvalue of A. The other factors $(1 - \lambda)$ and $(3 - \lambda)$ give the other eigenvalues 1 and 3, adding to 4 (the trace). Each eigenvalue 0, 1, 3 corresponds to an eigenvector:

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad A\boldsymbol{x}_1 = \mathbf{0}\boldsymbol{x}_1 \qquad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad A\boldsymbol{x}_2 = \mathbf{1}\boldsymbol{x}_2 \qquad \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad A\boldsymbol{x}_3 = \mathbf{3}\boldsymbol{x}_3\,.$$

I notice again that eigenvectors are perpendicular when A is symmetric.

The 3 by 3 matrix produced a third-degree (cubic) polynomial for $\det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 3\lambda$. We were lucky to find simple roots $\lambda = 0, 1, 3$. Normally we would use a command like **eig**(A), and the computation will never even use determinants (Section 9.3 shows a better way for large matrices).

The full command $[S, D] =$ **eig**(A) will produce unit eigenvectors in the columns of the **eigenvector matrix** S. The first one happens to have three minus signs, reversed from $(1, 1, 1)$ and divided by $\sqrt{3}$. The eigenvalues of A will be on the diagonal of the ***eigenvalue matrix*** (typed as D but soon called Λ).

2a

$$\begin{vmatrix} 5 & -2 \\ -2 & 5 \end{vmatrix}$$

$$\left.\begin{array}{l} D_1 = 5 > 0 \\ D_2 = 25-4 = 20 > 0 \end{array}\right\} \text{Pos. definite}$$

2b

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

$$\left.\begin{array}{l} D_1 = -2 < 0 \\ D_2 = 4-1 = 3 > 0 \end{array}\right\} \text{Neg. definite}$$

2c

$$\begin{pmatrix} 1 & -4 \\ -4 & 5 \end{pmatrix}$$

$$\left.\begin{array}{l} D_1 = 1 \\ D_2 = -5-16 = -21 < 0 \end{array}\right\} \underline{\text{Indef.}}$$

2d

$$\begin{pmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 3 & -2 & 8 \end{pmatrix}$$

$$\left.\begin{array}{l} D_1 = 3 > 0 \\ D_2 = 3-0 = 3 > 0 \\ D_3 = 24-9-12 = 3 > 0 \end{array}\right\} \underline{\text{Pos. def}}$$

2e

$$\begin{pmatrix} -4 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$D_1 = -4 < 0$$

$$D_2 = -8-4 = -12 < 0 \quad \text{Indef}$$

$$D_3 = -16+2+2-2+4-8 = -18 < 0$$

2f

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

$$\left.\begin{array}{l} D_1 = -2 < 0 \\ D_2 = 1 > 0 \\ D_3 = -4+2+2 = 0 \end{array}\right\} \text{We do not know.}$$

2g

$$\begin{pmatrix} -3 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

$$\left.\begin{array}{l} D_1 = -3 < 0 \\ D_2 = 2 > 0 \\ D_3 = -6+3+2 = -1 < 0 \end{array}\right\} \text{Neg. def}$$