Solution The contour at level c is given by

$$f(x,y) = \sqrt{x^2+y^2} = c.$$

For $c > 0$ this is a circle, just as in the previous example, but here the radius is c instead of $\sqrt{c}$. For $c = 0$, it is the origin. Thus, if the level c increases by 1, the radius of the contour increases by 1. This means the contours are equally spaced concentric circles (see Figure 12.41) which do not become more closely packed further from the origin. Thus, the graph of f has the same constant slope as we move away from the origin (see Figure 12.42), making it a cone rather than a bowl.

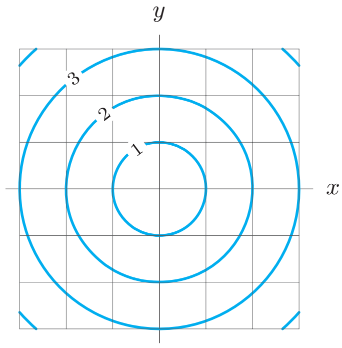


Figure 12.41: A contour diagram for $f(x,y) = \sqrt{x^2+y^2}$

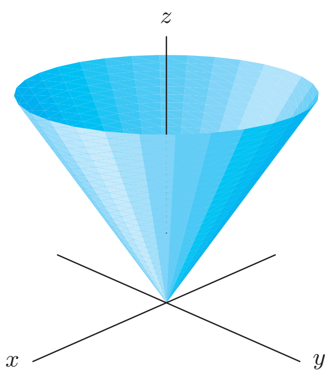


Figure 12.42: The graph of $f(x,y) = \sqrt{x^2+y^2}$

In both of the previous examples the level curves are concentric circles because the surfaces have circular symmetry. Any function of two variables which depends only on the quantity (x^2+y^2) has such symmetry: for example, $G(x,y) = e^{-(x^2+y^2)}$ or $H(x,y) = \sin(\sqrt{x^2+y^2})$.

Example 5 Draw a contour diagram for $f(x,y) = 2x + 3y + 1$.

1a

Solution The contour at level c has equation $2x + 3y + 1 = c$. Rewriting this as $y = -(2/3)x + (c-1)/3$, we see that the contours are parallel lines with slope $-2/3$. The y-intercept for the contour at level c is $(c-1)/3$; each time c increases by 3, the y-intercept moves up by 1. The contour diagram is shown in Figure 12.43.

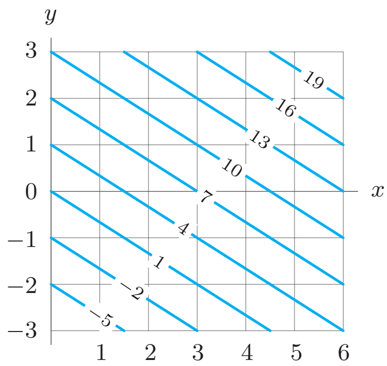


Figure 12.43: A contour diagram for $f(x,y) = 2x + 3y + 1$

> Contour lines, or level curves, are obtained from a surface by slicing it with horizontal planes. A contour diagram is a collection of level curves labeled with function values.

Finding Contours Algebraically

Algebraic equations for the contours of a function f are easy to find if we have a formula for $f(x, y)$. Suppose the surface has equation

$$z = f(x, y).$$

A contour is obtained by slicing the surface with a horizontal plane with equation $z = c$. Thus, the equation for the contour at height c is given by:

$$f(x, y) = c.$$

Example 3 Find equations for the contours of $f(x, y) = x^2 + y^2$ and draw a contour diagram for f. Relate the contour diagram to the graph of f.

Solution The contour at height c is given by

$$f(x, y) = x^2 + y^2 = c.$$

This is a contour only for $c \geq 0$, For $c > 0$ it is a circle of radius $\sqrt{c}$. For $c = 0$, it is a single point (the origin). Thus, the contours at an elevation of $c = 1, 2, 3, 4, \ldots$ are all circles centered at the origin of radius $1, \sqrt{2}, \sqrt{3}, 2, \ldots$. The contour diagram is shown in Figure 12.39. The bowl–shaped graph of f is shown in Figure 12.40. Notice that the graph of f gets steeper as we move further away from the origin. This is reflected in the fact that the contours become more closely packed as we move further from the origin; for example, the contours for $c = 6$ and $c = 8$ are closer together than the contours for $c = 2$ and $c = 4$.

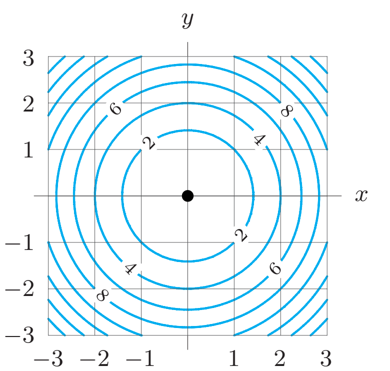


Figure 12.39: Contour diagram for $f(x, y) = x^2 + y^2$ (even values of c only)

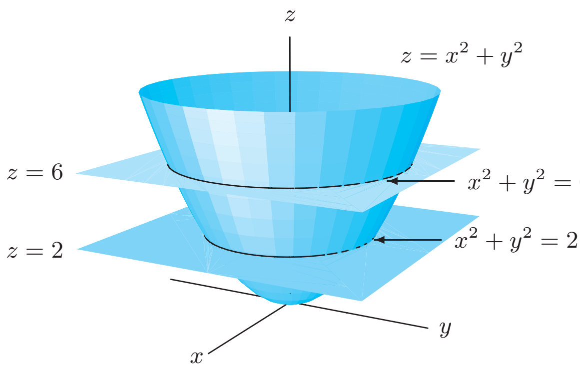


Figure 12.40: The graph of $f(x, y) = x^2 + y^2$

Example 4 Draw a contour diagram for $f(x, y) = \sqrt{x^2 + y^2}$ and relate it to the graph of f.

Solution The contour at level c is given by

$$f(x,y) = \sqrt{x^2+y^2} = c.$$

For $c > 0$ this is a circle, just as in the previous example, but here the radius is c instead of $\sqrt{c}$. For $c = 0$, it is the origin. Thus, if the level c increases by 1, the radius of the contour increases by 1. This means the contours are equally spaced concentric circles (see Figure 12.41) which do not become more closely packed further from the origin. Thus, the graph of f has the same constant slope as we move away from the origin (see Figure 12.42), making it a cone rather than a bowl.

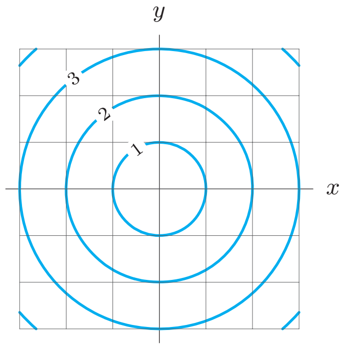


Figure 12.41: A contour diagram for $f(x,y) = \sqrt{x^2+y^2}$

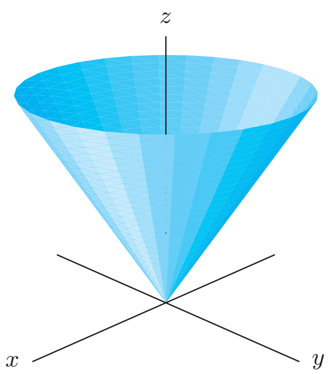


Figure 12.42: The graph of $f(x,y) = \sqrt{x^2+y^2}$

In both of the previous examples the level curves are concentric circles because the surfaces have circular symmetry. Any function of two variables which depends only on the quantity (x^2+y^2) has such symmetry: for example, $G(x,y) = e^{-(x^2+y^2)}$ or $H(x,y) = \sin(\sqrt{x^2+y^2})$.

Example 5 Draw a contour diagram for $f(x,y) = 2x + 3y + 1$.

Solution The contour at level c has equation $2x + 3y + 1 = c$. Rewriting this as $y = -(2/3)x + (c-1)/3$, we see that the contours are parallel lines with slope $-2/3$. The y-intercept for the contour at level c is $(c-1)/3$; each time c increases by 3, the y-intercept moves up by 1. The contour diagram is shown in Figure 12.43.

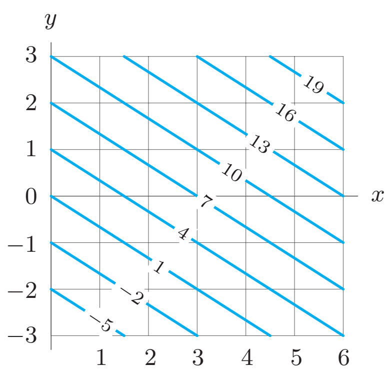


Figure 12.43: A contour diagram for $f(x,y) = 2x + 3y + 1$

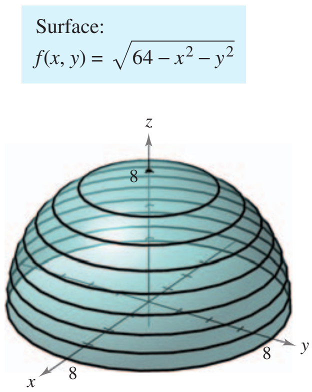
EXAMPLE 3 Sketching a Contour Map

The hemisphere given by $f(x, y) = \sqrt{64 - x^2 - y^2}$ is shown in Figure 13.9. Sketch a contour map for this surface using level curves corresponding to $c = 0, 1, 2, \ldots, 8$.

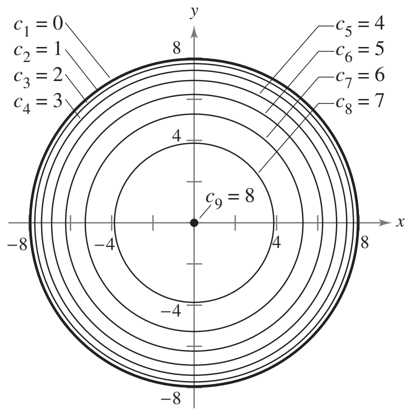
Solution For each value of c, the equation given by $f(x, y) = c$ is a circle (or point) in the xy-plane. For example, when $c_1 = 0$, the level curve is

$$x^2 + y^2 = 64 \qquad \text{Circle of radius 8}$$

which is a circle of radius 8. Figure 13.10 shows the nine level curves for the hemisphere.



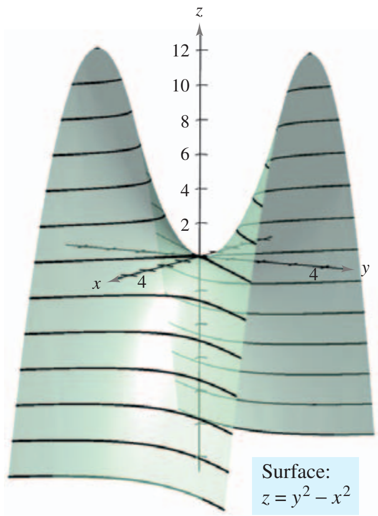
Hemisphere
Figure 13.9



Contour map
Figure 13.10

Animation

Try It Exploration A



Hyperbolic paraboloid
Figure 13.11

Animation

EXAMPLE 4 Sketching a Contour Map

The hyperbolic paraboloid given by

$$z = y^2 - x^2$$

is shown in Figure 13.11. Sketch a contour map for this surface.

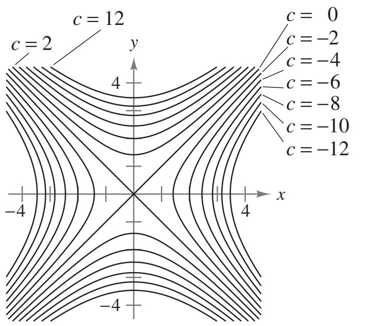
Solution For each value of c, let $f(x, y) = c$ and sketch the resulting level curve in the xy-plane. For this function, each of the level curves $(c \neq 0)$ is a hyperbola whose asymptotes are the lines $y = \pm x$. If $c < 0$, the transverse axis is horizontal. For instance, the level curve for $c = -4$ is given by

$$\frac{x^2}{2^2} - \frac{y^2}{2^2} = 1. \qquad \text{Hyperbola with horizontal transverse axis}$$

If $c > 0$, the transverse axis is vertical. For instance, the level curve for $c = 4$ is given by

$$\frac{y^2}{2^2} - \frac{x^2}{2^2} = 1. \qquad \text{Hyperbola with vertical transverse axis}$$

If $c = 0$, the level curve is the degenerate conic representing the intersecting asymptotes, as shown in Figure 13.12.



Hyperbolic level curves (at increments of 2)
Figure 13.12

Try It Open Exploration

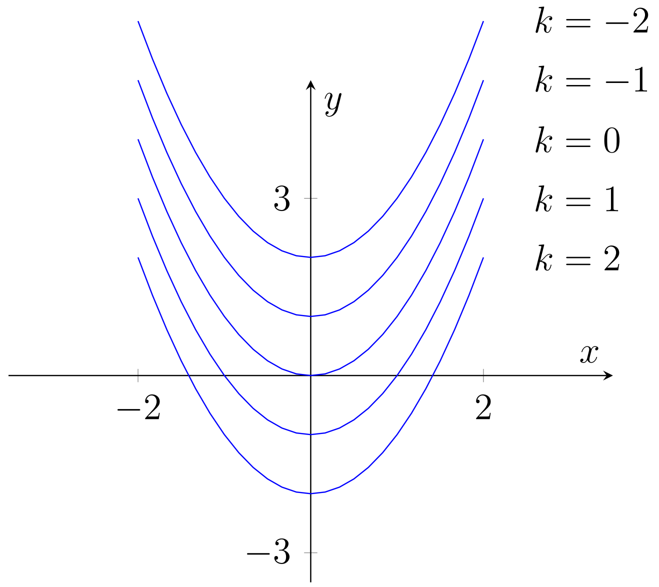
1ℓ

a. $f(x, y) = x^2 - y$, $z = -2, -1, 0, 1, 2$

The level curves of $z = f(x, y) = x^2 - y$ are

$$k = x^2 - y \implies y = x^2 - k$$

for some constant k. We sketch below the level curves corresponding to $k = -2, -1, 0, 1, 2$.



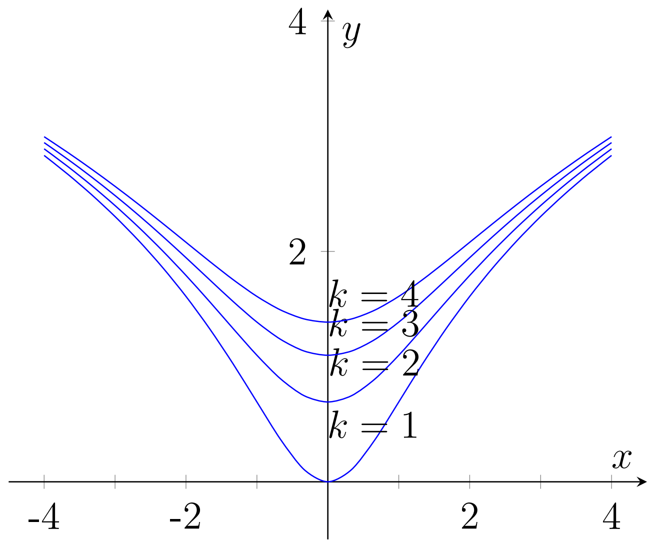
1g

a. $f(x, y) = e^y - x^2$, $z = 1, 2, 3, 4$

The level curves of $z = f(x, y) = e^y - x^2$ are

$$k = e^y - x^2 \implies y = \ln(x^2 + k)$$

for some constant k. For $k = 1, 2, 3, 4$, we therefore have



In Problems 35–36, give an example of:

35. A table of values, with three rows and three columns, for a nonlinear function that is linear in each row and in each column.

36. A linear function whose contours are lines with slope 2.

Are the statements in Problems 37–48 true or false? Give reasons for your answer.

37. The planes $z = 3 + 2x + 4y$ and $z = 5 + 2x + 4y$ intersect.

38. The function represented in Table 12.12 is linear.

Table 12.12

$u\backslash v$	1.1	1.2	1.3	1.4
3.2	11.06	12.06	13.06	14.06
3.4	11.75	12.82	13.89	14.96
3.6	12.44	13.58	14.72	15.86
3.8	13.13	14.34	15.55	16.76
4.0	13.82	15.10	16.38	17.66

39. Contours of $f(x, y) = 3x + 2y$ are lines with slope 3.

40. If f is linear, then the contours of f are parallel lines.

41. If $f(0, 0) = 1$, $f(0, 1) = 4$, $f(0, 3) = 5$, then f cannot be linear.

42. The graph of a linear function is always a plane.

43. The cross-section $x = c$ of a linear function $f(x, y)$ is always a line.

44. There is no linear function $f(x, y)$ with a graph parallel to the xy-plane.

45. There is no linear function $f(x, y)$ with a graph parallel to the xz-plane.

46. A linear function $f(x, y) = 2x+3y-5$, has exactly one point (a, b) satisfying $f(a, b) = 0$.

47. In a table of values of a linear function, the columns have the same slope as the rows.

48. There is exactly one linear function $f(x, y)$ whose $f = 0$ contour is $y = 2x + 1$.

12.5 FUNCTIONS OF THREE VARIABLES

In applications of calculus, functions of any number of variables can arise. The density of matter in the universe is a function of three variables, since it takes three numbers to specify a point in space. Models of the US economy often use functions of ten or more variables. We need to be able to apply calculus to functions of arbitrarily many variables.

One difficulty with functions of more than two variables is that it is hard to visualize them. The graph of a function of one variable is a curve in 2-space, the graph of a function of two variables is a surface in 3-space, so the graph of a function of three variables would be a solid in 4-space. Since we can't easily visualize 4-space, we won't use the graphs of functions of three variables. On the other hand, it is possible to draw contour diagrams for functions of three variables, only now the contours are surfaces in 3-space.

Representing a Function of Three Variables Using a Family of Level Surfaces

A function of two variables, $f(x, y)$, can be represented by a family of level curves of the form $f(x, y) = c$ for various values of the constant, c.

> A **level surface**, or **level set** of a function of three variables, $f(x, y, z)$, is a surface of the form $f(x, y, z) = c$, where c is a constant. The function f can be represented by the family of level surfaces obtained by allowing c to vary.

2a

The value of the function, f, is constant on each level surface.

Example 1 The temperature, in °C, at a point (x, y, z) is given by $T = f(x, y, z) = x^2 + y^2 + z^2$. What do the level surfaces of the function f look like and what do they mean in terms of temperature?

Solution The level surface corresponding to $T = 100$ is the set of all points where the temperature is 100°C. That is, where $f(x, y, z) = 100$, so

$$x^2 + y^2 + z^2 = 100.$$

This is the equation of a sphere of radius 10, with center at the origin. Similarly, the level surface corresponding to $T = 200$ is the sphere with radius $\sqrt{200}$. The other level surfaces are concentric spheres. The temperature is constant on each sphere. We may view the temperature distribution as a set of nested spheres, like concentric layers of an onion, each one labeled with a different temperature, starting from low temperatures in the middle and getting hotter as we go out from the center. (See Figure 12.69.) The level surfaces become more closely spaced as we move farther from the origin because the temperature increases more rapidly the farther we get from the origin.

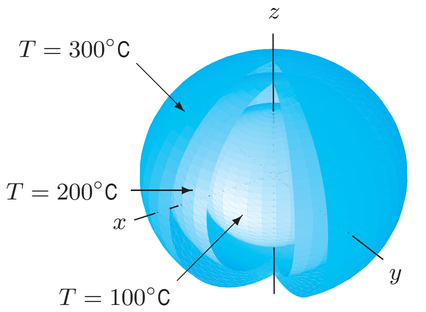


Figure 12.69: Level surfaces of $T = f(x, y, z) = x^2 + y^2 + z^2$, each one having a constant temperature

Example 2 What do the level surfaces of $f(x, y, z) = x^2 + y^2$ and $g(x, y, z) = z - y$ look like?

Solution The level surface of f corresponding to the constant c is the surface consisting of all points satisfying the equation

$$x^2 + y^2 = c.$$

Since there is no z-coordinate in the equation, z can take any value. For $c > 0$, this is a circular cylinder of radius $\sqrt{c}$ around the z-axis. The level surfaces are concentric cylinders; on the narrow ones near the z-axis, f has small values; on the wider ones, f has larger values. See Figure 12.70.

The level surface of g corresponding to the constant c is the plane

$$z - y = c.$$

Since there is no x variable in the equation, these plane are parallel to the x-axis and cut the yz-plane in the line $z - y = c$. See Figure 12.71.

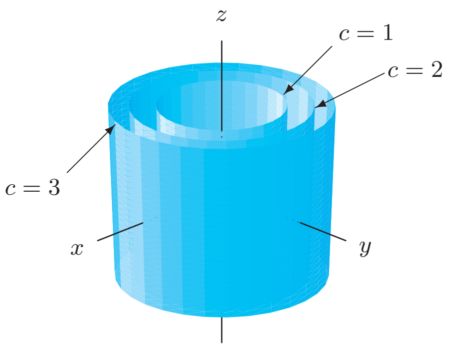


Figure 12.70: Level surfaces of $f(x, y, z) = x^2 + y^2$

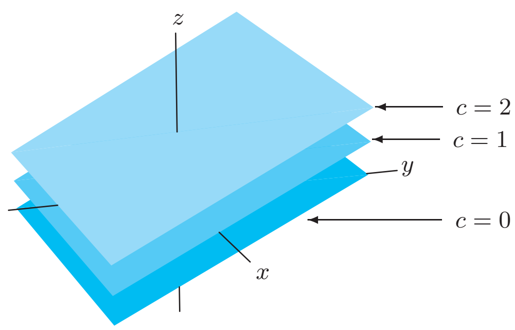


Figure 12.71: Level surfaces of $g(x, y, z) = z - y$

Example 3 What do the level surfaces of $f(x, y, z) = x^2 + y^2 - z^2$ look like?

Solution In Section 12.3, we saw that the two-variable quadratic function $g(x, y) = x^2 - y^2$ has a saddle-shaped graph and three types of contours. The contour equation $x^2 - y^2 = c$ gives a hyperbola opening right-left when $c > 0$, a hyperbola opening up-down when $c < 0$, and a pair of intersecting lines when $c = 0$. Similarly, the three-variable quadratic function $f(x, y, z) = x^2 + y^2 - z^2$ has three types of level surfaces depending on the value of c in the equation $x^2 + y^2 - z^2 = c$.

EXAMPLE 6 Level Surfaces

Describe the level surfaces of the function

$$f(x, y, z) = 4x^2 + y^2 + z^2.$$

Solution Each level surface has an equation of the form

$$4x^2 + y^2 + z^2 = c. \qquad \text{Equation of level surface}$$

So, the level surfaces are ellipsoids (whose cross sections parallel to the yz-plane are circles). As c increases, the radii of the circular cross sections increase according to the square root of c. For example, the level surfaces corresponding to the values $c = 0$, $c = 4$, and $c = 16$ are as follows.

$$4x^2 + y^2 + z^2 = 0 \qquad \text{Level surface for } c = 0 \text{ (single point)}$$

$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{4} = 1 \qquad \text{Level surface for } c = 4 \text{ (ellipsoid)}$$

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1 \qquad \text{Level surface for } c = 16 \text{ (ellipsoid)}$$

These level surfaces are shown in Figure 13.16.

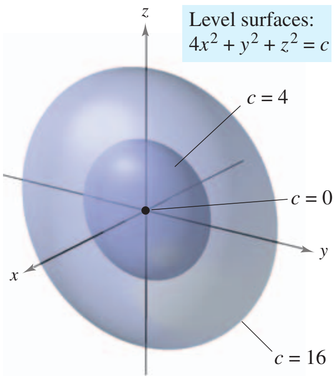


Figure 13.16

Rotatable Graph

Try It

Exploration A

NOTE If the function in Example 6 represented the *temperature* at the point (x, y, z), the level surfaces shown in Figure 13.16 would be called **isothermal surfaces.**

Computer Graphics

The problem of sketching the graph of a surface in space can be simplified by using a computer. Although there are several types of three-dimensional graphing utilities, most use some form of trace analysis to give the illusion of three dimensions. To use such a graphing utility, you usually need to enter the equation of the surface, the region in the xy-plane over which the surface is to be plotted, and the number of traces to be taken. For instance, to graph the surface given by

$$f(x, y) = (x^2 + y^2)e^{1-x^2-y^2}$$

you might choose the following bounds for x, y, and z.

$$-3 \le x \le 3 \qquad \text{Bounds for } x$$

$$-3 \le y \le 3 \qquad \text{Bounds for } y$$

$$0 \le z \le 3 \qquad \text{Bounds for } z$$

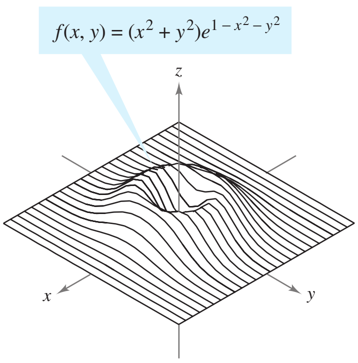


Figure 13.17

Rotatable Graph

Figure 13.17 shows a computer-generated graph of this surface using 26 traces taken parallel to the yz-plane. To heighten the three-dimensional effect, the program uses a "hidden line" routine. That is, it begins by plotting the traces in the foreground (those corresponding to the largest x-values), and then, as each new trace is plotted, the program determines whether all or only part of the next trace should be shown.

The graphs on page 891 show a variety of surfaces that were plotted by computer. If you have access to a computer drawing program, use it to reproduce these surfaces. Remember also that the three-dimensional graphics in this text can be viewed and rotated.

This is the equation of a sphere of radius 10, with center at the origin. Similarly, the level surface corresponding to $T = 200$ is the sphere with radius $\sqrt{200}$. The other level surfaces are concentric spheres. The temperature is constant on each sphere. We may view the temperature distribution as a set of nested spheres, like concentric layers of an onion, each one labeled with a different temperature, starting from low temperatures in the middle and getting hotter as we go out from the center. (See Figure 12.69.) The level surfaces become more closely spaced as we move farther from the origin because the temperature increases more rapidly the farther we get from the origin.

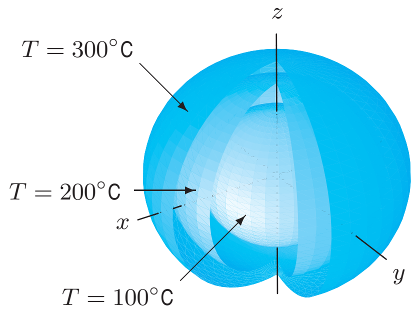


Figure 12.69: Level surfaces of $T = f(x, y, z) = x^2 + y^2 + z^2$, each one having a constant temperature

Example 2 What do the level surfaces of $f(x, y, z) = x^2 + y^2$ and $g(x, y, z) = z - y$ look like?

Solution The level surface of f corresponding to the constant c is the surface consisting of all points satisfying the equation

$$x^2 + y^2 = c.$$

Since there is no z-coordinate in the equation, z can take any value. For $c > 0$, this is a circular cylinder of radius $\sqrt{c}$ around the z-axis. The level surfaces are concentric cylinders; on the narrow ones near the z-axis, f has small values; on the wider ones, f has larger values. See Figure 12.70.

The level surface of g corresponding to the constant c is the plane

$$z - y = c.$$

Since there is no x variable in the equation, these plane are parallel to the x-axis and cut the yz-plane in the line $z - y = c$. See Figure 12.71.

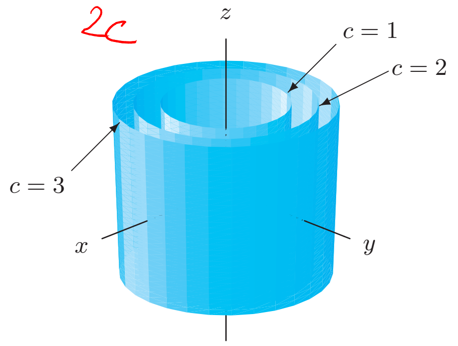


Figure 12.70: Level surfaces of $f(x, y, z) = x^2 + y^2$

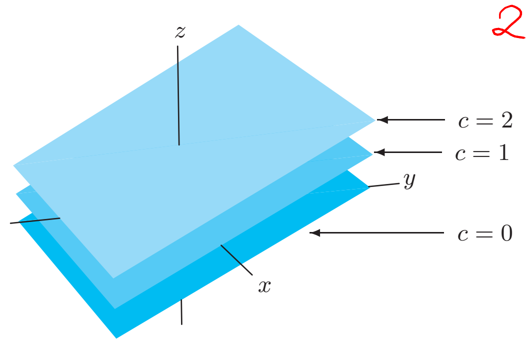


Figure 12.71: Level surfaces of $g(x, y, z) = z - y$

Example 3 What do the level surfaces of $f(x, y, z) = x^2 + y^2 - z^2$ look like?

Solution In Section 12.3, we saw that the two-variable quadratic function $g(x, y) = x^2 - y^2$ has a saddle-shaped graph and three types of contours. The contour equation $x^2 - y^2 = c$ gives a hyperbola opening right-left when $c > 0$, a hyperbola opening up-down when $c < 0$, and a pair of intersecting lines when $c = 0$. Similarly, the three-variable quadratic function $f(x, y, z) = x^2 + y^2 - z^2$ has three types of level surfaces depending on the value of c in the equation $x^2 + y^2 - z^2 = c$.

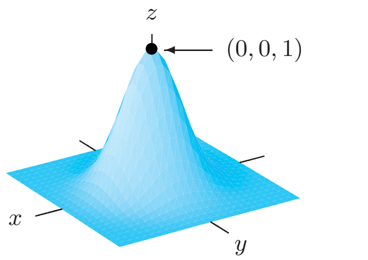


Figure 12.17: Graph of $G(x, y) = e^{-(x^2+y^2)}$

Now consider a point (x, y) on the circle $x^2 + y^2 = r^2$. Since

$$G(x, y) = e^{-(x^2+y^2)} = e^{-r^2},$$

the value of the function G is the same at all points on this circle. Thus, we say the graph of G has *circular symmetry*.

Cross-Sections and the Graph of a Function

We have seen that a good way to analyze a function of two variables is to let one variable vary while the other is kept fixed.

3a

> For a function $f(x, y)$, the function we get by holding x fixed and letting y vary is called a **cross-section** of f with x fixed. The graph of the cross-section of $f(x, y)$ with $x = c$ is the curve, or cross-section, we get by intersecting the graph of f with the plane $x = c$. We define a cross-section of f with y fixed similarly.

For example, the cross-section of $f(x, y) = x^2 + y^2$ with $x = 2$ is $f(2, y) = 4 + y^2$. The graph of this cross-section is the curve we get by intersecting the graph of f with the plane perpendicular to the x-axis at $x = 2$. (See Figure 12.18.)

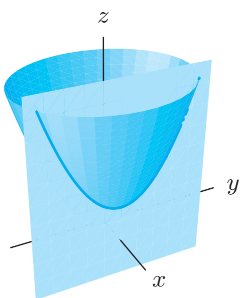


Figure 12.18: Cross-section of the surface $z = f(x, y)$ by the plane $x = 2$

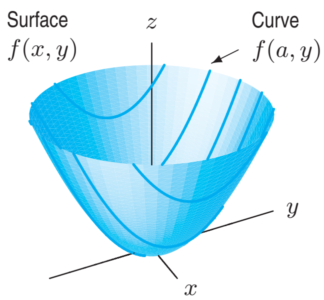


Figure 12.19: The curves $z = f(a, y)$ with a constant: cross-sections with x fixed

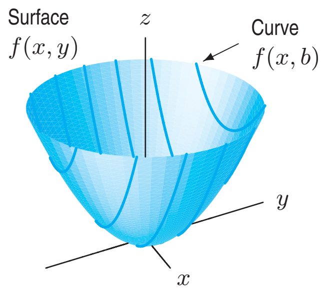


Figure 12.20: The curves $z = f(x, b)$ with b constant: cross-sections with y fixed

Figure 12.19 shows graphs of other cross-sections of f with x fixed; Figure 12.20 shows graphs of cross-sections with y fixed.

Example 3 Describe the cross-sections of the function $g(x, y) = x^2 - y^2$ with y fixed and then with x fixed. Use these cross-sections to describe the shape of the graph of g.

3b

Solution The cross-sections with y fixed at $y = b$ are given by

$$z = g(x, b) = x^2 - b^2.$$

Thus, each cross-section with y fixed gives a parabola opening upward, with minimum $z = -b^2$. The cross-sections with x fixed are of the form

$$z = g(a, y) = a^2 - y^2,$$

which are parabolas opening downward with a maximum of $z = a^2$. (See Figures 12.21 and 12.22.) The graph of g is shown in Figure 12.23. Notice the upward-opening parabolas in the x-direction and the downward-opening parabolas in the y-direction. We say that the surface is *saddle-shaped*.

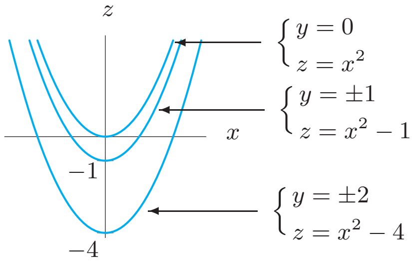


Figure 12.21: Cross-sections of $g(x, y) = x^2 - y^2$ with y fixed

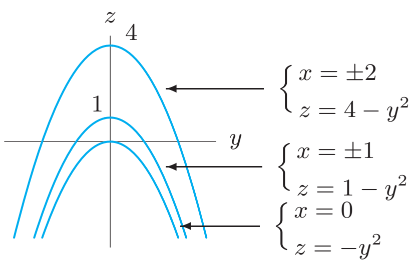


Figure 12.22: Cross-sections of $g(x, y) = x^2 - y^2$ with x fixed

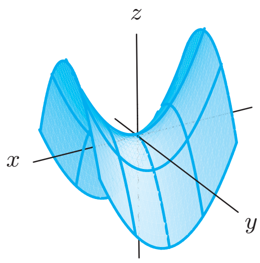


Figure 12.23: Graph of $g(x, y) = x^2 - y^2$ showing cross sections

Linear Functions

Linear functions are central to single-variable calculus; they are equally important in multivariable calculus. You may be able to guess the shape of the graph of a linear function of two variables. (It's a plane.) Let's look at an example.

Example 4 Describe the graph of $f(x, y) = 1 + x - y$.

Solution The plane $x = a$ is vertical and parallel to the yz-plane. Thus, the cross-section with $x = a$ is the line $z = 1 + a - y$ which slopes downward in the y-direction. Similarly, the plane $y = b$ is parallel to the xz-plane. Thus, the cross-section with $y = b$ is the line $z = 1 + x - b$ which slopes upward in the x-direction. Since all the cross-sections are lines, you might expect the graph to be a flat plane, sloping down in the y-direction and up in the x-direction. This is indeed the case. (See Figure 12.24.)

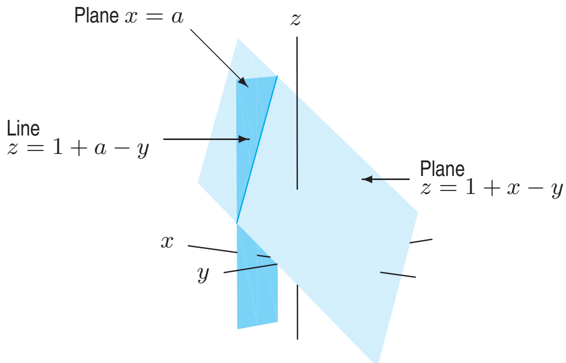


Figure 12.24: Graph of the plane $z = 1 + x - y$ showing cross-section with $x = a$

When One Variable is Missing: Cylinders

Suppose we graph an equation like $z = x^2$ which has one variable missing. What does the surface look like? Since y is missing from the equation, the cross-sections with y fixed are all the same parabola, $z = x^2$. Letting y vary up and down the y-axis, this parabola sweeps out the trough-

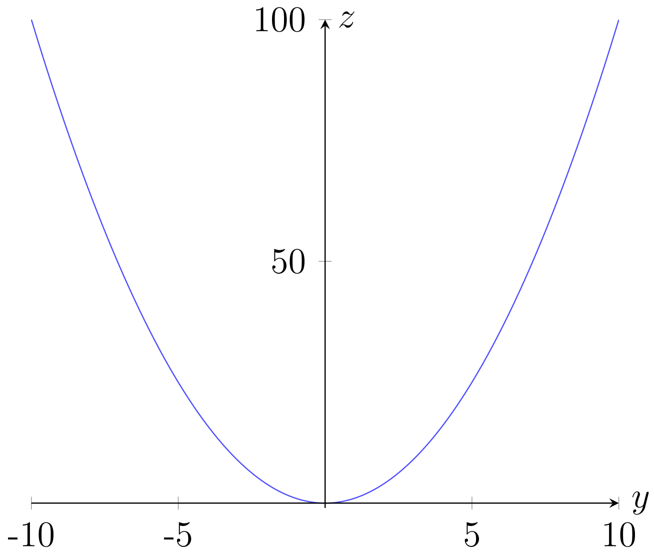
3c

a. $f(x,y) = (x-y)^2$

Let $z = f(x,y) = (x-y)^2$, and consider the following cross-sections. Taking a slice along the y-axis, we have

$$z = f(0,y) = (0-y)^2 = (-y)^2 = y^2.$$

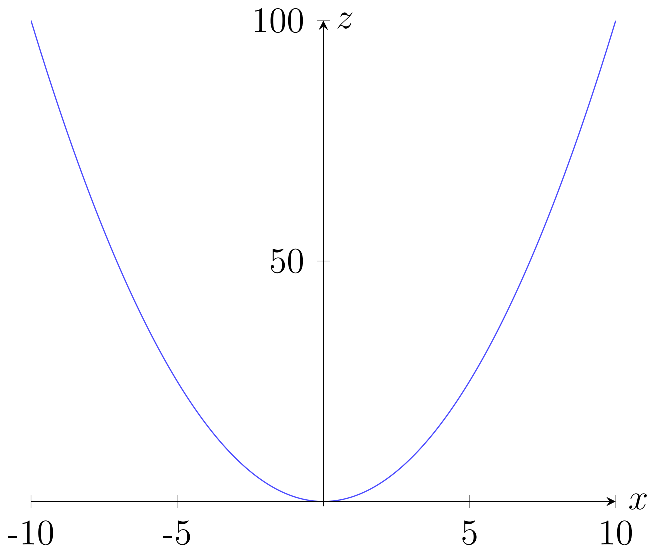
This cross-section will then look like:



Taking a slice along the x-axis, we have

$$z = f(x,0) = x^2,$$

and graphically,

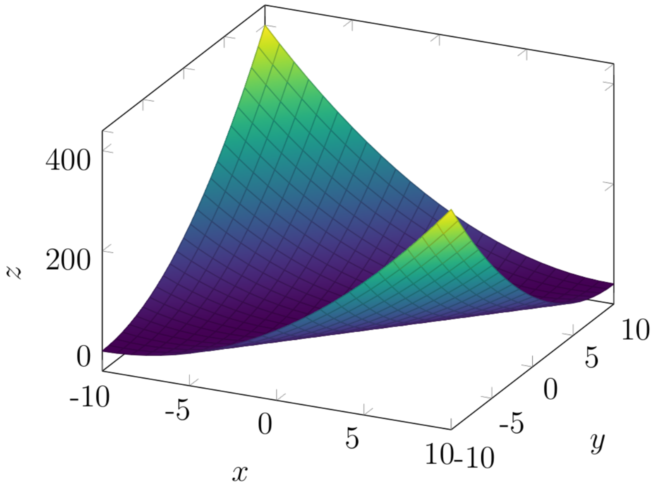


Finally, along the line $x = y$,

$$z = f(x,y) = (0)^2 = 0,$$

and so f is constant (and zero) along this cross-section. The surface

$z = f(x, y)$ is plotted below for $(x, y) \in [-5, 5] \times [-5, 5]$.



$z = y^2$, $z = x^2$, $z = 0$.

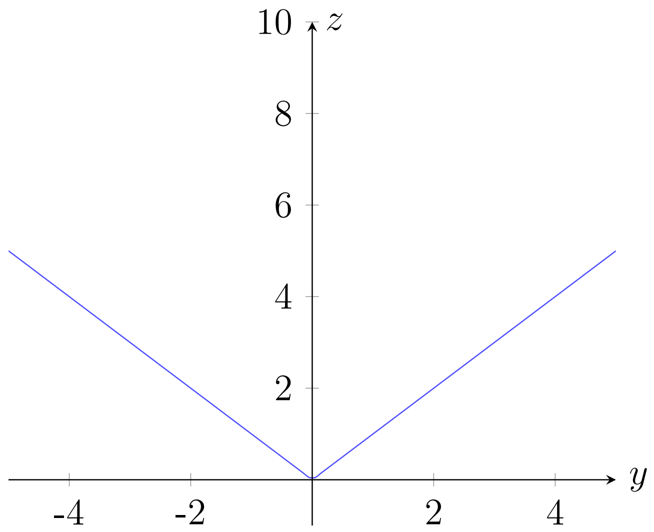
3d

b. $f(x, y) = |x| + |y|$

Let $z = f(x, y) = |x| + |y|$, and consider the following cross-sections. Taking a slice along the y-axis, we have

$$z = f(0, y) = |y|.$$

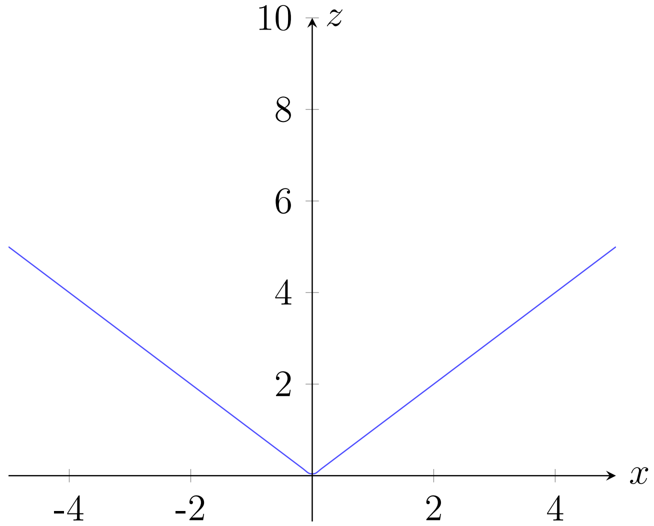
This cross-section will then look like:



Taking a slice along the x-axis, we have

$$z = f(x, 0) = |x|,$$

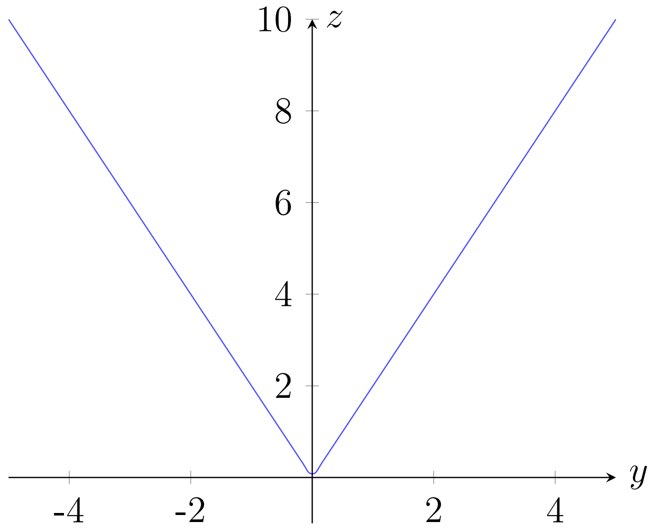
and graphically,



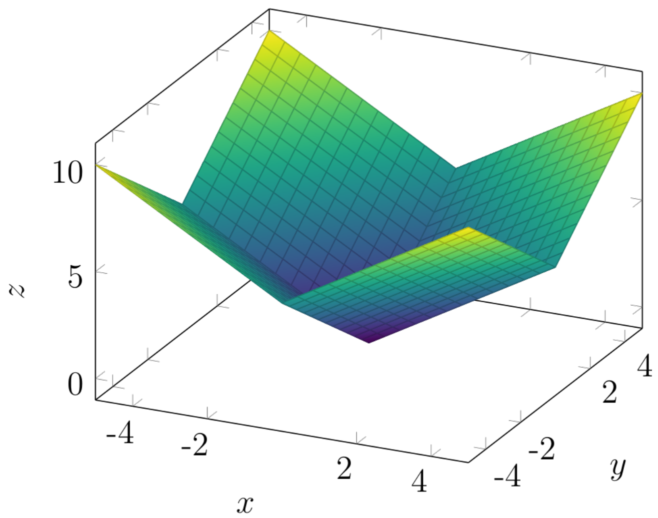
Finally, along the line $x = y$,

$$z = f(x, y) = |y| + |y| = 2|y|,$$

and graphically,



The surface $z = f(x, y)$ is plotted below for $(x, y) \in [-5, 5] \times [-5, 5]$.



$z = |y|$, $z = |x|$, $z = 2|y|$.

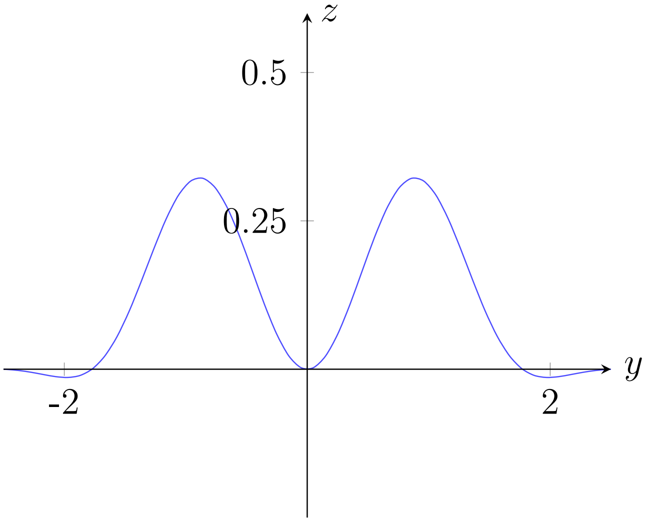
3e

c. $f(x,y) = e^{-(x^2+y^2)} \sin(x^2+y^2)$

Let $z = f(x,y) = e^{-(x^2+y^2)} \sin(x^2+y^2)$, and consider the following cross-sections. Taking a slice along the y-axis, we have

$$z = f(0,y) = e^{-(y^2)} \sin(y^2).$$

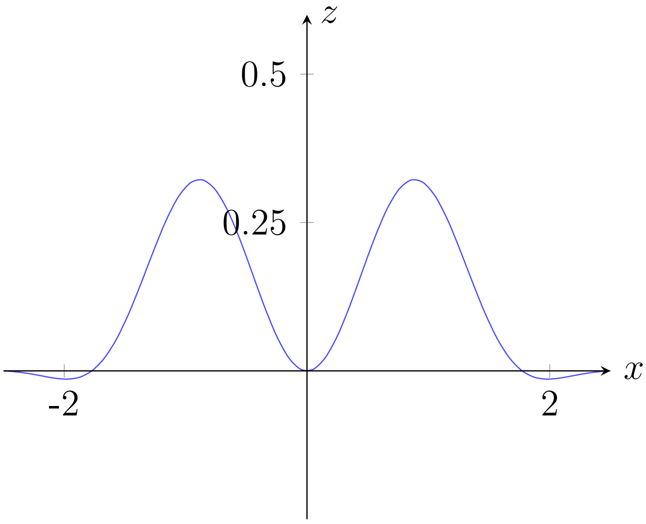
This cross-section will then look like:



Taking a slice along the x-axis, we have

$$z = f(x,0) = e^{-(x^2)} \sin(x^2),$$

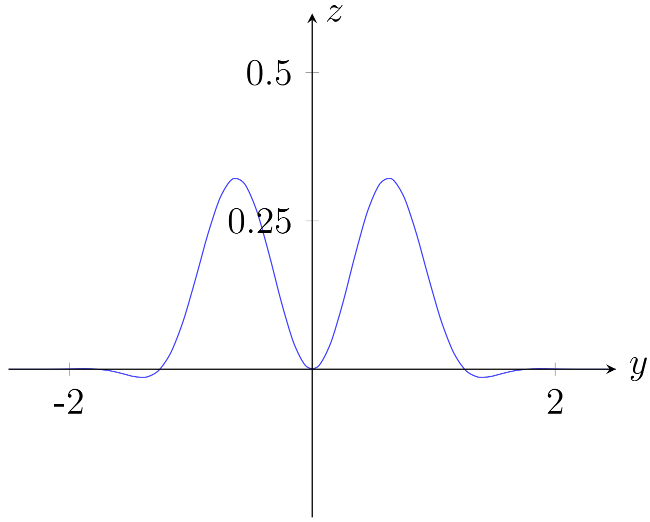
and graphically,



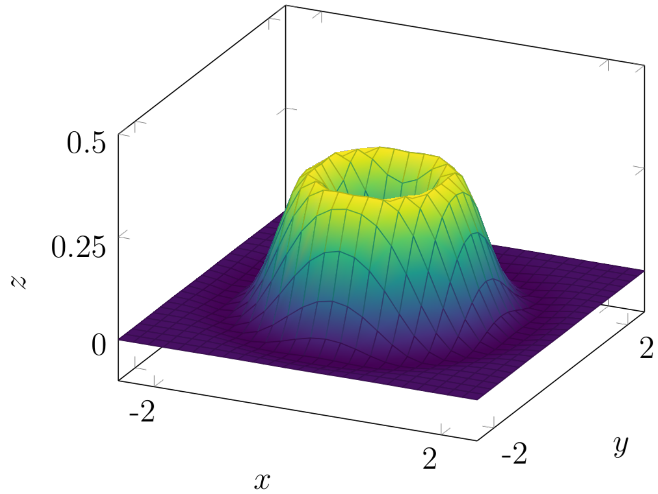
Finally, along the line $x = y$,

$$z = f(x,y) = e^{-(y^2+y^2)} \sin(y^2+y^2) = e^{-(2y^2)} \sin(2y^2),$$

and graphically,



The surface $z = f(x, y)$ is plotted below for $(x, y) \in [-5, 5] \times [-5, 5]$.

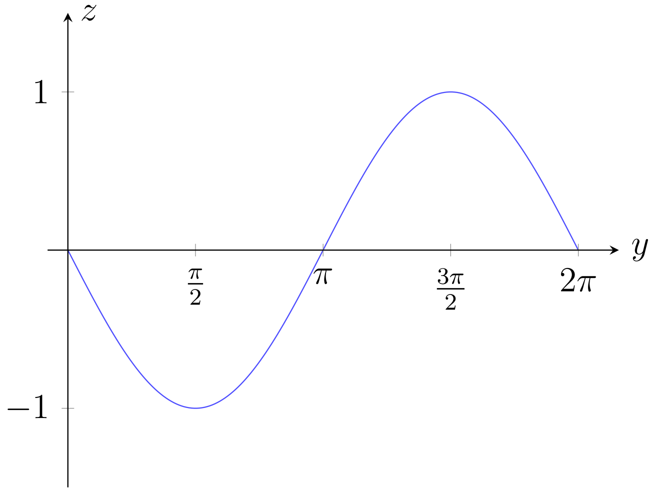


d. $f(x, y) = \sin(x - y)$

Let $z = f(x, y) = \sin(x - y)$, and consider the following cross-sections. Taking a slice along the y-axis, we have

$$z = f(0, y) = \sin(-y) = -\sin(y),$$

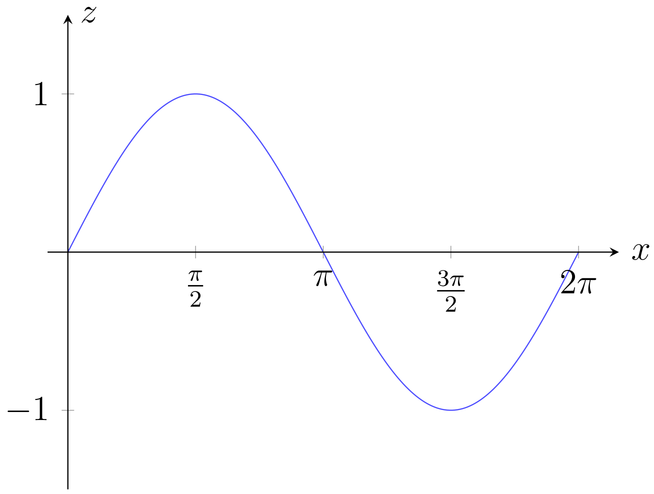
where we used the fact that sine is an even function. This cross-section will then look like:



Taking a slice along the x-axis, we have

$$z = f(x, 0) = \sin(x),$$

and graphically,



Finally, along the line $x = y$,

$$z = f(x, y) = \sin(0) = 0,$$

and so f is constant (and zero) along this cross-section. The surface $z = f(x, y)$ is plotted below for $(x, y) \in [-2\pi, 2\pi] \times [-2\pi, 2\pi]$.