10th lesson

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Theory

Definition 1. The *Euclidean metric (distance) on* $\mathbb{R}^n$ is the function $\rho\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ defined by

$$\rho(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

The number $\rho(\vec{x}, \vec{y})$ is called the *distance of the point $\vec{x}$ from the point $\vec{y}$*.

Definition 2. Let $\vec{x} \in \mathbb{R}^n$, $r \in \mathbb{R}, r > 0$. The set $B(\vec{x}, r)$ defined by

$$B(\vec{x}, r) = \{\vec{y} \in \mathbb{R}^n;\ \rho(\vec{x}, \vec{y}) < r\}$$

is called an *open ball with radius r centred at $\vec{x}$* or the *neighbourhood of $\vec{x}$*.

Definition 3. Let $M \subset \mathbb{R}^n$. We say that $\vec{x} \in \mathbb{R}^n$ is an *interior point of M*, if there exists $r > 0$ such that $B(\vec{x}, r) \subset M$.

The set of all interior points of M is called the *interior of M*, is denoted by $\operatorname{Int} M$.

The set $M \subset \mathbb{R}^n$ is *open in* $\mathbb{R}^n$, if each point of M is an interior point of M, i.e. if $M = \operatorname{Int} M$.

Remarks 4. A union of an arbitrary system of open sets is an open set.

An intersection of a finitely many open sets is an open set.

Definition 5. Let $M \subset \mathbb{R}^n$ and $\vec{x} \in \mathbb{R}^n$. We say that $\vec{x}$ is a *boundary point of M* if for each $r > 0$

$$B(\vec{x}, r) \cap M \neq \emptyset \quad \text{and} \quad B(\vec{x}, r) \cap (\mathbb{R}^n \setminus M) \neq \emptyset.$$

The *boundary of M* is the set of all boundary points of M (notation $\operatorname{bd} M$).

The *closure* of M is the set $M \cup \operatorname{bd} M$ (notation $\overline{M}$).

A set $M \subset \mathbb{R}^n$ is said to be *closed in* $\mathbb{R}^n$ if it contains all its boundary points, i.e. if $\operatorname{bd} M \subset M$, or in other words if $\overline{M} = M$.

Theorem 6 (convergence is coordinatewise)**.** Let $\vec{x}^j \in \mathbb{R}^n$ for each $j \in \mathbb{N}$ and let $\vec{x} \in \mathbb{R}^n$. The sequence $\{\vec{x}^j\}_{j=1}^{\infty}$ converges to $\vec{x}$ if and only if for each $i \in \{1, \ldots, n\}$ the sequence of real numbers $\{x_i^j\}_{j=1}^{\infty}$ converges to the real number x_i.

Theorem 7 (characterisation of closed sets)**.** Let $M \subset \mathbb{R}^n$. Then the following statements are equivalent:

1. M is closed in $\mathbb{R}^n$.
2. $\mathbb{R}^n \setminus M$ is open in $\mathbb{R}^n$.
3. Any $\vec{x} \in \mathbb{R}^n$ which is a limit of a sequence from M belongs to M.

Remarks 8. An intersection of an arbitrary system of closed sets is closed.

A union of finitely many closed sets is closed.

Definition 9. We say that the *set* $M \subset \mathbb{R}^n$ is *bounded* if there exists $r > 0$ such that $M \subset B(\vec{o}, r)$.

Definition 10. We say that a set $M \subset \mathbb{R}^n$ is *compact* if for each sequence of elements of M there exists a convergent subsequence with a limit in M.

Theorem 11 (characterisation of compact subsets of $\mathbb{R}^n$)**.** The set $M \subset \mathbb{R}^n$ is compact if and only if M is bounded and closed.

Exercises

1. Sketch the points into the coordinate system and find its distance

   (a) $A = (-1, 2)$, $B = (0, -4)$
   (b) $A = (-4, -1)$, $B = (0, 4)$
   (c) $A = (-2, -2)$, $B = (5, 3)$
   (d) $A = (-2, 1)$, $B = (0, 0)$
   (e) $A = (2, 3)$, $B = (2, -3)$

2. Which of the points $A = (1, -1, 0)$, $B = (0, 3, 4)$, $C = (2, 2, 1)$ $D = (0, -4, 0)$ lies closest to the xz-plane? Which points lies on the y-axis? Sketch them into the coordinate system.

3. You uare 2 units below the xy-plane and in the yz-plane. What are your coordinates?

4. You are standing at the point $(4, 5, 2)$, looking at the point $(0.5, 0, 3)$. Are you looking up or down?

5. Find the distance of the following points

   (a) $A = (0, 0, 0)$, $B = (3, 4, 12)$
   (b) $A = (6, 5, 4)$, $B = (10, 9, 8)$
   (c) $A = (3, 6, 1)$, $B = (1, 6, 3)$

6. Find the distance between Sokolovská 83 and V Holešovičkách 747 by

   (a) Euclidean distance
   (b) car
   (c) public transport
   (d) foot
   (e) bike
   (f) boat

   `https://en.mapy.cz/s/damedadeka`

7. Find the distance of the points $A = [-2, 3]$ and $B = [1, 7]$ in different metrics:

   (a) $\rho_2(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$
   (b) $\rho_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$
   (c) $\rho_\infty(x, y) = \max_{i=1,\dots,n} |x_i - y_i|$

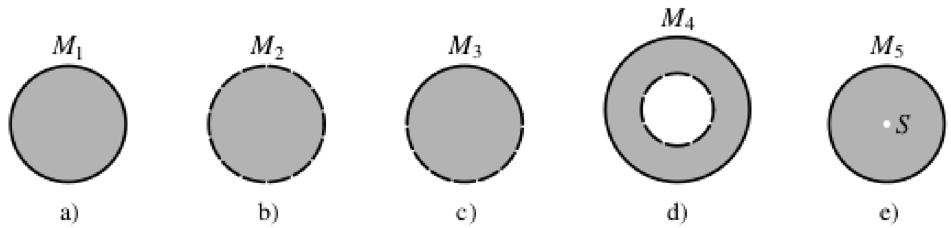
8. Decide about the boundedness of the sets:

(a) $M = \{[x,y] \in \mathbb{R}^2; \frac{x^2}{2} + y^2 \leq 2\}$
(b) $M = \{[x,y] \in \mathbb{R}^2; |x| \leq |y|\}$
(c) $M = \{[x,y] \in \mathbb{R}^2; |x| + |y| \leq 3\}$
(d) $M = \{[x,y] \in \mathbb{R}^2; |x-y| < 2\}$
(e) $M = \{(x,y,z) \in \mathbb{R}^3; 2 < xyz < 4\}$

9. Is the set M open or closed (or both or nothing)? Find its interior, closure, boundary:

(a) $M = (0,1)$
(b) $M = [0,1]$
(c) $M = (0,1]$
(d) $M = (0,\infty)$
(e) $M = [0,\infty)$
(f) $M = (-\infty,\infty)$
(g) $\mathbb{N}$
(h) $\mathbb{Q}$
(i) $\mathbb{R}$

10. Decide whether the set is open or closed (or both, or neither). Find the boundary.



11. True or false? $\overline{A \cap B} = \overline{A} \cap \overline{B}$

12. Find the limit $\lim_{n\to\infty} x_n$ of the following sequences:

(a) $x_n = \left(1 + \frac{1}{n}, 2 - \frac{1}{n}\right)$
(b) $x_n = \left(\frac{3n}{2n-1}, \pi, \cos\left(\operatorname{arccot} n\right)\right)$
(c) $x_n = \left(\frac{(-1)^n}{\ln n}, \sin\left(\pi - e^{-n}\right), \frac{1}{\sqrt{n}}, \arcsin\left(\ln\left(\frac{n}{n+3}\right)\right)\right)$
(d) $x_n = \left((-1)^n, \arctan\left(n^2\right)\right)$