Name________________________________

Distance Between Two Points (Pythagorean Theorem)

Date________________ Period____

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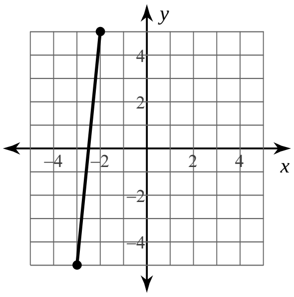
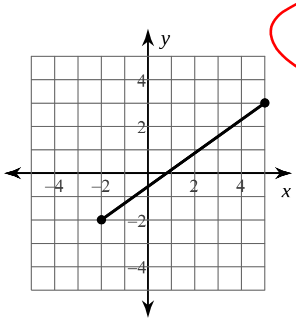
Using the Pythagorean Theorem, find the distance between each pair of points.

1)



10.1

2) 1c

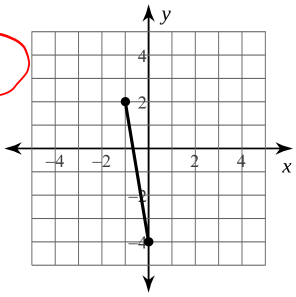


$\sqrt{25+49}$

$= \sqrt{74}$

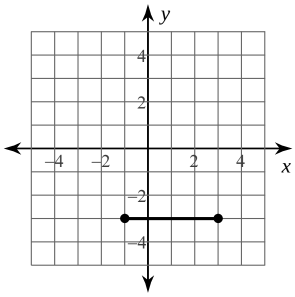
8.6

3) 1a



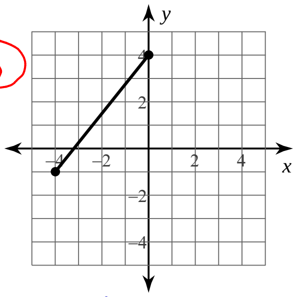
6.1 $\sqrt{1+36} = \sqrt{37}$

4)



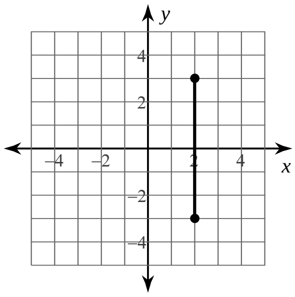
4

5) 1b



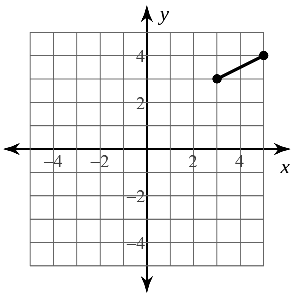
6.4 $\sqrt{16+25} = \sqrt{41}$

6) 1e



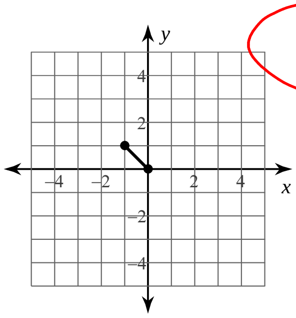
6

7)



2.2

8) 1d



1.4 $\approx \sqrt{2}$

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2

Example 7 Which of the points $A = (1, -1, 0)$, $B = (0, 3, 4)$, $C = (2, 2, 1)$, and $D = (0, -4, 0)$ lies closest to the xz-plane? Which point lies on the y-axis?

Solution The magnitude of the y-coordinate gives the distance to the xz-plane. The point A lies closest to that plane, because it has the smallest y-coordinate in magnitude. To get to a point on the y-axis, we move along the y-axis, but we don't move at all in the x- or the z-direction. Thus, a point on the y-axis has both its x- and z-coordinate equal to zero. The only point of the four that satisfies this is D. (See Figure 12.8.)

In general, if a point has one of its coordinates equal to zero, it lies in one of the coordinate planes. If a point has two of its coordinates equal to zero, it lies on one of the coordinate axes.

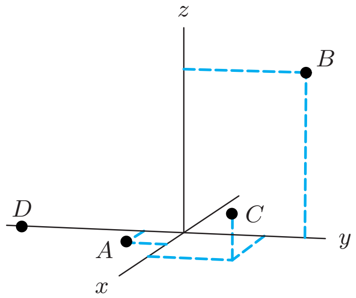

Figure 12.8: Which point lies closest to the xz-plane? Which point lies on the y-axis?

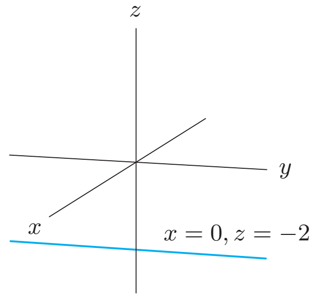

Figure 12.9: The line $x = 0$, $z = -2$

3

Example 8 You are 2 units below the xy-plane and in the yz-plane. What are your coordinates?

Solution Since you are 2 units below the xy-plane, your z-coordinate is -2. Since you are in the yz-plane, your x-coordinate is 0; your y-coordinate can be anything. Thus, you are at the point $(0, y, -2)$. The set of all such points forms a line parallel to the y-axis, 2 units below the xy-plane, and in the yz-plane. (See Figure 12.9.)

4

Example 9 You are standing at the point $(4, 5, 2)$, looking at the point $(0.5, 0, 3)$. Are you looking up or down?

Solution The point you are standing at has z-coordinate 2, whereas the point you are looking at has z-coordinate 3; hence you are looking up.

Example 10 Imagine that the yz-plane in Figure 12.7 is a page of this book. Describe the region behind the page algebraically.

Solution The positive part of the x-axis pokes out of the page; moving in the positive x-direction brings you out in front of the page. The region behind the page corresponds to negative values of x, so it is the set of all points in 3-space satisfying the inequality $x < 0$.

Distance Between Two Points

In 2-space, the formula for the distance between two points (x, y) and (a, b) is given by

$$\text{Distance} = \sqrt{(x-a)^2 + (y-b)^2}.$$

The distance between two points (x, y, z) and (a, b, c) in 3-space is represented by PG in Figure 12.10. The side PE is parallel to the x-axis, EF is parallel to the y-axis, and FG is parallel to the z-axis.

Distance and Midpoint with Three Dimensions (continued)

PRACTICE

For each pair of points, find the distance and the midpoint between the two given points.

5a **1.** (0, 0, 0) and (3, 4, 12)

$d = 13;$
midpoint $= (1.5, 2, 6)$

2. (4, −3, 6) and (1, 0, 17)

$d = \sqrt{139} \approx 11.79;$
midpoint $= (2.5, -1.5, 11.5)$

3. (6, 5, 4) and (10, 9, 8)

5b $d = \sqrt{48} \approx 6.93;$
midpoint $= (8, 7, 6)$

4. (1, 1, 1) and (10, 10, 10)

$d = \sqrt{243} \approx 15.59;$
midpoint $= (5.5, 5.5, 5.5)$

5. (−4, −5, −19) and (8, 3, 12)

$d = \sqrt{1169} \approx 34.19;$
midpoint $= (2, -1, -3.5)$

6. (5, 5, 18) and (5, 5, −8)

$d = \sqrt{676} = 26;$
midpoint $= (5, 5, 5)$

5c **7.** (3, 6, 1) and (1, 6, 3)

$d = \sqrt{8} \approx 2.83;$
midpoint $= (2, 6, 2)$

8. (9, 6, 3) and (−9, −6, −3)

$d = \sqrt{504} \approx 22.45;$
midpoint $= (0, 0, 0)$

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10th lesson

https://www2.karlin.mff.cuni.cz/~kuncova/en/teaching.php, kuncova@karlin.mff.cuni.cz

7. Find the distance of the points $A = [-2, 3]$ and $B = [1, 7]$ in different metrics:

   (a) $\rho_2(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$
   Solution:
   $$\sqrt{(-2-1)^2 + (3-7)^2} = \sqrt{25} = 5.$$

   (b) $\rho_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$
   Solution:
   $$|-2-1| + |3-7| = 3 + 4 = 7$$

   (c) $\rho_\infty(x, y) = \max_{i=1,\dots,n} |x_i - y_i|$
   Solution:
   $$\max\{|-2-1|, |3-7|\} = \max\{3, 4\} = 4$$

   https://www.geogebra.org/calculator/zbcpdav3

8. Decide about the boundedness of the sets:

   (a) $M = \{[x, y] \in \mathbb{R}^2; \frac{x^2}{2} + y^2 \leq 2\}$
   Solution: Yes, bounded. It is ellipse.

   (b) $M = \{[x, y] \in \mathbb{R}^2; |x| \leq |y|\}$
   Solution: No, unbounded. For example line $x = 0$ is in the set (and it is unbounded).

   (c) $M = \{[x, y] \in \mathbb{R}^2; |x| + |y| \leq 3\}$
   Solution: Yes, bounded. It is a rhombus.

   (d) $M = \{[x, y] \in \mathbb{R}^2; |x - y| < 2\}$
   Solution: No, unbounded. For example line $y = x$ is in the set (and it is unbounded).

   (e) $M = \{(x, y, z) \in \mathbb{R}^3;\ 2 < xyz < 4\}$
   Solution: No, unbounded. For example the curve $(x, 3/x, 1)$ is in the set (and it is unbounded).

9. Is the set M open or closed (or both or nothing)? Find its interior, closure, boundary:

   (a) $M = (0, 1)$
   Solution: Open. $\operatorname{Int} M = (0, 1)$ $\overline{M} = [0, 1]$. $\partial M = \{0, 1\}$.

   (b) $M = [0, 1]$
   Solution: Closed. $\operatorname{Int} M = (0, 1)$ $\overline{M} = [0, 1]$. $\partial M = \{0, 1\}$.

   (c) $M = (0, 1]$
   Solution: Neither open, nor closed. $\operatorname{Int} M = (0, 1)$ $\overline{M} = [0, 1]$. $\partial M = \{0, 1\}$.

(d) $M = (0, \infty)$

Solution: Open. $\operatorname{Int} M = (0, \infty)$ $\overline{M} = [0, \infty)$. $\partial M = \{0\}$.

(e) $M = [0, \infty)$

Solution: Closed. $\operatorname{Int} M = (0, \infty)$ $\overline{M} = [0, \infty)$. $\partial M = \{0\}$.

(f) $M = (-\infty, \infty)$

Solution: Both open and closed. $\operatorname{Int} M = (-\infty, \infty)$ $\overline{M} = (-\infty, \infty)$. $\partial M = \emptyset$.

(g) $\mathbb{N}$

Solution: Closed. $\operatorname{Int} M = \emptyset$. $\overline{M} = \mathbb{N}$. $\partial M = \mathbb{N}$.

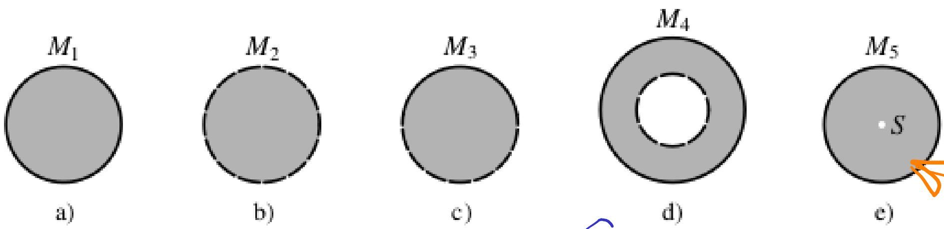
(h) $\mathbb{Q}$

Solution: Neither open nor closed. $\operatorname{Int} M = \emptyset$. $\overline{M} = \mathbb{R}$. $\partial M = \mathbb{R}$.

(i) $\mathbb{R}$

Solution: Both open and closed. $\operatorname{Int} M = \mathbb{R}$. $\overline{M} = \mathbb{R}$. $\partial M = \emptyset$.

10. Decide whether the set is open or closed (or both, or neither). Find the boundary.



closed

open

nothing

nothing

$\partial M =$

circle (just the border)

$\partial M =$

two circles

nothing

circle + point

11. True or false? $\overline{A \cap B} = \overline{A} \cap \overline{B}$

    Solution: False For example the intervals $A = (0, 1)$, $B = (1, 2)$. Then $\overline{A \cap B} = \overline{\emptyset} = \emptyset$, but $\overline{A} \cap \overline{B} = \overline{[0, 1] \cap [1, 2]} = \overline{\{1\}} = \{1\}$.

12. Find the limit of the following sequences:

    (a) $x_n = \left(1 + \frac{1}{n}, 2 - \frac{1}{n}\right)$
    Solution: Let us take the components separately.

    $$\lim_{n\to\infty} 1 + \frac{1}{n} = 1$$

    $$\lim_{n\to\infty} 2 - \frac{1}{n} = 2$$

    Together

    $$\lim_{n\to\infty} x_n = (1, 2)$$

    (b) $x_n = \left(\frac{3n}{2n-1}, \pi, \cos\left(\operatorname{arccot} n\right)\right)$
    Solution:

    $$\lim_{n\to\infty} x_n = (3/2, \pi, 1)$$

    (c) $x_n = \left(\frac{(-1)^n}{\ln n}, \sin\left(\pi - e^{-n}\right), \frac{1}{\sqrt{n}}, \arcsin\ \left(\ln\left(\frac{n}{n+3}\right)\right)\right)$
    Solution:

    $$\lim_{n\to\infty} x_n = (0, 0, 0, 0)$$

    (d) $x_n = \left((-1)^n, \arctan\ (n^2)\right)$
    Solution: $\lim_{n\to\infty}(-1)^n$ does not exist. Thus the whole limit does not exist.