# 11th lesson

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## Theory

**Theorem 1.** Functions $f(x, y) = x$, $f(x, y) = y$ are continuous at $\mathbb{R}^2$.

**Theorem 2.** Let $f$ and $g$ be continuous at a point $a$. Then

1. $f \pm g$ is continuous at $a$
2. $f \cdot g$ is continuous at $a$
3. if $g(a) \neq 0$, then $f/g$ is continuous at $a$.

Let $g$ be continuous at $a$, $f$ be continuous at $b = g(a)$. Then $f(g)$ is continuous at $a$.

**Definition 3.** We say that a function $f$ of $n$ variables has a limit at a point $\vec{a} \in \mathbb{R}^n$ equal to $A \in \mathbb{R}^*$ if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists \delta \in \mathbb{R}, \delta > 0\ \forall \vec{x} \in B(\vec{a}, \delta) \setminus \{\vec{a}\}\colon f(\vec{x}) \in B(A, \varepsilon).$$

**Remarks 4.**
- Each function has at a given point at most one limit. We write $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = A$.
- The function $f$ is continuous at $\vec{a}$ if and only if $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$.
- For limits of functions of several variables one can prove similar theorems as for limits of functions of one variable (arithmetics, the sandwich theorem, ...).

## Exercises

1. Where are continuous the following functions?

   (a) $f(x, y) = xy + \cos(y + e^x)$

   (b) $f(x, y) = \tan\,(x + y) + \operatorname{sgn}(xy)$

   (c) $f(x, y) = \frac{1}{\ln \sqrt{x^2+y^2}}$

2. Where is continuous $f(x, y) = \arctan \frac{y}{x}$?

   (a) Everywhere except at the origin
   (b) Everywhere except along the $x$-axis.
   (c) Everywhere except along the $y$-axis.
   (d) Everywhere except along the line $y = x$.

3. Find the following limits:

   (a) $\lim_{(x,y)\to(2,-1)} x^2 - 2xy + 3y^2 - 4x + 3y - 6$
   (b) $\lim_{(x,y)\to(2,-1)} \frac{2x+3y}{4x-3y}$

(c) $\lim_{(x,y)\to(4,1)} \sqrt{\frac{x^2-3xy}{x+y}}$

(d) $\lim_{(x,y)\to(-1,0)} 4\cos(3y) + \sin(x^2y^3)$

(e) $\lim_{(x,y)\to(1,4)} e^{\sqrt{x}-\sqrt{y}}$

4. Show that the following limits do not exist:

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

(b) $\lim_{(x,y)\to(0,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)^2$

(c) $\lim_{(x,y)\to(0,0)} -\frac{xy}{x^2+y^2}$

(d) $\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2}$

5. In the table there are values of a function $f(x,y)$. Does there exist the limit

$$\lim_{(x,y)\to(0,0)} f(x,y)?$$

| $x \backslash y$ | −1.0 | −0.5 | −0.2 | 0 | 0.2 | 0.5 | 1.0 |
|---|---|---|---|---|---|---|---|
| −1.0 | 0.00 | 0.60 | 0.92 | 1.00 | 0.92 | 0.60 | 0.00 |
| −0.5 | −0.60 | 0.00 | 0.72 | 1.00 | 0.72 | 0.00 | −0.6 |
| −0.2 | −0.92 | −0.72 | 0.00 | 1.00 | 0.00 | −0.72 | −0.92 |
| 0 | −1.00 | −1.00 | −1.00 | | −1.00 | −1.00 | −1.00 |
| 0.2 | −0.92 | −0.72 | 0.00 | 1.00 | 0.00 | −0.72 | −0.92 |
| 0.5 | −0.60 | 0.00 | 0.72 | 1.00 | 0.72 | 0.00 | −0.6 |
| 1.0 | 0.00 | 0.60 | 0.92 | 1.00 | 0.92 | 0.60 | 0.00 |

Source 1: https://www.cpp.edu/conceptests/question-library/mat214.shtml