11th lesson

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Exercises

1. Where are continuous the following functions?

   (a) $f(x,y) = xy + \cos(y + e^x)$

   (b) $f(x,y) = \tan\,(x+y) + \operatorname{sgn}(xy)$

   (c) $f(x,y) = \frac{1}{\ln\sqrt{x^2+y^2}}$

   (a) $[x,y] \in \mathbb{R}^2$

   (b) $[x,y] \in \mathbb{R}^2 : x + y \neq \frac{\pi}{2}, xy \neq 0$

   (c) $[x,y] \in \mathbb{R}^2 \setminus [0,0] : x^2 + y^2 \neq 1$

2. Where is continuous $f(x,y) = \arctan \frac{y}{x}$?

   (a) Everywhere except at the origin

   (b) Everywhere except along the x-axis.

   (c) Everywhere except along the y-axis.

   (d) Everywhere except along the line $y = x$.

   C - because $x \neq 0$, which is the y axis.

2a

a.

Next, use the constant multiple law on the second, third, fourth, and fifth limits:

$$= \left(\lim_{(x,y)\to(2,-1)} x^2 \right) - 2\left(\lim_{(x,y)\to(2,-1)} xy \right) + 3\left(\lim_{(x,y)\to(2,-1)} y^2 \right) - 4\left(\lim_{(x,y)\to(2,-1)} x \right)$$
$$+3\left(\lim_{(x,y)\to(2,-1)} y \right) - \lim_{(x,y)\to(2,-1)} 6.$$

Now, use the power law on the first and third limits, and the product law on the second limit:

T, a, R_p

Last, use the identity laws on the first six limits and the constant law on the last limit:

$$\begin{aligned} \lim_{(x,y)\to(2,-1)} \left(x^2 - 2xy + 3y^2 - 4x + 3y - 6\right) &= (2)^2 - 2(2)(-1) + 3(-1)^2 - 4(2) + 3(-1) - 6 \\ &= -6. \end{aligned}$$

2b

b. Before applying the quotient law, we need to verify that the limit of the denominator is nonzero. Using the difference law, constant multiple law, and identity law,

$$\begin{aligned} \lim_{(x,y)\to(2,-1)} (4x - 3y) &= \lim_{(x,y)\to(2,-1)} 4x - \lim_{(x,y)\to(2,-1)} 3y \\ &= 4\left(\lim_{(x,y)\to(2,-1)} x \right) - 3\left(\lim_{(x,y)\to(2,-1)} y \right) \\ &= 4(2) - 3(-1) = 11. \end{aligned}$$

Since the limit of the denominator is nonzero, the quotient law applies. We now calculate the limit of the numerator using the difference law, constant multiple law, and identity law:

$$\begin{aligned} \lim_{(x,y)\to(2,-1)} (2x + 3y) &= \lim_{(x,y)\to(2,-1)} 2x + \lim_{(x,y)\to(2,-1)} 3y \\ &= 2\left(\lim_{(x,y)\to(2,-1)} x \right) + 3\left(\lim_{(x,y)\to(2,-1)} y \right) \\ &= 2(2) + 3(-1) \\ &= 1. \end{aligned}$$

Therefore, according to the quotient law we have

$$\lim_{(x,y)\to(2,-1)} \frac{2x+3y}{4x-3y} = \frac{\lim_{(x,y)\to(2,-1)} (2x+3y)}{\lim_{(x,y)\to(2,-1)} (4x-3y)} = \frac{1}{11}.$$

2c

$x^2 - 3xy \to$ continuous

$x+y \to$ continuous and $\neq 0$ at $(4,1)$

$\Rightarrow \dfrac{x^2-3xy}{x+y}$ is cont. at $(4,1)$

$\sqrt{t}$ is continuous for $t \geq 0$

together $\sqrt{\dfrac{x^2-3xy}{x+y}}$ is continuous

and

$$\lim_{(x,y)\to(4,1)} \sqrt{\frac{x^2-3xy}{x+y}} = \sqrt{\frac{4^2-3\cdot 4\cdot 1}{4+1}}$$

$$= \sqrt{\frac{4}{5}} = \underline{\underline{\frac{2}{\sqrt{5}}}}$$

2d Function is continuous, hence

$$\lim_{(x,y)\to(-1,0)} 4\cos(3y) + \sin(x^2y^3) = 4\cos 0 + \sin 0$$

$$= \underline{\underline{4}}$$

2e Function is continuous for $x,y>0$:

$$\lim_{(x,y)\to(1,4)} e^{\sqrt{x}-\sqrt{y}} = e^{\sqrt{1}-\sqrt{4}} = e^{-1} = \underline{\underline{\frac{1}{e}}}$$

4a

(2) The simplest paths to try when you suspect a limit does not exist are below. Either find one where a limit does not exist or two with different limits. If you expect the limit does exist, use one of these paths to find a value for the limit, then establish that limit by methods to be given below.

(a) vertical lines: $x = a, y \longrightarrow b$

(b) horizontal lines: $y = b, x \longrightarrow a$

(c) $y = g(x)$ and $x \longrightarrow a$ while $b = g(a)$

(d) $x = g(y)$ and $y \longrightarrow b$ where $a = g(b)$

Example. Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$.

For $x = 0$ (approaching the origin along the y-axis),

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = \lim_{y\to 0}(-1) = -1.$$

But for $y = 0$ (approaching the origin along the x-axis),

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{y\to 0}(1) = 1.$$

Since the limits along the two paths are different, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist (DNE).

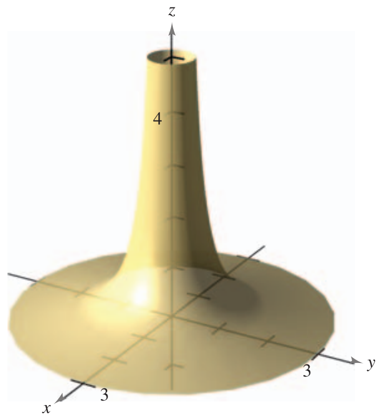
4d

For some functions, it is easy to recognize that a limit does not exist. For instance, it is clear that the limit

$$\lim_{(x,\,y)\to(0,\,0)} \frac{1}{x^2 + y^2}$$

does not exist because the values of $f(x, y)$ increase without bound as (x, y) approaches $(0, 0)$ along *any* path (see Figure 13.22).

For other functions, it is not so easy to recognize that a limit does not exist. For instance, the next example describes a limit that does not exist because the function approaches different values along different paths.



$\displaystyle\lim_{(x,\,y)\to(0,\,0)} \frac{1}{x^2 + y^2}$ does not exist.

Figure 13.22

Rotatable Graph

EXAMPLE 4 A Limit That Does Not Exist

Show that the following limit does not exist.

$$\lim_{(x,\,y)\to(0,\,0)} \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$$

4b

Solution The domain of the function given by

$$f(x, y) = \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$$

consists of all points in the xy-plane except for the point $(0, 0)$. To show that the limit as (x, y) approaches $(0, 0)$ does not exist, consider approaching $(0, 0)$ along two different "paths," as shown in Figure 13.23. Along the x-axis, every point is of the form $(x, 0)$, and the limit along this approach is

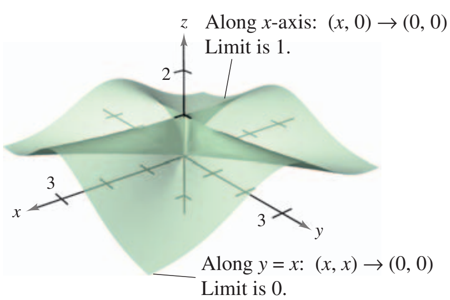
$$\lim_{(x,\,0)\to(0,\,0)} \left(\frac{x^2 - 0^2}{x^2 + 0^2}\right)^2 = \lim_{(x,\,0)\to(0,\,0)} 1^2 = 1. \qquad \text{Limit along } x\text{-axis}$$

However, if (x, y) approaches $(0, 0)$ along the line $y = x$, you obtain

$$\lim_{(x,\,x)\to(0,\,0)} \left(\frac{x^2 - x^2}{x^2 + x^2}\right)^2 = \lim_{(x,\,x)\to(0,\,0)} \left(\frac{0}{2x^2}\right)^2 = 0. \qquad \text{Limit along line } y = x$$

This means that in any open disk centered at $(0, 0)$ there are points (x, y) at which f takes on the value 1, and other points at which f takes on the value 0. For instance, $f(x, y) = 1$ at the points $(1, 0)$, $(0.1, 0)$, $(0.01, 0)$, and $(0.001, 0)$ and $f(x, y) = 0$ at the points $(1, 1)$, $(0.1, 0.1)$, $(0.01, 0.01)$, and $(0.001, 0.001)$. So, f does not have a limit as $(x, y) \to (0, 0)$.

NOTE In Example 4, you could conclude that the limit does not exist because you found two approaches that produced different limits. If two approaches had produced the same limit, you still could not have concluded that the limit exists. To form such a conclusion, you must show that the limit is the same along *all* possible approaches.



$\displaystyle\lim_{(x,\,y)\to(0,\,0)} \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$ does not exist.

Figure 13.23

Rotatable Graph

Try It

Exploration A

Open Exploration

4e

Example -1.1 Consider the function $f(x,y)$ of two variables x and y defined as

$$f(x,y) = -\frac{x\,y}{x^2 + y^2}.$$

Find the limit along the following curves as $(x,y) \to (0,0)$.

(a) the x-axis (b) the y-axis the line $y = x$

(d) the line $y = -x$ (e) the parabola $y = x^2$.

Solution (a) The x-axis has parametric equations $x = t$, $y = 0$, with $(0,0)$ corresponding to $t = 0$, so

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} f(t,0) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0.$$

Solution (b) The y-axis has parametric equations $x = 0$, $y = t$, with $(0,0)$ corresponding to $t = 0$, so

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} f(0,t) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0.$$

Solution (c) The line $y = x$ has parametric equations $x = t$, $y = t$, with $(0,0)$ corresponding to $t = 0$, so

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} f(t,t) = \lim_{t\to 0}\left(-\frac{t^2}{2t^2}\right) = \lim_{t\to 0}\left(-\frac{1}{2}\right) = -\frac{1}{2}.$$

Solution (d) The line $y = -x$-axis has parametric equations $x = t$, $y = -t$, with $(0,0)$ corresponding to $t = 0$, so

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} f(t,-t) = \lim_{t\to 0}\left(\frac{t^2}{2t^2}\right) = \lim_{t\to 0}\frac{1}{2} = \frac{1}{2}.$$

Solution (e) The parabola $y = x^2$ has parametric equations $x = t$, $y = t^2$, with $(0,0)$ corresponding to $t = 0$, so

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} f(t,t^2) = \lim_{t\to 0}\left(-\frac{t^3}{t^2 + t^4}\right) = \lim_{t\to 0}\left(-\frac{t}{1 + t^2}\right) = 0.$$

4d

For some functions, it is easy to recognize that a limit does not exist. For instance, it is clear that the limit

$$\lim_{(x, y)\to(0, 0)} \frac{1}{x^2 + y^2}$$

does not exist because the values of $f(x, y)$ increase without bound as (x, y) approaches $(0, 0)$ along *any* path (see Figure 13.22).

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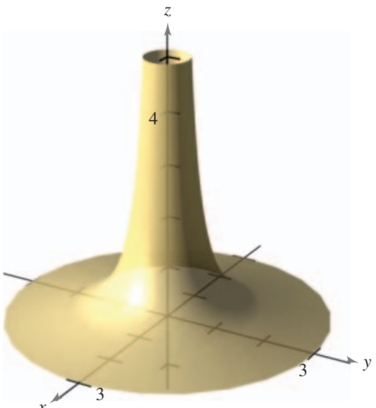

$\displaystyle\lim_{(x, y)\to(0, 0)} \frac{1}{x^2 + y^2}$ does not exist.

Figure 13.22

Rotatable Graph

EXAMPLE 4 A Limit That Does Not Exist

Show that the following limit does not exist.

$$\lim_{(x, y)\to(0, 0)} \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$$

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Solution The domain of the function given by

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consists of all points in the xy-plane except for the point $(0, 0)$. To show that the limit as (x, y) approaches $(0, 0)$ does not exist, consider approaching $(0, 0)$ along two different "paths," as shown in Figure 13.23. Along the x-axis, every point is of the form $(x, 0)$, and the limit along this approach is

$$\lim_{(x, 0)\to(0, 0)} \left(\frac{x^2 - 0^2}{x^2 + 0^2}\right)^2 = \lim_{(x, 0)\to(0, 0)} 1^2 = 1. \qquad \text{Limit along } x\text{-axis}$$

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This means that in any open disk centered at $(0, 0)$ there are points (x, y) at which f takes on the value 1, and other points at which f takes on the value 0. For instance, $f(x, y) = 1$ at the points $(1, 0)$, $(0.1, 0)$, $(0.01, 0)$, and $(0.001, 0)$ and $f(x, y) = 0$ at the points $(1, 1)$, $(0.1, 0.1)$, $(0.01, 0.01)$, and $(0.001, 0.001)$. So, f does not have a limit as $(x, y) \to (0, 0)$.

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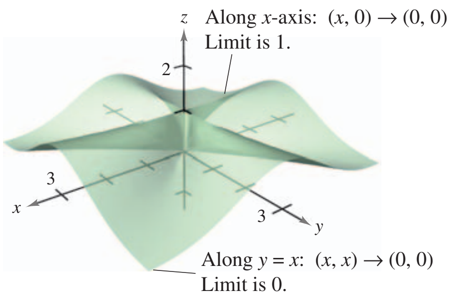

$\displaystyle\lim_{(x, y)\to(0, 0)} \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$ does not exist.

Figure 13.23

Rotatable Graph

Try It Exploration A Open Exploration

(c) $\lim_{(x,y)\to(4,1)} \sqrt{\frac{x^2-3xy}{x+y}}$

(d) $\lim_{(x,y)\to(-1,0)} 4\cos(3y) + \sin(x^2y^3)$

(e) $\lim_{(x,y)\to(1,4)} e^{\sqrt{x}-\sqrt{y}}$

4. Show that the following limits do not exist:

(a) $\lim_{(x,y)\to(0,0)} -\frac{xy}{x^2+y^2}$ (c) $\lim_{(x,y)\to(0,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)^2$ (d) $\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2}$

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

5. In the table there are values of a function $f(x,y)$. Does there exist the limit NO

$$\lim_{(x,y)\to(0,0)} f(x,y)?$$

lim = 1

$x\backslash y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.00	0.60	0.92	1.00	0.92	0.60	0.00
-0.5	-0.60	0.00	0.72	1.00	0.72	0.00	-0.6
-0.2	-0.92	-0.72	0.00	1.00	0.00	-0.72	-0.92
0	-1.00	-1.00	-1.00		-1.00	-1.00	-1.00
0.2	-0.92	-0.72	0.00	1.00	0.00	-0.72	-0.92
0.5	-0.60	0.00	0.72	1.00	0.72	0.00	-0.6
1.0	0.00	0.60	0.92	1.00	0.92	0.60	0.00

lim = -1

Source 1: https://www.cpp.edu/conceptests/question-library/mat214.shtml