13th lesson

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Theory

Set $\vec{e^j} = [0, \dots, 0, \underbrace{1}_{j\text{th coordinate}}, 0, \dots, 0]$.

Definition 1. Let f be a function of n variables, $j \in \{1, \dots, n\}$, $\vec{a} \in \mathbb{R}^n$. Then the number

$$\frac{\partial f}{\partial x_j}(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{e^j}) - f(\vec{a})}{t}$$
$$= \lim_{t \to 0} \frac{f(a_1, \dots, a_{j-1}, a_j + t, a_{j+1}, \dots, a_n) - f(a_1, \dots, a_n)}{t}$$

is called the *partial derivative (of first order) of function f according to jth variable at the point $\vec{a}$* (if the limit exists).

Definition 2. Let $G \subset \mathbb{R}^n$ be an open set, $f\colon G \to \mathbb{R}$, $i, j \in \{1, \dots, n\}$, and suppose that $\frac{\partial f}{\partial x_i}(\vec{x})$ exists finite for each $\vec{x} \in G$. Then the *partial derivative of the second order* of the function f according to ith and jth variable at a point $\vec{a} \in G$ is defined by

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) = \frac{\partial \left(\frac{\partial f}{\partial x_i} \right)}{\partial x_j}(\vec{a})$$

If $i = j$ then we use the notation $\frac{\partial^2 f}{\partial x_i^2}(\vec{a})$.

Similarly we define higher order partial derivatives.

Remarks 3. In general it is not true that $\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\vec{a})$.

Theorem 4 (interchanging of partial derivatives)**.** Let $i, j \in \{1, \dots, n\}$ and suppose that a function f has both partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_j \partial x_i}$ on a neighbourhood of a point $\vec{a} \in \mathbb{R}^n$ and that these functions are continuous at $\vec{a}$. Then

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\vec{a}).$$

Definition 5. Let $G \subset \mathbb{R}^n$ be an open set, $\vec{a} \in G$, and $f \in C^1(G)$. Then the graph of the function

$$T\colon \vec{x} \mapsto f(\vec{a}) + \frac{\partial f}{\partial x_1}(\vec{a})(x_1 - a_1) + \frac{\partial f}{\partial x_2}(\vec{a})(x_2 - a_2) + \cdots + \frac{\partial f}{\partial x_n}(\vec{a})(x_n - a_n), \quad \vec{x} \in \mathbb{R}^n,$$

is called the *tangent hyperplane* to the graph of the function f at the point $[\vec{a}, f(\vec{a})]$.

Exercises

1. Find the partial derivatives:

   (a) $f(x,y) = x^2y^2 + y^2 + 2x^3y$

   (b) $f(x,y) = e^{x^2y}$

   (c) $f(x,y) = xe^{x^2y}$

   (d) $f(x,y,z) = \frac{yze^x}{x^2 \sin y}$

   (e) $f(x,y,z) = 2x^2y + e^yz + \sqrt{z}\ln x$

   (f) $f(x,y,z) = ze^{x^2+xy}$

   (g) $f(x,y,z) = \frac{x}{(xy-z)^2}$

   (h) $f(x,y) = e^{2x^2+y^2+2xy+2y}$

   (i) $f(x,y) = \frac{x}{x+y^2}$

   (j) $f(x,y,z) = \sin(x^3+z)\ln(z)+x^2y^2z$

   (k) $f(x,y,z) = \frac{x^2e^{5y+3z}}{\sin z} + x^y$

2. Find the second order partial derivatives:

   (a) $f(x,y) = 4x^2 - 8xy^4 + 7y^5 - 3$

   (b) $f(x,y) = \sin(xy)$

   (c) $f(x,y) = \arctan \frac{y}{x}$

   (d) $f(x,y,z) = \sin(x^2 + xy)$

   (e) $f(x,y,z) = x\sqrt{y+2z}$

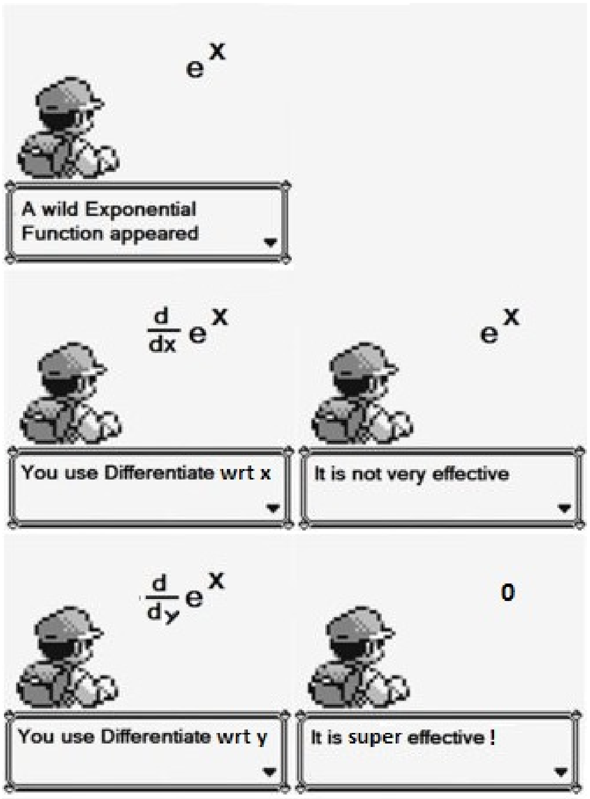
3. Find the tangent plane at the point

   (a) $f(x,y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$ at $(2,-1)$

   (b) $f(x,y) = \sin(2x)\cos(3y)$ at $(\pi/3, \pi/4)$

   (c) $f(x,y) = \frac{2x+y}{3y^2}$ at $(-2,3)$

   (d) $f(x,y,z) = x^2y^3z^4$ at $(2,1,-2)$



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