Limit Definition

As with derivatives in calculus I, there is a limit definition for partial derivatives:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y}{h}$$
$$\frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y}{h}$$

We won't be using the limit definition to find partial derivatives in this class, but we would need it if we wanted to go through a later example.

Example 1. Find all first partial derivatives of the following functions:

1a

1. $f(x,y)=x^2y^2+y^2+2x^3y$ $\quad \frac{\partial f}{\partial x}=2xy^2+6x^2y,\ \frac{\partial f}{\partial y}=2x^2y+2y+2x^3,$
   $\frac{\partial^2 f}{\partial x^2}=12xy+2y^2,\ \frac{\partial^2 f}{\partial y\,\partial x}=6x^2+4xy,\ \frac{\partial^2 f}{\partial y^2}=2+2x^2,\ \frac{\partial^2 f}{\partial x\,\partial y}=6x^2+4xy,$
   $\frac{\partial^3 f}{\partial x^3}=12y,\ \frac{\partial^3 f}{\partial y\,\partial x^2}=12x+4y=\frac{\partial^3 f}{\partial x^2\,\partial y}=\frac{\partial^3 f}{\partial x\,\partial y\,\partial x},\ \frac{\partial^3 f}{\partial y^3}=0,\ \frac{\partial^3 f}{\partial x\,\partial y^2}=4x=\frac{\partial^3 f}{\partial y^2\,\partial x}=\frac{\partial^3 f}{\partial y\,\partial x\,\partial y}$

1b

2. $f(x,y)=e^{x^2y}$ $\quad \frac{\partial f}{\partial x}=2xye^{x^2y},\ \frac{\partial f}{\partial y}=x^2e^{x^2y},$
   $\frac{\partial^2 f}{\partial x^2}=4x^2y^2e^{x^2y}+2ye^{x^2y},\ \frac{\partial^2 f}{\partial y\,\partial x}=2xe^{x^2y}+2x^3ye^{x^2y},\ \frac{\partial^2 f}{\partial y^2}=x^4e^{x^2y},\ \frac{\partial^2 f}{\partial x\,\partial y}=2xe^{x^2y}+2x^3ye^{x^2y}$

1c

3. $f(x,y)=xe^{x^2y}$ $\quad \frac{\partial f}{\partial x}=e^{x^2y}+2x^2ye^{x^2y},\ \frac{\partial f}{\partial y}=x^3e^{x^2y},$
   $\frac{\partial^2 f}{\partial x^2}=6xye^{x^2y}+4x^3y^2e^{x^2y},\ \frac{\partial^2 f}{\partial y\,\partial x}=3x^2e^{x^2y}+2x^4ye^{x^2y},\ \frac{\partial^2 f}{\partial y^2}=x^5e^{x^2y},\ \frac{\partial^2 f}{\partial x\,\partial y}=3x^2e^{x^2y}+2x^4ye^{x^2y}$

1d

4. $h(x,y,z)=\frac{yze^x}{x^2\sin(y)}$ $\quad \frac{\partial h}{\partial x}=\frac{yze^xx^2\sin(y)-yze^x2x\sin(y)}{(x^2\sin(y))^2}=\ldots=\frac{yze^x(x-1)}{x^3\sin(y)},$
   $\frac{\partial h}{\partial y}=\frac{ze^xx^2\sin(y)-yze^xx^2\cos(y)}{(x^2\sin(y))^2}=\frac{ze^x\left(\sin(y)-y\cos(y)\right)}{x^4\sin^2(y)},\ \frac{\partial h}{\partial z}=\frac{ye^x}{x^2\sin(y)}$

Some additional examples we'll look at if time permits:

1e

5. $f(x,y,z)=2x^2y+e^yz+\sqrt{z}\ln(x)$ $\quad \frac{\partial f}{\partial x}=4xy+\frac{\sqrt{z}}{x},\ \frac{\partial f}{\partial y}=2x^2+e^yz,\ \frac{\partial f}{\partial z}=e^y+\frac{\ln(x)}{2\sqrt{z}}$

1f

6. $f(x,y,z)=ze^{x^2+xy}$ $\quad \frac{\partial f}{\partial x}=z(2x+y)e^{x^2+xy},\ \frac{\partial f}{\partial y}=xze^{x^2+xy},\ \frac{\partial f}{\partial z}=e^{x^2+xy}$

1g

7. $f(x,y,z)=\frac{x}{(xy-z)^2}$ $\quad \frac{\partial f}{\partial x}=-\frac{xy+z}{(xy-z)^3},\ \frac{\partial f}{\partial y}=-\frac{2x^2}{(xy-z)^3},\ \frac{\partial f}{\partial z}=\frac{2x}{(xy-z)^3}$

Higher Order Derivatives

We can find second order derivatives by simply differentiating the first order partial derivatives again. We can find third or higher order derivatives in a similar manner.

MA2: Practice problems—Derivatives, geometry
Brief solutions

1. $\frac{\partial f}{\partial x} = (4x+2y)e^{2x^2+y^2+2xy+2y}$, $\qquad \frac{\partial f}{\partial y} = (2y+2x+2)e^{2x^2+y^2+2xy+2y}$.

2. $\frac{\partial f}{\partial x} = \frac{(x+y^2)-x}{(x+y^2)^2} = \frac{y^2}{(x+y^2)^2}$, $\qquad \frac{\partial f}{\partial y} = -\frac{2xy}{(x+y^2)^2}$.

3. $\frac{\partial f}{\partial x} = \cos(x^3+z)3x^2\ln(z) + 2xy^2z$, $\qquad \frac{\partial f}{\partial y} = x^2 2yz$, $\qquad \frac{\partial f}{\partial z} = \cos(x^3+z)\ln(z) + \sin(x^3+z)\frac{1}{z} + x^2y^2$.

4. $\frac{\partial f}{\partial x} = 2x\frac{e^{5y+3z}}{\sin(z)} + y\,x^{y-1}$, $\qquad \frac{\partial f}{\partial y} = 5e^{5y+3z}\frac{x^2}{\sin(z)} + \ln(x)x^y$, $\qquad \frac{\partial f}{\partial z} = x^2\frac{3e^{5y+3z}\sin(z)-e^{5y+3z}\cos(z)}{\sin^2(z)}$.

5. $\frac{\partial f}{\partial x} = (2x+y)\cos(x^2+xy)$, $\qquad \frac{\partial f}{\partial y} = x\,\cos(x^2+xy)$;
$\frac{\partial^2 f}{\partial x^2} = 2\cos(x^2+xy) - (2x+y)^2\sin(x^2+xy)$, $\qquad \frac{\partial^2 f}{\partial x\partial y} = \cos(x^2+xy) - x(2x+y)\sin(x^2+xy)$,
$\frac{\partial^2 f}{\partial y^2} = -x^2\,\sin(x^2+xy)$.

6. $\frac{\partial f}{\partial x} = \sqrt{y+2z}$, $\qquad \frac{\partial f}{\partial y} = \frac{x}{2\sqrt{y+2z}}$, $\qquad \frac{\partial f}{\partial z} = \frac{x}{\sqrt{y+2z}}$;
$\frac{\partial^2 f}{\partial x^2} = 0$, $\qquad \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x} = \frac{1}{2\sqrt{y+2z}}$, $\qquad \frac{\partial^2 f}{\partial x\partial z} = \frac{\partial^2 f}{\partial z\partial x} = \frac{1}{\sqrt{y+2z}}$, $\qquad \frac{\partial^2 f}{\partial y^2} = \frac{-x}{4[\sqrt{y+2z}]^3}$,
$\frac{\partial^2 f}{\partial y\partial z} = \frac{\partial^2 f}{\partial z\partial y} = \frac{-x}{2[\sqrt{y+2z}]^3}$, $\qquad \frac{\partial^2 f}{\partial z^2} = \frac{-x}{[\sqrt{y+2z}]^3}$.
Derivatives are simpler with $\frac{1}{\sqrt{y+2z}} = (y+2z)^{-1/2}$.

7. $\frac{\partial f}{\partial x} = \ln(xy+1) + \frac{xy}{xy+1}$, $\qquad \frac{\partial f}{\partial y} = \frac{x^2}{xy+1}$;
therefore $\nabla f(1,0) = (0,1)$ and $D_{\vec{u}}f(1,0) = \nabla f(1,0)\bullet\vec{u} = \frac{1}{\sqrt{5}}$.

8. a) We need the direction of the maximal descent, which is $-\nabla f(1,2)$. We have
$\nabla f = \big(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\big) = \big(\frac{-6x}{(3x^2+y^2+1)^2}, \frac{-2y}{(3x^2+y^2+1)^2}\big)$. Thus $-\nabla f(1,2) = \big(\frac{3}{32}, \frac{2}{32}\big)$, we can take $\vec{d} = (3,2)$.
b) We need the directional derivative $D_{\vec{u}}f(1,2)$, where $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{5}(-3,4)$, therefore
$D_{\vec{u}}f(1,2) = \nabla f(1,2)\bullet\vec{u} = \frac{1}{32}(3,2)\bullet\frac{1}{5}(-3,4) = \frac{-1}{160}$. The ground goes down in this direction.

9. There are several possible ways to approach this problem.
1) One can place the rectangle so that the upper left corner is at the origin, then the lower right corner is at $(10,-5)$. The area is given by $A(x,y) = -xy$. We want to know the rate of change of A when the point $(10,-5)$ changes in direction (and magnitude) $\vec{v} = (2,2)$.
$\nabla A = (-y,-x) \implies \nabla A(10,-5) = (5,-10)$, therefore $D_{\vec{v}}A(10,-5) = -10$. The area starts getting smaller at the rate 10 cm^2/sec.
2) We may simply consider the area $A(x,y) = xy$ and consider the case when $x = x(t)$, $y = y(t)$ depend on time. We differentiate with respect to t: $A'(x,y) = \frac{\partial A}{\partial x}x'(t) + \frac{\partial A}{\partial y}y'(t) = y\,x'(t) + x\,y'(t)$.
We have $x = 10$, $y = 5$ and the given data are $x' = 2$, $y' = -2$. Thus $A'(10,5) = 5\cdot 2 + 10\cdot(-2) = -10$.
We can also use the total differential, $\mathrm{d}A(10,5) = 5\mathrm{d}x + 10\mathrm{d}y$.

10. We interpret it as a level curve problem for $f(x,y) = \frac{x^2}{4} + y^2 = 1$. We check that the given point P satisfies $f(x,y) = 1$, so it indeed lies on this curve.
$\frac{\partial f}{\partial x} = \frac{1}{2}x$, $\frac{\partial f}{\partial y} = 2y$, therefore $\nabla f\big(\sqrt{3},-\frac{1}{2}\big) = \big(\frac{1}{2}\sqrt{3},-1\big)$.
This vector is perpendicular (normal) to the curve, hence also to the tangent line. Thus its equation is
$\nabla f(P)\bullet((x,y)-P) = 0 \implies \frac{1}{2}\sqrt{3}\big(x-\sqrt{3}\big) - \big(y+\frac{1}{2}\big) = 0 \implies y = \frac{1}{2}\sqrt{3}x - 1$.
To find the normal line, we can use $\nabla f(P)$ as its directional vector, obtaining parametric equations
$x = \sqrt{3} + \frac{1}{2}\sqrt{3}\,t$, $y = -\frac{1}{2}t$. To get classical equations we elliminate t, obtaining $x + \frac{1}{2}\sqrt{3}y = \frac{3}{4}\sqrt{3}$.
One can also find a vector perpendicular to $\nabla f(P)$, for instance vector $\big(1,\frac{1}{2}\sqrt{3}\big)$, and find the equation of the normal line using $\big(1,\frac{1}{2}\sqrt{3}\big)\bullet((x,y)-P) = 0$, again we end up with $x + \frac{1}{2}\sqrt{3}y = \frac{3}{4}\sqrt{3}$.

11. We interpret it as a level curve problem, $f(x,y,z) = \frac{(x-1)^2}{2} + \frac{y^2}{3} + \frac{z^2}{6} = 1$. We check that the given point P satisfies $f(x,y,z) = 1$, so it indeed lies on this curve.
$\frac{\partial f}{\partial x} = x-1$, $\frac{\partial f}{\partial y} = \frac{2}{3}y$, $\frac{\partial f}{\partial y} = \frac{1}{3}z$, therefore $\nabla f(0,1,-1) = \big(-1,\frac{2}{3},-\frac{1}{3}\big)$.
This vector is perpendicular (normal) to the curve, hence also to the tangent plane. Thus its equation is
$\nabla f(P)\bullet((x,y,z)-P) = 0 \implies -x+\frac{2}{3}(y-1)-\frac{1}{3}(z+1) = 0 \implies -x+\frac{2}{3}y-\frac{1}{3}z = 1 \implies 3x-2y+z+3 = 0$.

Solution.

2a

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 8x - 8y^4 \\
\frac{\partial z}{\partial y} &= -8x(4y^3) + 35y^4 = -32xy^3 + 35y^4 \\
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = 8 \\
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \\
&= \frac{\partial}{\partial y}(-32xy^3 + 35y^4) = -32x(3y^2) + 140y^3 \\
&= -96xy^2 + 140y^3 \\
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(-32xy^3 + 35y^4) = -32y^3 \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(8x - 8y^4) = -32y^3
\end{aligned}$$

0.9 Example

2b

Find all the first and second order partial derivatives of the function $z = \sin xy$.
Solution.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= y\cos xy \\
\frac{\partial z}{\partial y} &= x\cos xy \\
\frac{\partial^2 z}{\partial x^2} &= -y^2 \sin xy \\
\frac{\partial^2 z}{\partial y^2} &= -x^2 \sin xy \\
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(x\cos xy) = x(-y\sin xy) + \cos xy = -xy\sin xy + \cos xy \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(y\cos xy) = y(-x\sin xy) + \cos xy = -xy\sin xy + \cos xy
\end{aligned}$$

0.10 Subscript notation for second order partial derivatives

If $z = f(x, y)$ then

- z_{xx} means $\dfrac{\partial^2 z}{\partial x^2}$
- z_{yy} means $\dfrac{\partial^2 z}{\partial y^2}$

2e

Solution: Observe that the function is undefined along the line $x = 0$. Its graph is the portion of the *helicoid*[2] surface shown in figure 4.

The first partial derivatives are:

$$f_x(x,y) = \frac{-y}{x^2}\frac{1}{1+y^2/x^2} = \frac{-y}{x^2+y^2}\,,\ x \neq 0\,,$$

$$f_y(x,y) = \frac{1}{x}\frac{1}{1+y^2/x^2} = \frac{x}{x^2+y^2}\,,\ x \neq 0\,.$$

To compute the second partial derivatives, we merely differentiate the above functions with respect to either x or y:

$$f_{xx}(x,y) = \partial_x\left(\frac{-y}{x^2+y^2}\right) = \frac{2xy}{(x^2+y^2)^2}\,,\ x \neq 0\,,$$

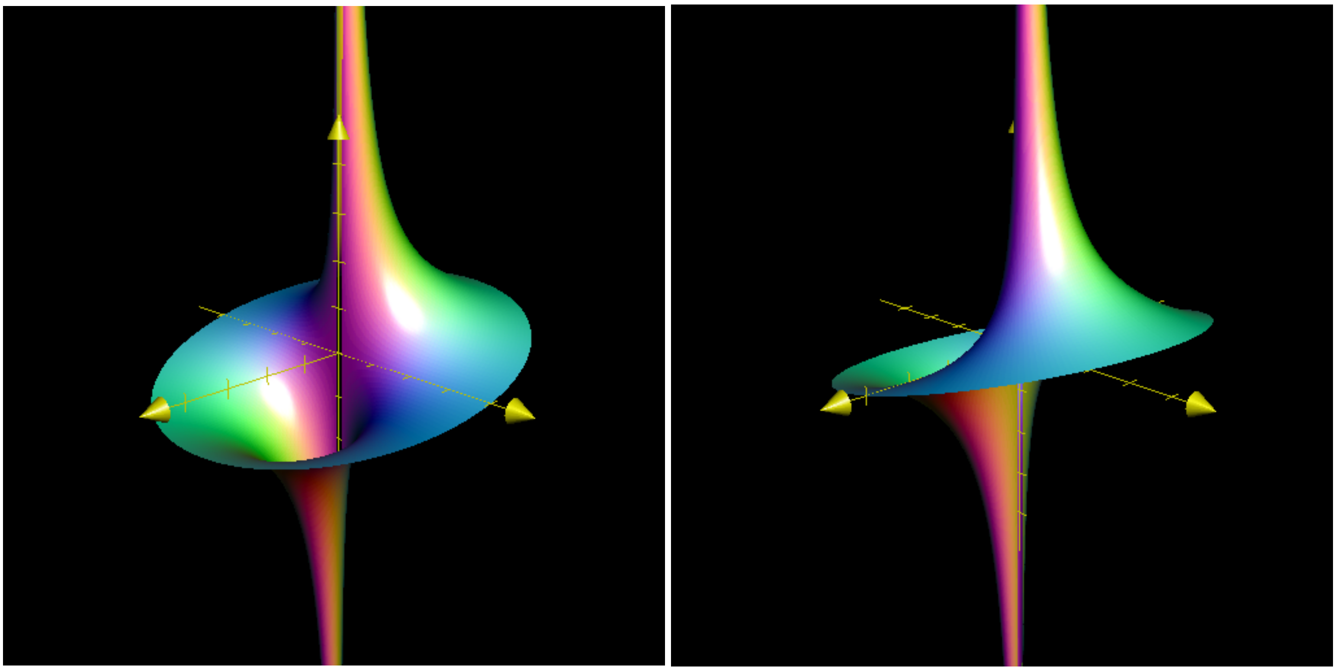
$$f_{xy}(x,y) = \partial_y\left(\frac{-y}{x^2+y^2}\right) = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},\ x \neq 0\,,$$

$$f_{yy}(x,y) = \partial_y\left(\frac{x}{x^2+y^2}\right) = \frac{-2xy}{(x^2+y^2)^2}\,,\ x \neq 0\,,$$

$$f_{yx}(x,y) = \partial_x\left(\frac{x}{x^2+y^2}\right) = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},\ x \neq 0\,.$$

Observe that $f_{xy}(x,y) = f_{yx}(x,y)$.

The graphs of the first and second partial derivative functions are shown below in figures 5, 6 and 7.



(a) (b)

FIGURE 5. (A) – The graph of $f_x(x,y) = -y/r^2$ for $f(x,y) = \arctan(y/x)$. (B) – The graph of $f_y(x,y) = x/r^2$ for $f(x,y) = \arctan(y/x)$.

[2] A helicoid is a surface swept out by revolving a line around an axis as you slide it along the axis. Stacking the graphs of functions $z_k = \arctan y/x + k\pi$ for $k \in \mathbb{Z}$, and filling in the z - axis and lines $x = 0, z = k\pi$ gives an entire helicoid. It can also be parameterized as the surface $\boldsymbol{\sigma}(u,v) = \langle u\cos v, u\sin v, v\rangle$, for $u \in \mathbb{R}$ and $v \in \mathbb{R}$.

MA2: Practice problems—Derivatives, geometry
Brief solutions

1. $\frac{\partial f}{\partial x} = (4x+2y)e^{2x^2+y^2+2xy+2y}$, $\quad \frac{\partial f}{\partial y} = (2y+2x+2)e^{2x^2+y^2+2xy+2y}$.

2. $\frac{\partial f}{\partial x} = \frac{(x+y^2)-x}{(x+y^2)^2} = \frac{y^2}{(x+y^2)^2}$, $\quad \frac{\partial f}{\partial y} = -\frac{2xy}{(x+y^2)^2}$.

3. $\frac{\partial f}{\partial x} = \cos(x^3+z)3x^2\ln(z) + 2xy^2z$, $\quad \frac{\partial f}{\partial y} = x^2 2yz$, $\quad \frac{\partial f}{\partial z} = \cos(x^3+z)\ln(z) + \sin(x^3+z)\frac{1}{z} + x^2y^2$.

4. $\frac{\partial f}{\partial x} = 2x\frac{e^{5y+3z}}{\sin(z)} + y\,x^{y-1}$, $\quad \frac{\partial f}{\partial y} = 5e^{5y+3z}\frac{x^2}{\sin(z)} + \ln(x)x^y$, $\quad \frac{\partial f}{\partial z} = x^2\frac{3e^{5y+3z}\sin(z)-e^{5y+3z}\cos(z)}{\sin^2(z)}$.

2d **5.** $\frac{\partial f}{\partial x} = (2x+y)\cos(x^2+xy)$, $\quad \frac{\partial f}{\partial y} = x\,\cos(x^2+xy)$;
$\frac{\partial^2 f}{\partial x^2} = 2\cos(x^2+xy) - (2x+y)^2\sin(x^2+xy)$, $\quad \frac{\partial^2 f}{\partial x\partial y} = \cos(x^2+xy) - x(2x+y)\sin(x^2+xy)$,
$\frac{\partial^2 f}{\partial y^2} = -x^2\,\sin(x^2+xy)$.

2e **6.** $\frac{\partial f}{\partial x} = \sqrt{y+2z}$, $\quad \frac{\partial f}{\partial y} = \frac{x}{2\sqrt{y+2z}}$, $\quad \frac{\partial f}{\partial z} = \frac{x}{\sqrt{y+2z}}$;
$\frac{\partial^2 f}{\partial x^2} = 0$, $\quad \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x} = \frac{1}{2\sqrt{y+2z}}$, $\quad \frac{\partial^2 f}{\partial x\partial z} = \frac{\partial^2 f}{\partial z\partial x} = \frac{1}{\sqrt{y+2z}}$, $\quad \frac{\partial^2 f}{\partial y^2} = \frac{-x}{4[\sqrt{y+2z}]^3}$,
$\frac{\partial^2 f}{\partial y\partial z} = \frac{\partial^2 f}{\partial z\partial y} = \frac{-x}{2[\sqrt{y+2z}]^3}$, $\quad \frac{\partial^2 f}{\partial z^2} = \frac{-x}{[\sqrt{y+2z}]^3}$.
Derivatives are simpler with $\frac{1}{\sqrt{y+2z}} = (y+2z)^{-1/2}$.

7. $\frac{\partial f}{\partial x} = \ln(xy+1) + \frac{xy}{xy+1}$, $\quad \frac{\partial f}{\partial y} = \frac{x^2}{xy+1}$;
therefore $\nabla f(1,0) = (0,1)$ and $D_{\vec{u}}f(1,0) = \nabla f(1,0)\bullet\vec{u} = \frac{1}{\sqrt{5}}$.

8. a) We need the direction of the maximal descent, which is $-\nabla f(1,2)$. We have $\nabla f = \big(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\big) = \big(\frac{-6x}{(3x^2+y^2+1)^2},\frac{-2y}{(3x^2+y^2+1)^2}\big)$. Thus $-\nabla f(1,2) = \big(\frac{3}{32},\frac{2}{32}\big)$, we can take $\vec{d} = (3,2)$.
b) We need the directional derivative $D_{\vec{u}}f(1,2)$, where $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{5}(-3,4)$, therefore $D_{\vec{u}}f(1,2) = \nabla f(1,2)\bullet\vec{u} = \frac{1}{32}(3,2)\bullet\frac{1}{5}(-3,4) = \frac{-1}{160}$. The ground goes down in this direction.

9. There are several possible ways to approach this problem.
1) One can place the rectangle so that the upper left corner is at the origin, then the lower right corner is at $(10,-5)$. The area is given by $A(x,y) = -xy$. We want to know the rate of change of A when the point $(10,-5)$ changes in direction (and magnitude) $\vec{v} = (2,2)$.
$\nabla A = (-y,-x) \implies \nabla A(10,-5) = (5,-10)$, therefore $D_{\vec{v}}A(10,-5) = -10$. The area starts getting smaller at the rate 10 cm^2/sec.
2) We may simply consider the area $A(x,y) = xy$ and consider the case when $x = x(t)$, $y = y(t)$ depend on time. We differentiate with respect to t: $A'(x,y) = \frac{\partial A}{\partial x}x'(t) + \frac{\partial A}{\partial y}y'(t) = y\,x'(t) + x\,y'(t)$.
We have $x = 10$, $y = 5$ and the given data are $x' = 2$, $y' = -2$. Thus $A'(10,5) = 5\cdot 2 + 10\cdot(-2) = -10$.
We can also use the total differential, $\mathrm{d}A(10,5) = 5\mathrm{d}x + 10\mathrm{d}y$.

10. We interpret it as a level curve problem for $f(x,y) = \frac{x^2}{4} + y^2 = 1$. We check that the given point P satisfies $f(x,y) = 1$, so it indeed lies on this curve.
$\frac{\partial f}{\partial x} = \frac{1}{2}x$, $\frac{\partial f}{\partial y} = 2y$, therefore $\nabla f\big(\sqrt{3},-\frac{1}{2}\big) = \big(\frac{1}{2}\sqrt{3},-1\big)$.
This vector is perpendicular (normal) to the curve, hence also to the tangent line. Thus its equation is $\nabla f(P)\bullet((x,y)-P) = 0 \implies \frac{1}{2}\sqrt{3}\big(x-\sqrt{3}\big) - \big(y+\frac{1}{2}\big) = 0 \implies y = \frac{1}{2}\sqrt{3}x - 1$.
To find the normal line, we can use $\nabla f(P)$ as its directional vector, obtaining parametric equations $x = \sqrt{3} + \frac{1}{2}\sqrt{3}\,t$, $y = -\frac{1}{2}t$. To get classical equations we elliminate t, obtaining $x + \frac{1}{2}\sqrt{3}y = \frac{3}{4}\sqrt{3}$.
One can also find a vector perpendicular to $\nabla f(P)$, for instance vector $\big(1,\frac{1}{2}\sqrt{3}\big)$, and find the equation of the normal line using $\big(1,\frac{1}{2}\sqrt{3}\big)\bullet((x,y)-P) = 0$, again we end up with $x + \frac{1}{2}\sqrt{3}y = \frac{3}{4}\sqrt{3}$.

11. We interpret it as a level curve problem, $f(x,y,z) = \frac{(x-1)^2}{2} + \frac{y^2}{3} + \frac{z^2}{6} = 1$. We check that the given point P satisfies $f(x,y,z) = 1$, so it indeed lies on this curve.
$\frac{\partial f}{\partial x} = x-1$, $\frac{\partial f}{\partial y} = \frac{2}{3}y$, $\frac{\partial f}{\partial y} = \frac{1}{3}z$, therefore $\nabla f(0,1,-1) = \big(-1,\frac{2}{3},-\frac{1}{3}\big)$.
This vector is perpendicular (normal) to the curve, hence also to the tangent plane. Thus its equation is $\nabla f(P)\bullet((x,y,z)-P) = 0 \implies -x+\frac{2}{3}(y-1)-\frac{1}{3}(z+1) = 0 \implies -x+\frac{2}{3}y-\frac{1}{3}z = 1 \implies 3x-2y+z+3 = 0$.

To see why this formula is correct, let's first find two tangent lines to the surface S. The equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $x = x_0$ is $z = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$. Similarly, the equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $y = y_0$ is $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0)$. A parallel vector to the first tangent line is $\vec{\mathbf{a}} = \hat{\mathbf{j}} + f_y(x_0, y_0)\,\hat{\mathbf{k}}$; a parallel vector to the second tangent line is $\vec{\mathbf{b}} = \hat{\mathbf{i}} + f_x(x_0, y_0)\,\hat{\mathbf{k}}$. We can take the cross product of these two vectors:

$$\begin{aligned}\vec{\mathbf{a}} \times \vec{\mathbf{b}} &= (\,\hat{\mathbf{j}} + f_y(x_0, y_0)\,\hat{\mathbf{k}}) \times (\,\hat{\mathbf{i}} + f_x(x_0, y_0)\,\hat{\mathbf{k}}) \\ &= \begin{vmatrix} \hat{\mathbf{i}}\,\hat{\mathbf{j}}\,\hat{\mathbf{k}} \\ 0\,1\,f_y(x_0, y_0) \\ 1\,0\,f_x(x_0, y_0) \end{vmatrix} \\ &= f_x(x_0, y_0)\,\hat{\mathbf{i}} + f_y(x_0, y_0)\,\hat{\mathbf{j}} - \hat{\mathbf{k}}.\end{aligned}$$

This vector is perpendicular to both lines and is therefore perpendicular to the tangent plane. We can use this vector as a normal vector to the tangent plane, along with the point $P_0 = (x_0, y_0, f(x_0, y_0))$ in the equation for a plane:

$$\begin{aligned}\vec{\mathbf{n}} \cdot ((x - x_0)\,\hat{\mathbf{i}} + (y - y_0)\,\hat{\mathbf{j}} + (z - f(x_0, y_0))\,\hat{\mathbf{k}}) &= 0 \\ (f_x(x_0, y_0)\,\hat{\mathbf{i}} + f_y(x_0, y_0)\,\hat{\mathbf{j}} - \hat{\mathbf{k}}) \cdot ((x - x_0)\,\hat{\mathbf{i}} + (y - y_0)\,\hat{\mathbf{j}} + (z - f(x_0, y_0))\,\hat{\mathbf{k}}) &= 0 \\ f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) &= 0.\end{aligned}$$

Solving this equation for z gives Equation 14.4.1.

Example 14.4.1: Finding a Tangent Plane

3a Find the equation of the tangent plane to the surface defined by the function $f(x, y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$ at point $(2, -1)$.

Solution

First, we must calculate $f_x(x, y)$ and $f_y(x, y)$, then use Equation with $x_0 = 2$ and $y_0 = -1$:

$$\begin{aligned}f_x(x, y) &= 4x - 3y + 2 \\ f_y(x, y) &= -3x + 16y - 4 \\ f(2, -1) = 2(2)^2 - 3(2)(-1) + 8(-1)^2 + 2(2) - 4(-1) + 4 &= 34 \\ f_x(2, -1) = 4(2) - 3(-1) + 2 &= 13 \\ f_y(2, -1) = -3(2) + 16(-1) - 4 &= -26.\end{aligned}$$

Then Equation 14.4.1becomes

$$\begin{aligned}z &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ z &= 34 + 13(x - 2) - 26(y - (-1)) \\ z &= 34 + 13x - 26 - 26y - 26 \\ z &= 13x - 26y - 18.\end{aligned}$$

(See the following figure).

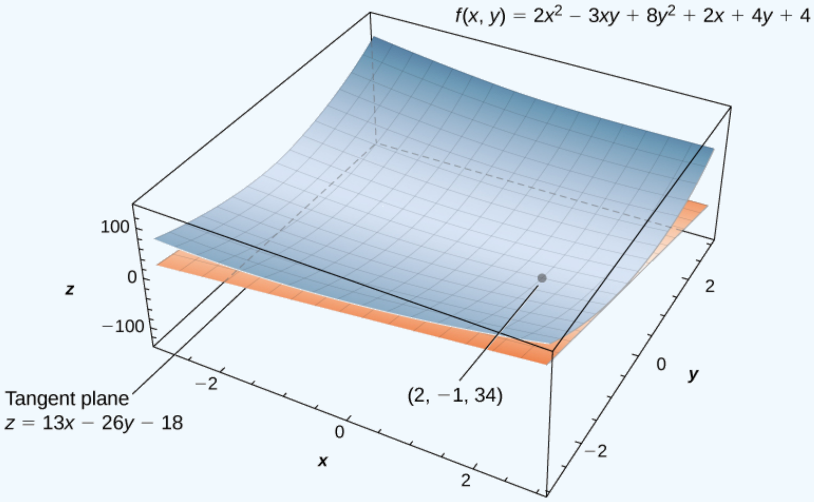


Figure 14.4.2: Calculating the equation of a tangent plane to a given surface at a given point.

Exercise 14.4.1

Find the equation of the tangent plane to the surface defined by the function $f(x,y)=x^3-x^2y+y^2-2x+3y-2$ at point $(-1,3)$.

Hint

First, calculate $f_x(x,y)$ and $f_y(x,y)$, then use Equation 14.4.1.

Answer

$z=7x+8y-3$

طك

Example 14.4.2: Finding Another Tangent Plane

Find the equation of the tangent plane to the surface defined by the function $f(x,y)=\sin(2x)\cos(3y)$ at the point $(\pi/3,\pi/4)$.

Solution

First, calculate $f_x(x,y)$ and $f_y(x,y)$, then use Equation 14.4.1with $x_0=\pi/3$ and $y_0=\pi/4$:

$$f_x(x,y)=2\cos(2x)\cos(3y)$$
$$f_y(x,y)=-3\sin(2x)\sin(3y)$$
$$f\left(\frac{\pi}{3},\frac{\pi}{4}\right)=\sin\left(2\left(\frac{\pi}{3}\right)\right)\cos(3(\frac{\pi}{4}))=(\frac{\sqrt{3}}{2})(-\frac{\sqrt{2}}{2})=-\frac{\sqrt{6}}{4}$$
$$f_x\left(\frac{\pi}{3},\frac{\pi}{4}\right)=2\cos\left(2\left(\frac{\pi}{3}\right)\right)\cos\left(3\left(\frac{\pi}{4}\right)\right)=2\left(-\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)=\frac{\sqrt{2}}{2}$$
$$f_y\left(\frac{\pi}{3},\frac{\pi}{4}\right)=-3\sin\left(2\left(\frac{\pi}{3}\right)\right)\sin\left(3\left(\frac{\pi}{4}\right)\right)=-3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)=-\frac{3\sqrt{6}}{4}.$$

Then Equation 14.4.1becomes

$$\begin{aligned}z&=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)\\&=-\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{3}\right)-\frac{3\sqrt{6}}{4}\left(y-\frac{\pi}{4}\right)\\&=\frac{\sqrt{2}}{2}x-\frac{3\sqrt{6}}{4}y-\frac{\sqrt{6}}{4}-\frac{\pi\sqrt{2}}{6}+\frac{3\pi\sqrt{6}}{16}\end{aligned}$$

A tangent plane to a surface does not always exist at every point on the surface. Consider the piecewise function

3c

Example 27.2: Find the equation of the tangent plane to $z = g(x, y) = \frac{2x+y}{3y^2}$ when $x_0 = -2$ and $y_0 = 3$.

Solution: The point of tangency is $(x_0, y_0, z_0) = \left(-2,3,-\frac{1}{27}\right)$, where $z_0 = \frac{2(-2)+(3)}{3(3)^2} = -\frac{1}{27}$.

The partial derivatives are

$$g_x(x, y) = \frac{2}{3y^2} \quad \text{and} \quad g_y(x, y) = -\frac{1+4x}{3y^3}.$$

Evaluated at $x_0 = -2$ and $y_0 = 3$, we have $g_x(-2,3) = \frac{2}{27}$ and $g_y(-2,3) = \frac{7}{81}$. Thus, the equation of the plane of tangency is

$$\frac{2}{27}(x-(-2)) + \frac{7}{81}(y-3) - \left(z - \left(-\frac{1}{27}\right)\right) = 0.$$

Multiplying by 81 to clear fractions and then distributing to clear parentheses, the equation is simplified to $6x + 7y - 81z = 12$.

♦ • ♦ • ♦ • ♦ • ♦

This process can be extended to surfaces in higher dimension.

3d

Example 27.3: Find the equation of the tangent plane to $w = f(x, y, z) = x^2y^3z^4$ at $(2,1,-2,64)$.

Solution: The partial derivatives are evaluated at $x_0 = 2$, $y_0 = 1$ and $z_0 = -2$:

$$\begin{aligned} f_x(x,y,z) &= 2xy^3z^4 \to f_x(2,1,-2) = 64, \\ f_y(x,y,z) &= 3x^2y^2z^4 \to f_y(2,1,-2) = 192, \\ f_z(x,y,z) &= 4x^2y^3z^3 \to f_z(2,1,-2) = -128. \end{aligned}$$

Thus, the plane of tangency is

$$64(x-2) + 192(y-1) - 128(z-(-2)) - 1(w-64) = 0.$$

Solving for w, we have

$$w = 64(x-2) + 192(y-1) - 128(z+2) + 64.$$

Simplified, we have $w = 64x + 192y - 128z - 512$.

♦ • ♦ • ♦ • ♦ • ♦