13th lesson

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Theory

Set $\vec{e^j} = [0, \dots, 0, \underbrace{1}_{j\text{th coordinate}}, 0, \dots, 0]$.

Definition 1. Let f be a function of n variables, $j \in \{1, \dots, n\}$, $\vec{a} \in \mathbb{R}^n$. Then the number

$$\begin{aligned}\frac{\partial f}{\partial x_j}(\vec{a}) &= \lim_{t \to 0} \frac{f(\vec{a} + t\vec{e^j}) - f(\vec{a})}{t} \\ &= \lim_{t \to 0} \frac{f(a_1, \dots, a_{j-1}, a_j + t, a_{j+1}, \dots, a_n) - f(a_1, \dots, a_n)}{t}\end{aligned}$$

is called the *partial derivative (of first order) of function f according to jth variable at the point $\vec{a}$* (if the limit exists).

Definition 2. Let $G \subset \mathbb{R}^n$ be an open set, $\vec{a} \in G$, and $f \in C^1(G)$. The *gradient of f at the point $\vec{a}$* is the vector

$$\operatorname{grad} f(\vec{a}) = \left[\frac{\partial f}{\partial x_1}(\vec{a}), \frac{\partial f}{\partial x_2}(\vec{a}), \dots, \frac{\partial f}{\partial x_n}(\vec{a})\right].$$

Exercises

1. Find the partial derivatives using the limit definition:
   (a) $f(x, y) = xy$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at a point (x_0, y_0)
   (b) $f(x, y) = x^2y + 2x + y^3$, find $\frac{\partial f}{\partial x}$ at a point (x_0, y_0)
   (c) $f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at a point (x_0, y_0)
   (d) $f(x, y) = \begin{cases} \frac{x^3 + x^4 - y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$, find $\frac{\partial f}{\partial x}(0, 0)$.
2. Find the gradient
   (a) $f(x, y) = x^2 + y^2$ at $(1, 2)$, $(2, 1)$,$(0, 0)$,
   (b) $f(x, y) = 2xy + x^2 + y$ at $(1, 1)$, $(0, -1)$,$(0, 0)$,
3. Find the directional derivative of
   (a) $f(x, y) = \frac{x}{x^2 + y^2}$ at $(1, 2)$ in the direction $\vec{v} = (3, 5)$
   (b) $f(x, y, z) = \sqrt{xyz}$ at $(3, 2, 6)$ in the direction $\vec{v} = (-1, -2, 2)$