Or we could try to evaluate the limit

$$\lim_{k \to 0} \frac{f(x, y+k) - f(x,y)}{k}.$$

We can perform both of these operations, and you should verify that the first one gives us

$$2x$$

and the second

$$3y^2.$$

But the first answer is exactly what you get when you take f, hold y as a constant, and just differentiate the function, thinking of it as a function only of x. Indeed, we could write

$$\frac{dy}{dx}(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}.$$

Indeed we shall perform this process to differentiate functions of two variables, but we shall use a slightly different notation instead to remind us of the fact that there are several variables in our function. We actually use a 'curly d' and write

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h},$$

and also

$$\frac{\partial f}{\partial y}(x,y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x,y)}{h}.$$

We do, however, read the symbol ∂ as a 'd'. A more compact notation is also used for partial derivative and the symbols

$$f_x(x,y) \text{ and } f_y(x,y)$$

are used in place of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

1a

Example 3 *Given $f(x,y) = xy$, find $f_x(x,y)$ and $f_y(x,y)$.*

Using the definition, we find

$$f_x(x,y) = \lim_{h \to 0} \frac{(x+h)y - xy}{h} = y,$$

and

$$f_y(x,y) = \lim_{k \to 0} \frac{x(y+k) - xy}{k} = x.$$

with similar notations for $f_y(x,y)$. For ease of notation, $f_x(x,y)$ is often abbreviated f_x.

Example 12.3.1: Computing partial derivatives with the limit definition

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Let $f(x,y) = x^2y + 2x + y^3$. Find $f_x(x,y)$ using the limit definition.

Solution

Using Definition 83, we have:

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2y + 2(x+h) + y^3 - (x^2y + 2x + y^3)}{h} \\ &= \lim_{h\to 0} \frac{(x^2y + 2xhy + h^2y + 2x + 2h + y^3 - (x^2y + 2x + y^3)}{h} \\ &= \lim_{h\to 0} \frac{2xhy + h^2y + 2h}{h} \\ &= \lim_{h\to 0} 2xy + hy + 2 \\ &= 2xy + 2. \end{aligned}$$

We have found $f_x(x,y) = 2xy + 2$.

Example $12.3.1$ found a partial derivative using the formal, limit--based definition. Using limits is not necessary, though, as we can rely on our previous knowledge of derivatives to compute partial derivatives easily. When computing $f_x(x,y)$, we hold y fixed -- it does not vary. Therefore we can compute the derivative with respect to x by treating y as a constant or coefficient.

Just as $\frac{d}{dx}\left(5x^2\right) = 10x$, we compute $\frac{\partial}{\partial x}\left(x^2y\right) = 2xy$. Here we are treating y as a coefficient.

Just as $\frac{d}{dx}\left(5^3\right) = 0$, we compute $\frac{\partial}{\partial x}\left(y^3\right) = 0$. Here we are treating y as a constant. More examples will help make this clear.

Example 12.3.2: Finding partial derivatives

Find $f_x(x,y)$ and $f_y(x,y)$ in each of the following.

1. $f(x,y) = x^3y^2 + 5y^2 - x + 7$
2. $f(x,y) = \cos(xy^2) + \sin x$
3. $f(x,y) = e^{x^2y^3}\sqrt{x^2+1}$

Solution

1. We have $f(x,y) = x^3y^2 + 5y^2 - x + 7$. Begin with $f_x(x,y)$. Keep y fixed, treating it as a constant or coefficient, as appropriate:

$$f_x(x,y) = 3x^2y^2 - 1. \tag{12.3.4}$$

Note how the $5y^2$ and 7 terms go to zero. To compute $f_y(x,y)$, we hold x fixed:

$$f_y(x,y) = 2x^3y + 10y. \tag{12.3.5}$$

Note how the $-x$ and 7 terms go to zero.

2. We have $f(x,y) = \cos(xy^2) + \sin x$.
Begin with $f_x(x,y)$. We need to apply the Chain Rule with the cosine term; y^2 is the coefficient of the x-term inside the cosine function.

$$f_x(x,y) = -\sin(xy^2)(y^2) + \cos x = -y^2\sin(xy^2) + \cos x. \tag{12.3.6}$$

13.3: Partial Derivatives

Learning Objectives

- Calculate the partial derivatives of a function of two variables.
- Calculate the partial derivatives of a function of more than two variables.
- Determine the higher-order derivatives of a function of two variables.
- Explain the meaning of a partial differential equation and give an example.

Now that we have examined limits and continuity of functions of two variables, we can proceed to study derivatives. Finding derivatives of functions of two variables is the key concept in this chapter, with as many applications in mathematics, science, and engineering as differentiation of single-variable functions. However, we have already seen that limits and continuity of multivariable functions have new issues and require new terminology and ideas to deal with them. This carries over into differentiation as well.

Derivatives of a Function of Two Variables

When studying derivatives of functions of one variable, we found that one interpretation of the derivative is an instantaneous rate of change of y as a function of x. Leibniz notation for the derivative is dy/dx, which implies that y is the dependent variable and x is the independent variable. For a function $z = f(x, y)$ of two variables, x and y are the independent variables and z is the dependent variable. This raises two questions right away: How do we adapt Leibniz notation for functions of two variables? Also, what is an interpretation of the derivative? The answer lies in partial derivatives.

Definition: Partial Derivatives

Let $f(x, y)$ be a function of two variables. Then the *partial derivative* of f with respect to x, written as $\partial f/\partial x$, or f_x, is defined as

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \tag{13.3.1}$$

The partial derivative of f with respect to y, written as $\partial f/\partial y$, or f_y, is defined as

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}. \tag{13.3.2}$$

This definition shows two differences already. First, the notation changes, in the sense that we still use a version of Leibniz notation, but the d in the original notation is replaced with the symbol ∂. (This rounded "d" is usually called "partial," so $\partial f/\partial x$ is spoken as the "partial of f with respect to x.") This is the first hint that we are dealing with partial derivatives. Second, we now have two different derivatives we can take, since there are two different independent variables. Depending on which variable we choose, we can come up with different partial derivatives altogether, and often do.

1e

Example 13.3.1: Calculating Partial Derivatives from the Definition

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12.$$

Solution

First, calculate $f(x+h, y)$.

$$\begin{aligned} f(x+h, y) &= (x+h)^2 - 3(x+h)y + 2y^2 - 4(x+h) + 5y - 12 \\ &= x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12. \end{aligned}$$

Next, substitute this into Equation 13.3.1 and simplify:

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$$\begin{aligned}\frac{\partial f}{\partial x} &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}\\ &= \lim_{h\to 0}\frac{(x^2+2xh+h^2-3xy-3hy+2y^2-4x-4h+5y-12)-(x^2-3xy+2y^2-4x+5y-12)}{h}\\ &= \lim_{h\to 0}\frac{x^2+2xh+h^2-3xy-3hy+2y^2-4x-4h+5y-12-x^2+3xy-2y^2+4x-5y+12}{h}\\ &= \lim_{h\to 0}\frac{2xh+h^2-3hy-4h}{h}\\ &= \lim_{h\to 0}\frac{h(2x+h-3y-4)}{h}\\ &= \lim_{h\to 0}(2x+h-3y-4)\\ &= 2x-3y-4.\end{aligned}$$

To calculate $\frac{\partial f}{\partial y}$, first calculate $f(x,y+h)$:

$$\begin{aligned}f(x+h,y) &= x^2-3x(y+h)+2(y+h)^2-4x+5(y+h)-12\\ &= x^2-3xy-3xh+2y^2+4yh+2h^2-4x+5y+5h-12.\end{aligned}$$

Next, substitute this into Equation 13.3.2and simplify:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}\\ &= \lim_{h\to 0}\frac{(x^2-3xy-3xh+2y^2+4yh+2h^2-4x+5y+5h-12)-(x^2-3xy+2y^2-4x+5y-12)}{h}\\ &= \lim_{h\to 0}\frac{x^2-3xy-3xh+2y^2+4yh+2h^2-4x+5y+5h-12-x^2+3xy-2y^2+4x-5y+12}{h}\\ &= \lim_{h\to 0}\frac{-3xh+4yh+2h^2+5h}{h}\\ &= \lim_{h\to 0}\frac{h(-3x+4y+2h+5)}{h}\\ &= \lim_{h\to 0}(-3x+4y+2h+5)\\ &= -3x+4y+5\end{aligned}$$

Exercise 13.3.1

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x,y)=4x^2+2xy-y^2+3x-2y+5.$$

Hint

Use Equations 13.3.1and 13.3.2from the definition of partial derivatives.

Answer

$$\begin{aligned}\frac{\partial f}{\partial x} &= 8x+2y+3\\ \frac{\partial f}{\partial y} &= 2x-2y-2\end{aligned}$$

The idea to keep in mind when calculating partial derivatives is to treat all independent variables, other than the variable with respect to which we are differentiating, as constants. Then proceed to differentiate as with a function of a single variable. To see why this is true, first fix y and define $g(x)=f(x,y)$ as a function of x. Then

$$\begin{aligned}g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}\\ &= \frac{\partial f}{\partial x}.\end{aligned}$$

The same is true for calculating the partial derivative of f with respect to y. This time, fix x and define $h(y)=f(x,y)$ as a function of y. Then

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1c

Define $f(x,y)$ by

$$f(x,y) = \begin{cases} \dfrac{x^3+x^4-y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

If we want to calculate the partial derivative of $f(x,y)$ at any point away from the origin $(0,0)$, we can use the usual formulas. However, if we want to calculate $\dfrac{\partial f}{\partial x}(0,0)$, we have to use the definition of the partial derivative. (There are no formulas that apply at points around which a function definition is broken up in this way.)

So, we plug in the above limit definition for $\frac{\partial f}{\partial x}$. We use the fact that $f(0,0) = 0$ and

$$f(h,0) = \frac{h^3+h^4-0^3}{h^2+0^2} = \frac{h^3+h^4}{h^2} = h+h^2.$$

Then,

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{h+h^2-0}{h} \\ &= \lim_{h\to 0} 1+h \\ &= 1. \end{aligned}$$

2a

5.4.1 Examples

Example 5.4.1.1 *Find the gradient vector of*

$$f(x, y) = x^2 + y^2$$

What are the gradient vectors at $(1, 2), (2, 1)$ *and* $(0, 0)$*?*

We begin with the formula.

$$\nabla f = \langle f_x, f_y \rangle = \boxed{\langle 2x, 2y \rangle}$$

Now, let us find the gradient at the following points.

- $\nabla f(1, 2) = \langle 2, 4 \rangle$
- $\nabla f(2, 1) = \langle 4, 2 \rangle$
- $\nabla f(0, 0) = \langle 0, 0 \rangle$

Notice that at $(0, 0)$ the gradient vector is the zero vector. Since the gradient corresponds to the notion of slope at that point, this is the same as saying the slope is zero.

2b

Example 5.4.1.2 *Find the gradient vector of*

$$f(x, y) = 2xy + x^2 + y$$

What are the gradient vectors at $(1, 1), (0, -1)$ *and* $(0, 0)$*?*

$$\nabla f = \langle f_x, f_y \rangle = \boxed{\langle 2y + 2x, 2x + 1 \rangle}$$

Now, let us find the gradient at the following points.

- $\nabla f(1, 1) = \langle 4, 3 \rangle$
- $\nabla f(0, -1) = \langle -2, 1 \rangle$
- $\nabla f(0, 0) = \langle 0, 1 \rangle$

So far, we've learned the definition of the gradient vector and we know that it tells us the direction of steepest ascent. What if, however, we want to know the rate of ascent in another direction? For that, we use something called a directional derivative.

Definition 5.4.2 *The* **directional derivative***, denoted $D_v f(x,y)$, is a derivative of a multivariable function in the direction of a vector $\vec{v}$. It is the scalar projection of the gradient onto $\vec{v}$.*

$$D_v f(x,y) = comp_v \nabla f(x,y) = \frac{\nabla f(x,y) \cdot \vec{v}}{|\vec{v}|}$$

This produces a vector whose magnitude represents the rate a function ascends (how steep it is) at point (x,y) in the direction of $\vec{v}$.

If our function has three inputs, the math works out the same. Suppose $f(x,y,z) = w$. Then,

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$$

and the directional derivative in the direction of $\vec{u}$ is

$$D_v f(x,y,z) = \frac{\nabla f(x,y,z) \cdot \vec{v}}{|\vec{v}|}$$

Let's look at some examples.

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5.4.2 Examples

Example 5.4.2.1 *Find the directional derivative of*

$$f(x,y) = \frac{x}{x^2+y^2}$$

in the direction of $\vec{v} = \langle 3,5\rangle$ at the point $(1,2)$.

First, we find the gradient.

$$\begin{aligned} f_x(x,y) &= \frac{d}{dx}\left(\frac{x}{x^2+y^2}\right) \\ &= \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} \\ &= \frac{y^2-x^2}{(x^2+y^2)^2} \end{aligned}$$

$$f_y(x,y) = \frac{d}{dy}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{-2xy}{(x^2+y^2)^2}$$

The gradient is then

$$\nabla f(1,2) = \left\langle \frac{4-1}{(4+1)^2}, \frac{-4}{(4+1)^2} \right\rangle = \left\langle \frac{3}{25}, \frac{-4}{25} \right\rangle = \frac{1}{25}\langle 3, -4\rangle$$

We now find the magnitude of $\vec{v}$. We get

$$|\vec{v}| = \sqrt{9+25} = \sqrt{34}$$

The directional derivative is then

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$$D_v f(1,2) = \frac{\nabla f(1,2)\cdot\vec{v}}{|\vec{v}|} = \frac{1}{25\sqrt{34}}\langle 3,-4\rangle\cdot\langle 3,5\rangle = \frac{1}{25\sqrt{34}}(9-20) = \boxed{-\frac{11}{25\sqrt{34}}}$$

Example 5.4.2.2 *Find the directional derivative of*

$$f(x,y,z) = \sqrt{xyz}$$

in the direction of $\vec{v} = \langle -1,-2,2\rangle$ *at the point* $(3,2,6)$.

First, we find the partial derivatives to define the gradient.

$$f_x(x,y,z) = \frac{yz}{2\sqrt{xyz}}$$
$$f_y(x,y,z) = \frac{xz}{2\sqrt{xyz}}$$
$$f_z(x,y,z) = \frac{xy}{2\sqrt{xyz}}$$

The gradient is

$$\nabla f(3,2,6) = \left\langle \frac{12}{2(6)}, \frac{18}{2(6)}, \frac{6}{2(6)} \right\rangle = \left\langle 1, \frac{3}{2}, \frac{1}{2} \right\rangle = \frac{1}{2}\langle 2,3,1\rangle$$

The magnitude of $\vec{v} = \langle -1, -2, 2 \rangle$ is

$$|\vec{v}| = \sqrt{1+4+4} = 3$$

Therefore, the directional derivative is

$$D_v f(3,2,6) = \frac{\nabla f(3,2,6) \cdot \vec{v}}{|\vec{v}|} = \frac{1}{3(2)} \langle 2,3,1 \rangle \cdot \langle -1,-2,2 \rangle = \frac{1}{6}(-2-6+2) = \boxed{-1}$$

The next natural question is:

In what direction is the derivative maximum?

As we just saw, the directional derivative is calculated by taking the scalar projection of ∇f onto a vector $\vec{v}$. Define θ be the angle between $\vec{v}$ and ∇f. Then,

$$\frac{\nabla f \cdot \vec{v}}{|\vec{v}|} = \frac{|\nabla f||\vec{u}| \cos(\theta)}{|\vec{v}|} = |\nabla f| \cos(\theta)$$

This is maximized if $\theta = 0$. From this, we know the following:

- The maximum rate of change (the largest directional derivative) is $|\nabla f|$.
- This occurs when $\vec{v}$ is *parallel* to ∇f, i.e. when they point in the same direction.

That makes sense since ∇f is the vector pointing toward *steepest ascent*, so it should be the direction with the largest derivative.

You'll typically be asked for the unit vector, $\vec{u}$, that creates the maximum directional derivative. This is because unit vectors are thought of as vectors that just contain information about direction. Based on our discussion above, this will always be

$$\vec{u} = \frac{\nabla f}{|\nabla f|}$$

Let's look at two examples.

5.4.3 Examples

Example 5.4.3.1 1. *Find the maximum rate of change of f at the given point and the direction in which it occurs.*

$$f(s,t) = te^{st}, \quad (0,2)$$