15th lesson

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Theory

Theorem 1 (derivative of a composite function; chain rule)**.** Let $r, s \in \mathbb{N}$ and let $G \subset \mathbb{R}^s$, $H \subset \mathbb{R}^r$ be open sets. Let $\varphi_1, \ldots, \varphi_r \in C^1(G)$, $f \in C^1(H)$ and $[\varphi_1(\vec{x}), \ldots, \varphi_r(\vec{x})] \in H$ for each $\vec{x} \in G$. Then the compound function $F\colon G \to \mathbb{R}$ defined by

$$F(\vec{x}) = f\big(\varphi_1(\vec{x}), \varphi_2(\vec{x}), \ldots, \varphi_r(\vec{x})\big), \quad \vec{x} \in G,$$

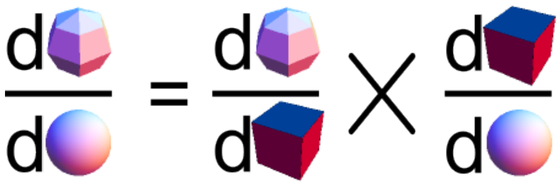
is of the class $\mathcal{C}^1$ on G. Let $\vec{a} \in G$ and $\vec{b} = [\varphi_1(\vec{a}), \ldots, \varphi_r(\vec{a})]$. Then for each $j \in \{1, \ldots, s\}$ we have

$$\frac{\partial F}{\partial x_j}(\vec{a}) = \sum_{i=1}^{r} \frac{\partial f}{\partial y_i}(\vec{b}) \frac{\partial \varphi_i}{\partial x_j}(\vec{a}).$$

Remarks 2 (Especially)**.** Let $f(x, y, z)$ be $\mathbf{C}^1$ function and $x = \phi(u, v)$, $y = \psi(u, v)$, $z = \chi(u, v)$, where ϕ, ψ, χ be $\mathbf{C}^1$. Then

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$



Source 1: http://mathinsight.org/media/image/image/chain_rule_geometric_objects.png

Exercises

1. (a) Find $\frac{\partial z}{\partial t} = \frac{\partial f}{\partial t}$ for $z = f(x, y) = 4x^2 + 3y^2$, $x(t) = \sin t$, $y(t) = \cos t$.
   (b) Find $\frac{\partial z}{\partial t} = \frac{\partial f}{\partial t}$ for $z = f(x, y) = \sqrt{x^2 - y^2}$, $x(t) = e^{2t}$, $y(t) = e^{-t}$.
   (c) Find $\frac{\partial z}{\partial u} = \frac{\partial f}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial f}{\partial v}$ for $z = f(x, y) = 3x^2 - 2xy + y^2$, $x(u, v) = 3u + 2v$, $y(u, v) = 4u - v$.
   (d) Find $\frac{\partial w}{\partial u} = \frac{\partial f}{\partial u}$ and $\frac{\partial w}{\partial v} = \frac{\partial f}{\partial v}$ for $w = f(x, y) = 3x^2 - 2xy + 4z^2$, $x(u, v) = e^u \sin v$, $y(u, v) = e^u \cos v$, $z(u, v) = e^u$.

2. (a) What is $\frac{\partial f}{\partial t}$ at $t = 1$, if $x(1) = 2$, $y(1) = 3$, $x'(1) = -4$, $y'(1) = 5$, $\frac{\partial f}{\partial x}(2, 3) = -6$, $\frac{\partial f}{\partial y}(2, 3) = 7$.

   (b) What is $G'(2)$ if $G(t) = h(t^2, t^3)$ and for $h(x, y)$ we have $\frac{\partial h}{\partial x}(4, 8) = 10$ and $\frac{\partial h}{\partial y}(4, 8) = -20$.

3. Compute $\frac{\partial^2 f}{\partial u^2}$ for $f(x, y)$ if $x = u^2 + 3v$, $y = uv$.

4. Compute $\frac{\partial^2 f}{\partial \theta^2}$ for $f(x, y)$ if $x = r \cos \theta$, $y = r \sin \theta$.

5. Let us consider a spherical balloon. Its radius $r(P, T)$ is a function of the atmospheric pressure P in atmospheres and the temperature T in degrees Celsius.

   When the radius is 10 meters, the rate of change of the radius with respect to the atmospheric pressure is -0.01 meters per atmosphere. The rate of change of the radius with respect to the temperature is 0.0002 meter per degree.

   What are the rates of change of the volume $V = \frac{4}{3}\pi r^3$ of the baloon with respect to P and T?

Source 2: https://commons.wikimedia.org/wiki/File:Hot_air_balloons_in_leon.jpg