$$\lim_{t\to t_0}\frac{z(t)-z(t_0)}{t-t_0}=f_x(x_0,y_0)\lim_{t\to t_0}\left(\frac{x(t)-x(t_0)}{t-t_0}\right)$$
$$+f_y(x_0,y_0)\lim_{t\to t_0}\left(\frac{y(t)-y(t_0)}{t-t_0}\right)$$
$$+\lim_{t\to t_0}\frac{E(x(t),y(t))}{t-t_0}.$$

The left-hand side of this equation is equal to $dz/dt$, which leads to

$$\frac{dz}{dt}=f_x(x_0,y_0)\frac{dx}{dt}+f_y(x_0,y_0)\frac{dy}{dt}+\lim_{t\to t_0}\frac{E(x(t),y(t))}{t-t_0}.$$

The last term can be rewritten as

$$\lim_{t\to t_0}\frac{E(x(t),y(t))}{t-t_0}=\lim_{t\to t_0}\frac{(E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0})$$
$$=\lim_{t\to t_0}\left(\frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\right)\lim_{t\to t_0}\left(\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0}\right).$$

As $t$ approaches $t_0$, $(x(t),y(t))$ approaches $(x(t_0),y(t_0))$, so we can rewrite the last product as

$$\lim_{(x,y)\to(x_0,y_0)}\frac{(E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\lim_{(x,y)\to(x_0,y_0)}(\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0}).$$

Since the first limit is equal to zero, we need only show that the second limit is finite:

$$\lim_{(x,y)\to(x_0,y_0)}\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t+0}=\lim_{(x,y)\to(x_0,y_0)}\sqrt{\frac{(x-x_0)^2+(y-y_0)^2}{(t-t_0)^2}}$$
$$=\lim_{(x,y)\to(x_0,y_0)}\sqrt{\left(\frac{x-x_0}{t-t_0}\right)^2+\left(\frac{y-y_0}{t-t_0}\right)^2}$$
$$=\sqrt{\left[\lim_{(x,y)\to(x_0,y_0)}\left(\frac{x-x_0}{t-t_0}\right)\right]^2+\left[\lim_{(x,y)\to(x_0,y_0)}\left(\frac{y-y_0}{t-t_0}\right)\right]^2}.$$

Since $x(t)$ and $y(t)$ are both differentiable functions of $t$, both limits inside the last radical exist. Therefore, this value is finite. This proves the chain rule at $t=t_0$; the rest of the theorem follows from the assumption that all functions are differentiable over their entire domains.

□

Closer examination of Equation 14.5.2reveals an interesting pattern. The first term in the equation is $\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}$ and the second term is $\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$. Recall that when multiplying fractions, cancelation can be used. If we treat these derivatives as fractions, then each product "simplifies" to something resembling $\partial f/dt$. The variables $x$ and $y$ that disappear in this simplification are often called **intermediate variables**: they are independent variables for the function $f$, but are dependent variables for the variable $t$. Two terms appear on the right-hand side of the formula, and $f$ is a function of two variables. This pattern works with functions of more than two variables as well, as we see later in this section.

## Example 14.5.1: Using the Chain Rule

Calculate $dz/dt$ for each of the following functions:

1a

a. $z=f(x,y)=4x^2+3y^2, x=x(t)=\sin t, y=y(t)=\cos t$

b. $z=f(x,y)=\sqrt{x^2-y^2}, x=x(t)=e^{2t}, y=y(t)=e^{-t}$

**Solution**

1a

a. To use the chain rule, we need four quantities—$\partial z/\partial x, \partial z/\partial y, dx/dt$, and $dy/dt$:

- $\frac{\partial z}{\partial x}=8x$
- $\frac{dx}{dt}=\cos t$
- $\frac{\partial z}{\partial y}=6y$
- $\frac{dy}{dt}=-\sin t$

Now, we substitute each of these into Equation 14.5.2:

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}=(8x)(\cos t)+(6y)(-\sin t)=\underline{8x\cos t-6y\sin t}.$$

This answer has three variables in it. To reduce it to one variable, use the fact that $x(t)=\sin t$ and $y(t)=\cos t$. We obtain

$$\frac{dz}{dt}=8x\cos t-6y\sin t=8(\sin t)\cos t-6(\cos t)\sin t=\underline{2\sin t\cos t}.$$

This derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x,y)$, then differentiating with respect to $t$:

$$z=f(x,y)=f(x(t),y(t))=4(x(t))^2+3(y(t))^2=4\sin^2 t+3\cos^2 t.$$

Then

$$\frac{dz}{dt}=2(4\sin t)(\cos t)+2(3\cos t)(-\sin t)=8\sin t\cos t-6\sin t\cos t=2\sin t\cos t,$$

which is the same solution. However, it may not always be this easy to differentiate in this form.

1b

b. To use the chain rule, we again need four quantities—$\partial z/\partial x, \partial z/dy, dx/dt$, and $dy/dt$ :

- $\frac{\partial z}{\partial x}=\frac{x}{\sqrt{x^2-y^2}}$
- $\frac{dx}{dt}=2e^{2t}$
- $\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{x^2-y^2}}$
- $\frac{dx}{dt}=-e^{-t}$.

We substitute each of these into Equation 14.5.2:

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$
$$=\left(\frac{x}{\sqrt{x^2-y^2}}\right)(2e^{2t})+\left(\frac{-y}{\sqrt{x^2-y^2}}\right)(-e^{-t})$$
$$=\underline{\frac{2xe^{2t}-ye^{-t}}{\sqrt{x^2-y^2}}}.$$

To reduce this to one variable, we use the fact that $x(t)=e^{2t}$ and $y(t)=e^{-t}$. Therefore,

$$\begin{aligned}\frac{dz}{dt} &= \frac{2xe^2t+ye^{-t}}{\sqrt{x^2-y^2}} \\ &= \frac{2(e^{2t})e^{2t}+(e^{-t})e^{-t}}{\sqrt{e^{4t}-e^{-2t}}} \\ &= \frac{2e^{4t}+e^{-2t}}{\sqrt{e^{4t}-e^{-2t}}}.\end{aligned}$$

To eliminate negative exponents, we multiply the top by $e^{2t}$ and the bottom by $\sqrt{e^{4t}}$:

$$\begin{aligned}\frac{dz}{dt} &= \frac{2e^{4t}+e^{-2t}}{\sqrt{e^{4t}-e^{-2t}}}\cdot\frac{e^{2t}}{\sqrt{e^{4t}}} \\ &= \frac{2e^{6t}+1}{\sqrt{e^{8t}-e^{2t}}} \\ &= \frac{2e^{6t}+1}{\sqrt{e^{2t}(e^{6t}-1)}} \\ &= \frac{2e^{6t}+1}{e^t\sqrt{e^{6t}-1}}.\end{aligned}$$

Again, this derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x,y)$, then differentiating with respect to $t$:

$$\begin{aligned}z &= f(x,y) \\ &= f(x(t),y(t)) \\ &= \sqrt{(x(t))^2-(y(t))^2} \\ &= \sqrt{e^{4t}-e^{-2t}} \\ &= (e^{4t}-e^{-2t})^{1/2}.\end{aligned}$$

Then

$$\begin{aligned}\frac{dz}{dt} &= \frac{1}{2}(e^{4t}-e^{-2t})^{-1/2}\left(4e^{4t}+2e^{-2t}\right) \\ &= \frac{2e^{4t}+e^{-2t}}{\sqrt{e^{4t}-e^{-2t}}}.\end{aligned}$$

This is the same solution.

## Exercise 14.5.1

Calculate $dz/dt$ given the following functions. Express the final answer in terms of $t$.

$$z = f(x,y) = x^2-3xy+2y^2$$
$$x = x(t) = 3\sin 2t, y = y(t) = 4\cos 2t$$

**Hint**

Calculate $\partial z/\partial x$, $\partial z/dy$, $dx/dt$,and $dy/dt$, then use Equation 14.5.2

**Answer**

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$$

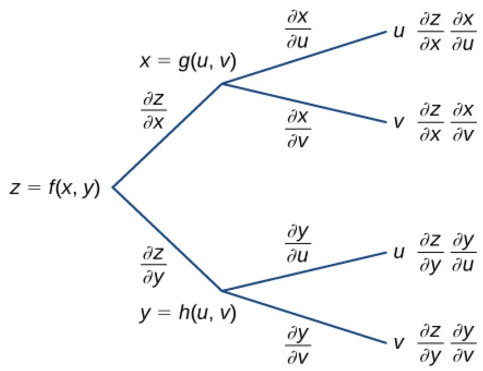


Figure 14.5.2: Tree diagram for $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$ .

To derive the formula for $\partial z/\partial u$, start from the left side of the diagram, then follow only the branches that end with $u$ and add the terms that appear at the end of those branches. For the formula for $\partial z/\partial v$, follow only the branches that end with $v$ and add the terms that appear at the end of those branches.

There is an important difference between these two chain rule theorems. In Note, the left-hand side of the formula for the derivative is not a partial derivative, but in Note it is. The reason is that, in Note, $z$ is ultimately a function of $t$ alone, whereas in Note, $z$ is a function of both $u$ and $v$.

### Example 14.5.2: Using the Chain Rule for Two Variables

1c

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ using the following functions:

$$z = f(x, y) = 3x^2 - 2xy + y^2, \ x = x(u, v) = 3u + 2v, \ y = y(u, v) = 4u - v.$$

**Solution**

To implement the chain rule for two variables, we need six partial derivatives—$\partial z/\partial x$, $\partial z/\partial y$, $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, and $\partial y/\partial v$:

$$\frac{\partial z}{\partial x} = 6x - 2y \frac{\partial z}{\partial y} = -2x + 2y$$

$$\frac{\partial x}{\partial u} = 3 \frac{\partial x}{\partial v} = 2$$

$$\frac{\partial y}{\partial u} = 4 \frac{\partial y}{\partial v} = -1.$$

To find $\partial z/\partial u$, we use Equation 14.5.3:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= 3(6x - 2y) + 4(-2x + 2y)$$

$$= 10x + 2y.$$

Next, we substitute $x(u, v) = 3u + 2v$ and $y(u, v) = 4u - v$:

$$\frac{\partial z}{\partial u} = 10x + 2y$$

$$= 10(3u + 2v) + 2(4u - v)$$

$$= 38u + 18v.$$

To find $\partial z/\partial v$, we use Equation ???:

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\ &= 2(6x-2y) + (-1)(-2x+2y) \\ &= 14x - 6y.\end{aligned}$$

Then we substitute $x(u,v) = 3u+2v$ and $y(u,v) = 4u - v$:

$$\begin{aligned}\frac{\partial z}{\partial v} &= 14x - 6y \\ &= 14(3u+2v) - 6(4u-v) \\ &= 18u + 34v\end{aligned}$$

## Exercise 14.5.2

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ given the following functions:

$$z = f(x,y) = \frac{2x-y}{x+3y},\ x(u,v) = e^{2u}\cos 3v,\ y(u,v) = e^{2u}\sin 3v.$$

**Hint**

Calculate $\partial z/\partial x,\ \partial z/\partial y,\ \partial x/\partial u,\ \partial x/\partial v,\ \partial y/\partial u$,and $\partial y/\partial v$, then use Equation 14.5.3and Equation ???.

**Answer**

$$\frac{\partial z}{\partial u} = 0, \frac{\partial z}{\partial v} = \frac{-21}{(3\sin 3v + \cos 3v)^2}$$

## The Generalized Chain Rule

Now that we've see how to extend the original chain rule to functions of two variables, it is natural to ask: Can we extend the rule to more than two variables? The answer is yes, as the *generalized chain rule* states.

## Generalized Chain Rule

Let $w = f(x_1, x_2, \ldots, x_m)$ be a differentiable function of $m$ independent variables, and for each $i \in 1, \ldots, m$, let $x_i = x_i(t_1, t_2, \ldots, t_n)$ be a differentiable function of $n$ independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j} \tag{14.5.5}$$

for any $j \in 1, 2, \ldots, n$.

In the next example we calculate the derivative of a function of three independent variables in which each of the three variables is dependent on two other variables.

## Example 14.5.3: Using the Generalized Chain Rule

Calculate $\partial w/\partial u$ and $\partial w/\partial v$ using the following functions:

$$\begin{aligned} w = f(x,y,z) &= 3x^2 - 2xy + 4z^2 \\ x = x(u,v) &= e^u \sin v \\ y = y(u,v) &= e^u \cos v \\ z = z(u,v) &= e^u. \end{aligned}$$

右

**Solution**

  

The formulas for $\partial w/\partial u$ and $\partial w/\partial v$ are

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$
$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}.$$

Therefore, there are nine different partial derivatives that need to be calculated and substituted. We need to calculate each of them:

$$\frac{\partial w}{\partial x}=6x-2y\frac{\partial w}{\partial y}=-2x\frac{\partial w}{\partial z}=8z$$
$$\frac{\partial x}{\partial u}=e^u\sin v\frac{\partial y}{\partial u}=e^u\cos v\frac{\partial z}{\partial u}=e^u$$
$$dfrac\partial x\partial v=e^u\cos v\frac{\partial y}{\partial v}=-e^u\sin v\frac{\partial z}{\partial v}=0.$$

Now, we substitute each of them into the first formula to calculate $\partial w/\partial u$:

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$
$$=(6x-2y)e^u\sin v-2xe^u\cos v+8ze^u,$$

then substitute $x(u,v)=e^u\sin v, y(u,v)=e^u\cos v,$ and $z(u,v)=e^u$ into this equation:

$$\frac{\partial w}{\partial u}=(6x-2y)e^u\sin v-2xe^u\cos v+8ze^u$$
$$=(6e^u\sin v-2eu\cos v)e^u\sin v-2(e^u\sin v)e^u\cos v+8e^{2u}$$
$$=6e^{2u}\sin^2 v-4e^{2u}\sin v\cos v+8e^{2u}$$
$$=2e^{2u}(3\sin^2 v-2\sin v\cos v+4).$$

Next, we calculate $\partial w/\partial v$:

$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$
$$=(6x-2y)e^u\cos v-2x(-e^u\sin v)+8z(0),$$

then we substitute $x(u,v)=e^u\sin v, y(u,v)=e^u\cos v,$ and $z(u,v)=e^u$ into this equation:

$$\frac{\partial w}{\partial v}=(6x-2y)e^u\cos v-2x(-e^u\sin v)$$
$$=(6e^u\sin v-2e^u\cos v)e^u\cos v+2(e^u\sin v)(e^u\sin v)$$
$$=2e^{2u}\sin^2 v+6e^{2u}\sin v\cos v-2e^{2u}\cos^2 v$$
$$=2e^{2u}(\sin^2 v+\sin v\cos v-\cos^2 v).$$

## Exercise 14.5.3

Calculate $\partial w/\partial u$ and $\partial w/\partial v$ given the following functions:

2a

***Example 5*** What is the $t$-derivative of $z = f(x(t), y(t))$ at $t = 1$ if $x(1) = 2, y(1) = 3$, $x'(1) = -4$, $y'(1) = 5$, $f_x(2,3) = -6$, and $f_y(2,3) = 7$?

SOLUTION By formula **(1)** with $t = 1$,

$$\begin{aligned}\left[\frac{d}{dt}\{f(x(t), y(t))\}\right]_{t=1} &= f_x(x(1), y(1))\, x'(1) + f_y(x(1), y(1))\, y'(1)\\ &= f_x(2,3)\, x'(1) + f_y(2,3)\, y'(1)\\ &= (-6)(-4) + (7)(5) = 59. \ \square\end{aligned}$$

2b

***Example 6*** Find $G'(2)$ where $G(t) = h(t^2, t^3)$ and $h = h(x,y)$ is such that $h_x(4,8) = 10$ and $h_y(4,8) = -20$.

SOLUTION Formula **(1)** gives

$$\begin{aligned}G'(t) &= \frac{d}{dt}[h(t^2, t^3)] = h_x(t^2, t^3)\frac{d}{dt}(t^2) + h_y(t^2, t^3)\frac{d}{dt}(t^3)\\ &= 2th_x(t^2, t^3) + 3t^2 h_y(t^2, t^3).\end{aligned}$$

Therefore,

$$\begin{aligned}G'(2) &= 2(2)h_x(2^2, 2^3) + 3(2^2)h_y(2^2, 2^3)\\ &= 4h_x(4,8) + 12h_y(4,8) = 4(10) + 12(-20) = -200. \ \square\end{aligned}$$

In applications it often helps to interpret the Chain Rule formula **(1)** in terms of rates of change. We write it in the form

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} \qquad \textbf{(6)}$$

without reference to where the derivatives are evaluated. Equation **(6)** states that the rate of change of $F$ with respect to $t$ equals the rate of change of $F$ with respect to $x$ multiplied by the rate of change of $x$ with respect to $t$, plus the rate of change of $F$ with respect to $y$ multiplied by the rate of change of $y$ with respect to $t$.

***Example 7*** A small plane uses gasoline at the rate of $r = r(h, v)$ gallons per hour when it is flying at an elevation of $h$ feet above the ground and its air speed is $v$ knots (nautical miles per hour). At a moment when the plane has an altitude of 8000 feet and a speed of 120 knots, its height is increasing 500 feet per minute and it is accelerating 3 knots per minute. At what rate is its gasoline consumption increasing or decreasing at that moment if at $h = 8000$ and $v = 120$ the function $r$ and its derivatives have the values $r = 7.2$ gallons per hour, $\partial r/\partial h = -2 \times 10^{-4}$ gallons per hour per foot, and $\partial r/\partial v = 0.13$ gallons per hour per knot?[(1)]

[(1)] Data adapted from *Cessna 172N Information Manual*, Wichita Kansas: Cessna Aircraft Company, 1978, p.5-16.

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\boxed{-\frac{z\mathbf{e}^{zy}-24xy^3z^3}{y\mathbf{e}^{zy}+2xz-18xy^4z^2}=\frac{24xy^3z^3-z\mathbf{e}^{zy}}{y\mathbf{e}^{zy}+2xz-18xy^4z^2}}$$

Note that in for the second form of each of the answers we simply moved the "-" in front of the fraction up to the numerator and multiplied it through. We could just have easily done this with the denominator instead if we'd wanted to.

---

3

11. Determine $f_{uu}$ for the following situation.

$$f=f\left(x,y\right) \qquad x=u^2+3v, \quad y=uv$$

Step 1

These kinds of problems always seem mysterious at first. That is probably because we don't actually know what the function itself is. This isn't really a problem. It simply means that the answers can get a little messy as we'll rarely be able to do much in the way of simplification.

So, the first step here is to get the first derivative and this will require the following chain rule formula.

$$f_u=\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

Here is the first derivative,

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\left[2u\right]+\frac{\partial f}{\partial y}\left[v\right]=2u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}$$

Do not get excited about the "unknown" derivatives in our answer here. They will always be present in these kinds of problems.

Step 2

Now, much as we did in the notes, let's do a little rewrite of the answer above as follows,

$$\frac{\partial}{\partial u}\left(f\right)=2u\frac{\partial}{\partial x}\left(f\right)+v\frac{\partial}{\partial y}\left(f\right)$$

With this rewrite we now have a "formula" for differentiating any function of *x* and *y* with respect to *u* whenever $x=u^2+3v$ and $y=uv$. In other words, whenever we have such a function all we need to do is replace the *f* in the parenthesis with whatever our function is. We'll need this eventually.

Step 3

Now, let's get the second derivative. We know that we find the second derivative as follows,

$$f_{uu} = \frac{\partial^2 f}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right) = \frac{\partial}{\partial u}\left(2u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}\right)$$

Step 4

Now, recall that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are functions of *x* and *y* which are in turn defined in terms of *u* and *v* as defined in the problem statement. This means that we'll need to do the product rule on the first term since it is a product of two functions that both involve *u*. We won't need to product rule the second term, in this case, because the first function in that term involves only *v*'s.

Here is that work,

$$f_{uu} = 2\frac{\partial f}{\partial x} + 2u\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) + v\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right)$$

Because the function is defined only in terms of *x* and *y* we cannot "merge" the *u* and *x* derivatives in the second term into a "mixed order" second derivative. For the same reason we cannot "merge" the *u* and *y* derivatives in the third term.

In each of these cases we are being asked to differentiate functions of *x* and *y* with respect to *u* where *x* and *y* are defined in terms of *u* and *v*.

Step 5

Now, recall the "formula" from Step 2,

$$\frac{\partial}{\partial u}(f) = 2u\frac{\partial}{\partial x}(f) + v\frac{\partial}{\partial y}(f)$$

Recall that this tells us how to differentiate any function of *x* and *y* with respect to *u* as long as *x* and *y* are defined in terms of *u* and *v* as they are in this problem.

Well luckily enough for us both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are functions of *x* and *y* which in turn are defined in terms of *u* and *v* as we need them to be. This means we can use this "formula" for each of the derivatives in the result from Step 4 as follows,

$$\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right)=2u\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=2u\frac{\partial^2 f}{\partial x^2}+v\frac{\partial^2 f}{\partial y\partial x}$$
$$\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right)=2u\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=2u\frac{\partial^2 f}{\partial x\partial y}+v\frac{\partial^2 f}{\partial y^2}$$

Step 6
Okay, all we need to do now is put the results from Step 5 into the result from Step 4 and we'll be done.

$$\begin{aligned}
f_{uu}&=2\frac{\partial f}{\partial x}+2u\left[2u\frac{\partial^2 f}{\partial x^2}+v\frac{\partial^2 f}{\partial y\partial x}\right]+v\left[2u\frac{\partial^2 f}{\partial x\partial y}+v\frac{\partial^2 f}{\partial y^2}\right]\\
&=2\frac{\partial f}{\partial x}+4u^2\frac{\partial^2 f}{\partial x^2}+2uv\frac{\partial^2 f}{\partial y\partial x}+2uv\frac{\partial^2 f}{\partial x\partial y}+v^2\frac{\partial^2 f}{\partial y^2}\\
&=\boxed{2\frac{\partial f}{\partial x}+4u^2\frac{\partial^2 f}{\partial x^2}+4uv\frac{\partial^2 f}{\partial x\partial y}+v^2\frac{\partial^2 f}{\partial y^2}}
\end{aligned}$$

Note that we assumed that the two mixed order partial derivative are equal for this problem and so combined those terms. If you can't assume this or don't want to assume this then the second line would be the answer.

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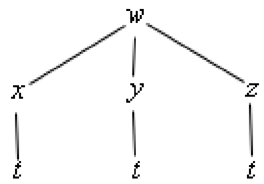


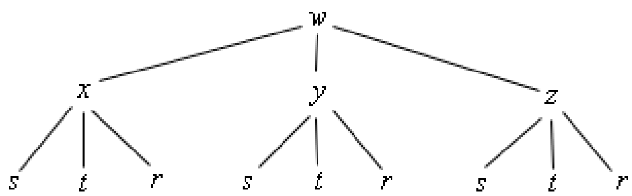

From this it looks like the chain rule for this case should be,

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

which is really just a natural extension to the two variable case that we saw above.

**(b)** $\frac{\partial w}{\partial r}$ for $w = f(x,y,z)$, $x = g_1(s,t,r)$, $y = g_2(s,t,r)$, and $z = g_3(s,t,r)$

Here is the tree diagram for this situation.




From this it looks like the derivative will be,

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

So, provided we can write down the tree diagram, and these aren't usually too bad to write down, we can do the chain rule for any set up that we might run across.

We've now seen how to take first derivatives of these more complicated situations, but what about higher order derivatives? How do we do those? It's probably easiest to see how to deal with these with an example.

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***Example 5*** Compute $\frac{\partial^2 f}{\partial \theta^2}$ for $f(x,y)$ if $x = r\cos\theta$ and $y = r\sin\theta$.

***Solution***

We will need the first derivative before we can even think about finding the second derivative so let's get that. This situation falls into the second case that we looked at above so we don't need a new tree diagram. Here is the first derivative.

$$\underline{\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}}$$
$$= -r\sin(\theta)\frac{\partial f}{\partial x} + r\cos(\theta)\frac{\partial f}{\partial y}$$

Okay, now we know that the second derivative is,

$$\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial \theta}\right) = \underline{\frac{\partial}{\partial \theta}\left(-r\sin(\theta)\frac{\partial f}{\partial x} + r\cos(\theta)\frac{\partial f}{\partial y}\right)}$$

The issue here is to correctly deal with this derivative. Since the two first order derivatives, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, are both functions of *x* and *y* which are in turn functions of *r* and $\theta$ both of these terms are products. So, the using the product rule gives the following,

$$\underline{\frac{\partial^2 f}{\partial \theta^2} = -r\cos(\theta)\frac{\partial f}{\partial x} - r\sin(\theta)\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right) - r\sin(\theta)\frac{\partial f}{\partial y} + r\cos(\theta)\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)}$$

We now need to determine what $\underline{\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)}$ and $\underline{\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)}$ will be. These are both chain rule problems again since both of the derivatives are functions of *x* and *y* and we want to take the derivative with respect to $\theta$.

Before we do these let's rewrite the first chain rule that we did above a little.

$$\frac{\partial}{\partial \theta}(f) = -r\sin(\theta)\frac{\partial}{\partial x}(f) + r\cos(\theta)\frac{\partial}{\partial y}(f) \qquad (1)$$

Note that all we've done is change the notation for the derivative a little. With the first chain rule written in this way we can think of (1) as a formula for differentiating any function of *x* and *y* with respect to $\theta$ provided we have $x = r\cos\theta$ and $y = r\sin\theta$.

This however is exactly what we need to do the two new derivatives we need above. Both of the first order partial derivatives, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, are functions of *x* and *y* and $x = r\cos\theta$ and $y = r\sin\theta$ so we can use (1) to compute these derivatives.

To do this we'll simply replace all the *f* 's in (1) with the first order partial derivative that we want to differentiate. At that point all we need to do is a little notational work and we'll get the formula that we're after.

Here is the use of (1) to compute $\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial x}\right)$.

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial x}\right) &= -r\sin(\theta)\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)+r\cos(\theta)\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\\ &= -r\sin(\theta)\frac{\partial^2 f}{\partial x^2}+r\cos(\theta)\frac{\partial^2 f}{\partial y\partial x}\end{aligned}$$

Here is the computation for $\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial y}\right)$.

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial y}\right) &= -r\sin(\theta)\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)+r\cos(\theta)\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\\ &= -r\sin(\theta)\frac{\partial^2 f}{\partial x\partial y}+r\cos(\theta)\frac{\partial^2 f}{\partial y^2}\end{aligned}$$

The final step is to plug these back into the second derivative and do some simplifying.

$$\begin{aligned}\frac{\partial^2 f}{\partial\theta^2} &= -r\cos(\theta)\frac{\partial f}{\partial x}-r\sin(\theta)\left(-r\sin(\theta)\frac{\partial^2 f}{\partial x^2}+r\cos(\theta)\frac{\partial^2 f}{\partial y\partial x}\right)-\\ &\qquad r\sin(\theta)\frac{\partial f}{\partial y}+r\cos(\theta)\left(-r\sin(\theta)\frac{\partial^2 f}{\partial x\partial y}+r\cos(\theta)\frac{\partial^2 f}{\partial y^2}\right)\\ &= -r\cos(\theta)\frac{\partial f}{\partial x}+r^2\sin^2(\theta)\frac{\partial^2 f}{\partial x^2}-r^2\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial y\partial x}-\\ &\qquad r\sin(\theta)\frac{\partial f}{\partial y}-r^2\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial x\partial y}+r^2\cos^2(\theta)\frac{\partial^2 f}{\partial y^2}\\ &= -r\cos(\theta)\frac{\partial f}{\partial x}-r\sin(\theta)\frac{\partial f}{\partial y}+r^2\sin^2(\theta)\frac{\partial^2 f}{\partial x^2}-\\ &\qquad 2r^2\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial y\partial x}+r^2\cos^2(\theta)\frac{\partial^2 f}{\partial y^2}\end{aligned}$$

It's long and fairly messy but there it is.

The final topic in this section is a revisiting of implicit differentiation. With these forms of the chain rule implicit differentiation actually becomes a fairly simple process. Let's start out with the implicit differentiation that we saw in a Calculus I course.

We will start with a function in the form $F(x,y)=0$ (if it's not in this form simply move everything to one side of the equal sign to get it into this form) where $y=y(x)$. In a Calculus I course we were then

## Section 14.4

(3/23/08)

## Chain Rules with two variables

OVERVIEW: *In this section we discuss procedures for differentiating composite functions with two variables. Then we consider second-order and higher-order derivatives of such functions.*

***Topics:***

- ***Using the Chain Rule for one variable***
- ***The general Chain Rule with two variables***
- ***Higher order partial derivatives***

### *Using the Chain Rule for one variable*

Partial derivatives of composite functions of the forms $z = F(g(x,y))$ can be found directly with the Chain Rule for one variable, as is illustrated in the following three examples.

***Example 1*** Find the $x$-and $y$-derivatives of $z = (x^2y^3 + \sin x)^{10}$.

SOLUTION To find the $x$-derivative, we consider $y$ to be constant and apply the one-variable Chain Rule formula $\dfrac{d}{dx}(f^{10}) = 10\, f^9 \dfrac{df}{dx}$ from Section 2.8. We obtain

$$\begin{aligned}\frac{\partial}{\partial x}[(x^2y^3 + \sin x)^{10}] &= 10(x^2y^3 + \sin x)^9 \frac{\partial}{\partial x}(x^2y^3 + \sin x)\\ &= 10(x^2y^3 + \sin x)^9(2xy^3 + \cos x).\end{aligned}$$

Similarly, we find the $y$-derivative by treating $x$ as a constant and using the same one-variable Chain Rule formula with $y$ as variable:

$$\begin{aligned}\frac{\partial}{\partial y}[(x^2y^3 + \sin x)^{10}] &= 10(x^2y^3 + \sin x)^9 \frac{\partial}{\partial y}(x^2y^3 + \sin x)\\ &= 10(x^2y^3 + \sin x)^9(3x^2y^2). \square\end{aligned}$$

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***Example 2*** The radius (meters) of a spherical balloon is given as a function $r = r(P,T)$ of the atmospheric pressure $P$ (atmospheres) and the temperature $T$ (degrees Celsius). At one moment the radius is ten meters, the rate of change of the radius with respect to atmospheric pressure is $-0.01$ meters per atmosphere, and the rate of change of the radius with respect to the temperature is 0.002 meter per degree. What are the rates of change of the volume $V = \frac{4}{3}\pi r^3$ of the balloon with respect to $P$ and $T$ at that time?

SOLUTION We first take the $P$-derivative with $T$ constant and then take the $T$-derivative with $P$ constant, using the Chain Rule for one variable in each case to differentiate $r^3$. We obtain

$$\begin{aligned}\frac{\partial V}{\partial P} &= \frac{\partial}{\partial P}\left(\tfrac{4}{3}\pi r^3\right) = \tfrac{1}{3}\pi r^2 \frac{\partial r}{\partial P}\\ \frac{\partial V}{\partial P} &= \frac{\partial}{\partial T}\left(\tfrac{4}{3}\pi r^3\right) = \tfrac{1}{3}\pi r^2 \frac{\partial r}{\partial T}.\end{aligned}$$

Setting $r = 10, \partial r/\partial P = -0.01$, and $\partial r/\partial T = 0.002$ then gives

$$\begin{aligned}\frac{\partial V}{\partial P} &= \tfrac{1}{3}\pi(10^2)(-0.01) = -\tfrac{1}{3}\pi \doteq -1.05\ \frac{\text{cubic meters}}{\text{atmosphere}}\\ \frac{\partial V}{\partial T} &= \tfrac{1}{3}\pi(10^2)(0.002) = \tfrac{1}{15}\pi \doteq 0.21\ \frac{\text{cubic meters}}{\text{degree}}. \square\end{aligned}$$