16th lesson

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Theory

Theorem 1 (Necessary condition of the existence of local extremum)**.** Let $n \in \mathbb{N}$, $G \subset \mathbb{R}^n$ be an open set, $a \in G$ and $i \in \{1, \cdots, n\}$. Let us suppose that a function $f : G \to \mathbb{R}$ has a local extremum at a point a. Then the partial derivative $\frac{\partial f}{\partial x_i}(a)$ does not exist or $\frac{\partial f}{\partial x_i}(a) = 0$.

Theorem 2 (Sufficient condition of the existence of local extremum)**.** Let $G \subset \mathbb{R}^n$ be an open set, $a \in G$ and let $f \in C^2(G)$. Let $\operatorname{grad} f(a) = 0$. Then

1. If (the matrix of second derivatives) $f''(a)$ is positive definite, then f has a strict local minimum at a.
2. If $f''(a)$ is negative definite, then f has a strict local maximum at a.
3. If $f''(a)$ is indefinite, then f does not have a local extrema at a (saddle point).

Exercises

1. Find the local extrema (= find all critical points and decide, whether it is loc. minimum, loc. maximum or the saddle point):

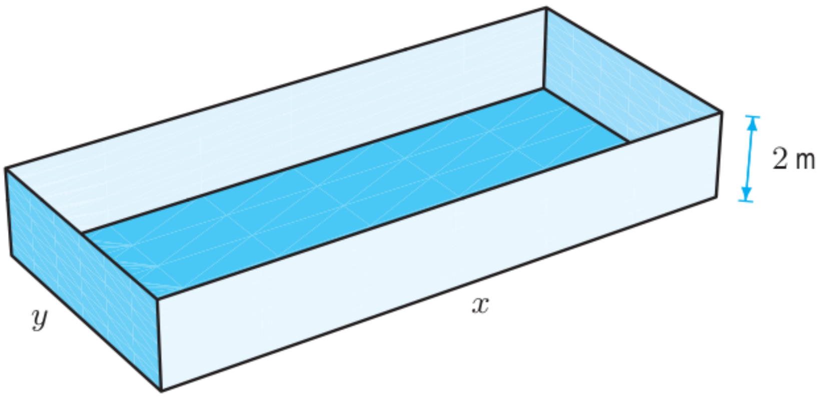
   (a) $f(x, y) = 3x^2 + 2y^3 - 6xy$
   (b) $f(x, y) = x^3y - 3xy^3 + 8y$
   (c) $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$
   (d) $f(x, y) = 2x^3 + 9xy^2 + 15x^2 + 27y^2$
   (e) $f(x, y) = x + 2y^4 - \ln(x^4y^8)$ for $x, y > 0$
   (f) $f(x, y) = e^{x^2-y}(5 - 2x + y)$
   (g) $f(x, y, z) = x^3 - 2x^2 + y^2 + z^2 - 2xy + xz - yz + 3z$

2. Find the minimum distance between a point on the plane $x + y + z = 1$ and the point $(2, -1, -2)$.

3. We should deliver 480 m^3 of kids pool balls. The trucker uses open-top boxes in numerous trips. The cost to the trucker is the cost of the box plus \$80 per trip. We know that the height of the box is fixed = 2 meters. But we can choose the length and width.

   About the box: the cost is $\$100/m^2$ for the ends, $\$50/m^2$ for the sides and $\$200/m^2$ for the bottom. (Smaller box is cheaper, but we need more trips.) What size of the box minimize our total cost?

   Source: Calculus: Single and Multivariable, Deborah Hughes-Hallett and col.



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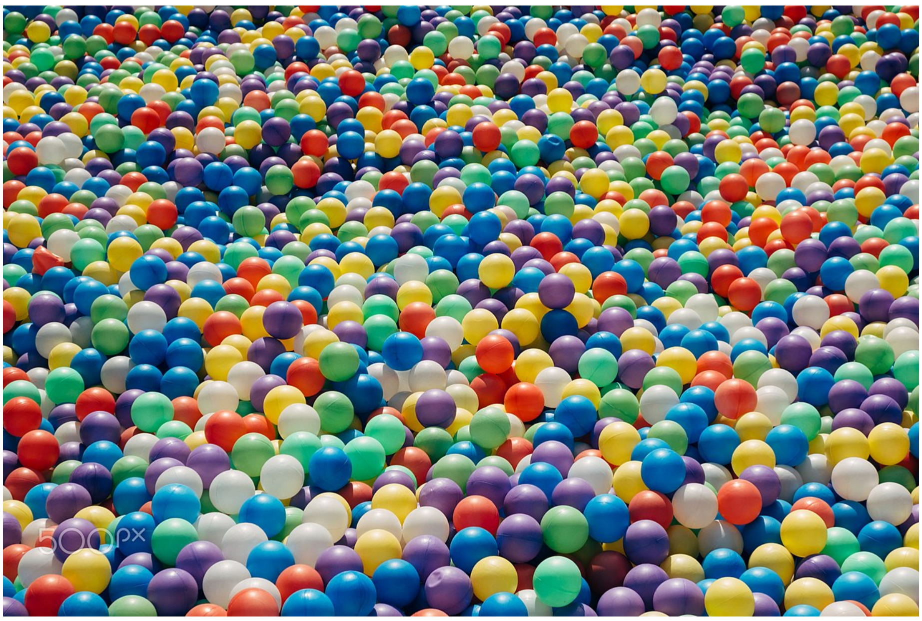

Source 2: `https://commons.wikimedia.org/wiki/Category:Ball_pits`