* It may require some algebraic manipulation to find the solutions: a basic technique is to solve one equation for one of the variables, and then plug the result into the other equation. Another technique is to try to factor one of the equations and then analyze cases.

- Step 3: At *each* critical point, evaluate $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$ and apply the Second Derivatives Test:

If $\boxed{D > 0 \text{ and } f_{xx} > 0\text{: local minimum}}$. If $\boxed{D > 0 \text{ and } f_{xx} < 0\text{: local maximum}}$. If $\boxed{D < 0\text{: saddle point}}$.

- Example: Verify that $f(x, y) = x^2 + y^2$ has only one critical point, a minimum at the origin.
  - First, we have $f_x = 2x$ and $f_y = 2y$. Since they are both defined everywhere, we need only find where they are both zero.
  - Setting both partial derivatives equal to zero yields $x = 0$ and $y = 0$, so the only critical point is $(0, 0)$.
  - To classify the critical points, we compute $f_{xx} = 2$, $f_{xy} = 0$, and $f_{yy} = 2$. Then $D = 2 \cdot 2 - 0^2 = 4$.
  - So, by the classification test, since $D > 0$ and $f_{xx} > 0$ at $(0, 0)$, we see that $\boxed{(0,0) \text{ is a local minimum}}$.

1a

- Example: For the function $f(x, y) = 3x^2 + 2y^3 - 6xy$, find the critical points and classify them as minima, maxima, or saddle points.
  - First, we have $f_x = 6x - 6y$ and $f_y = 6y^2 - 6x$. Since they are both defined everywhere, we need only find where they are both zero.
  - Next, we can see that f_x is zero only when $y = x$. Then the equation $f_y = 0$ becomes $6x^2 - 6x = 0$, which by factoring we can see has solutions $x = 0$ or $x = 1$. Since $y = x$, we conclude that $\boxed{(0,0), \text{ and } (1,1) \text{ are critical points}}$.
  - To classify the critical points, we compute $f_{xx} = 6$, $f_{xy} = -6$, and $f_{yy} = 12y$. Then $D(0, 0) = 6 \cdot 0 - (-6)^2 < 0$ and $D(1, 1) = 6 \cdot 12 - (-6)^2 > 0$.
  - So, by the classification test, $\boxed{(0,0) \text{ is a saddle point}}$ and $\boxed{(1,1) \text{ is a local minimum}}$.

1b

- Example: For the function $g(x, y) = x^3y - 3xy^3 + 8y$, find the critical points and classify them as minima, maxima, or saddle points.
  - First, we have $g_x = 3x^2y - 3y^3$ and $g_y = x^3 - 9xy^2 + 8$. Since they are both defined everywhere, we need only find where they are both zero.
  - Setting both partial derivatives equal to zero. Since $g_x = 3y(x^2 - y^2) = 3y(x + y)(x - y)$, we see that $g_x = 0$ precisely when $y = 0$ or $y = x$ or $y = -x$.
  - If $y = 0$, then $g_y = 0$ implies $x^3 + 8 = 0$, so that $x = -2$. This yields the point $(x, y) = (-2, 0)$.
  - If $y = x$, then $g_y = 0$ implies $-8x^3 + 8 = 0$, so that $x = 1$. This yields the point $(x, y) = (1, 1)$.
  - If $y = -x$, then $g_y = 0$ implies $-8x^3 + 8 = 0$, so that $x = 1$. This yields the point $(x, y) = (1, -1)$.
  - To summarize, we see that $\boxed{(-2,0),\ (1,1), \text{ and } (1,-1) \text{ are critical points}}$.
  - To classify the critical points, we compute $g_{xx} = 6xy$, $g_{xy} = 3x^2 - 9y^2$, and $g_{yy} = -18xy$.
  - Then $D(-2, 0) = 0 \cdot 0 - (12)^2 < 0$, $D(1, 1) = 6 \cdot (-18) - (-6)^2 < 0$, and $D(1, -1) = (-6) \cdot (18) - (-6)^2 < 0$.
  - So, by the classification test, $\boxed{(-2,0),\ (1,1), \text{ and } (1,-1) \text{ are all saddle points}}$.

- Example: Find the value of the function $h(x, y) = x + 2y^4 - \ln(x^4y^8)$ at its local minimum, for x and y positive.
  - To solve this problem, we will search for all critical points of $h(x, y)$ that are minima.
  - First, we have $h_x = 1 - \dfrac{4x^3y^8}{x^4y^8} = 1 - \dfrac{4}{x}$ and $h_y = 8y^3 - \dfrac{8x^4y^7}{x^4y^8} = 8y^3 - \dfrac{8}{y}$. Both partial derivatives are defined everywhere in the given domain.

for every (x, y) on the $\mathbb{R}^2$ plane. Both are positive numbers. You may be tempted to conclude that $(0,0)$ is a local maximum point. However, if one plots the graph of this function (see Figure 2.18), one can see easily that $(0,0)$ is neither a local maximum or a local minimum.

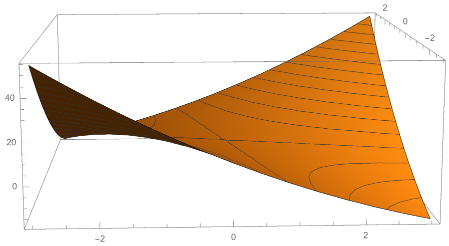

Figure 2.18: $(0,0)$ is neither a maximum or minimum

Around $(0,0)$, the graph is a concave up in some directions but concave down in other directions. We call this $(0,0)$ a **saddle**.

This example shows the signs of f_{xx} and f_{yy} alone could not conclude the nature of the critical point. In fact, the second derivative test for two-variable functions is slightly more complicated than that in single-variable calculus:

Theorem 2.5 — Second Derivative Test for Two-Variable Functions. Let $f(x, y)$ be a twice differentiable function and (x_0, y_0) is a critical point of f, i.e. $\nabla f(x_0, y_0) = \mathbf{0}$. Then the nature of this critical point (x_0, y_0) is determined by the following table:

$\left(f_{xx}f_{yy} - f_{xy}^2\right)\Big\|_{(x_0,y_0)}$	$f_{xx}(x_0, y_0)$	(x_0, y_0) is a:
> 0	> 0	local minimum
> 0	< 0	local maximum
< 0	anything	saddle

Any other cases are inconclusive.

For the function $f(x, y) = x^2 + 4xy + y^2$ in the above example, to determine the nature of $(0,0)$ we also need $f_{xy}(0,0)$, which can be found as equal to 4.

Therefore, we have:

$$\left(f_{xx}f_{yy} - f_{xy}^2\right)\Big|_{(0,0)} = 2 \times 2 - 4^2 < 0,$$
$$f_{xx}(0,0) = 2 > 0.$$

From the table in Theorem 2.5, we conclude $(0,0)$ is a saddle, as expected from the plot of the its graph.

Let's look at one more example before we learn the proof of the Second Derivative Test.

■ **Example 2.10** Let $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$. Find all critical points and determine the nature of each of them.

■ **Solution** To find all critical points, we set:

$$\frac{\partial f}{\partial x} = -6x + 6y = 0,$$
$$\frac{\partial f}{\partial y} = 6y - 6y^2 + 6x = 0.$$

From the first equation, we get $y = x$. Substitute this into the second equation, we yield:

$$6x - 6x^2 + 6x = 0, \text{ or equivalently } 2x - x^2 = 0.$$

By factorization, we get $x(2-x) = 0$. Therefore

$$x = 0 \text{ or } x = 2.$$

By noting that $y = x$, we have two critical points: $(0,0)$ and $(2,2)$.

Next we compute the second derivatives of f:

$$f_{xx} = -6 \qquad f_{xy} = 6$$
$$f_{yx} = 6 \qquad f_{yy} = 6 - 12y$$

Critical point P	$f_{xx}(P)$	$f_{yy}(P)$	$f_{xy}(P)$	$\left(f_{xx}f_{yy} - f_{xy}^2\right)(P)$	Nature of P
$(0,0)$	-6	6	6	-72	saddle
$(2,2)$	-6	-18	6	72	local maximum

Explanation of the Second Derivative Test

In single-variable, the second derivative test can be explained using convexity of the graph $y = f(x)$. However, this approach can hardly be generalized to higher dimensions.

Before we explain why the above second derivative test works for two-variable functions $f(x,y)$, we first seek an alternative explanation of the single-variable second derivative test using Taylor's series.

Recall that the Taylor's series of a given function $f(x)$ about $x = a$ is given by:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

If $f(x)$ has a critical point at $x = a$, then $f'(a) = 0$. Also, when x is very close to a, the higher-order terms $(x-a)^3$, $(x-a)^4$, etc. are significantly smaller than the quadratic term $(x-a)^2$. Therefore, the function $f(x)$ is approximately given by:

$$f(x) \simeq f(a) + \frac{f''(a)}{2!}(x-a)^2 \quad \text{when } x \text{ is near } a.$$

The right-hand side $f(a) + \frac{f''(a)}{2!}(x-a)^2$ is a quadratic function. If $f''(a) > 0$, then the graph $y = f(a) + \frac{f''(a)}{2!}(x-a)^2$ is a concave up parabola and so $f(a) + \frac{f''(a)}{2!}(x-a)^2 \geq f(a)$. Therefore, $f(x)$, which is approximately $f(a) + \frac{f''(a)}{2!}(x-a)^2$, is also $\geq f(a)$ when x is near a. This explains $f(x)$ has a local minimum at $x = a$.

On the other hand, if $f''(a) < 0$, then the graph $y = f(a) + \frac{f''(a)}{2!}(x-a)^2$ is a concave down parabola. Similar argument as above shows $f(x)$ has a local maximum at $x = a$.

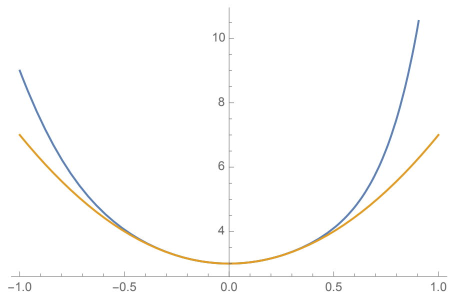

Figure 2.19: blue graph shows $y = f(x)$ where $f'(0) = 0$; yellow graph shows $y = f(0) + \frac{f''(0)}{2!}x^2$ where $f''(0) > 0$

Back to multivariable calculus, we now explain the second derivative test using the Taylor's series approach. Given a function $f(x,y)$, the multivariable Taylor's series about $(x,y) =$

MA2: Solved problems—Functions of more variables: Extrema

1. Find and identify local extrema of $f(x,y) = 2x^3 + 9xy^2 + 15x^2 + 27y^2$.

2. Find and identify local extrema of $f(x,y,z) = x^3 - 2x^2 + y^2 + z^2 - 2xy + xz - yz + 3z$.

3. Find the global extrema of $f(x,y) = x^2 + 2y^2$ given the condition
$$x^2 - 2x + 2y^2 + 4y = 0.$$

4. Find the point on the plane given by $x + y - z = 1$ that is closest to the point $P = (0,-3,2)$ and calculate their distance. Use Lagrange multipliers.

5. A certain line in 3D is given by the equations
$$x + y + z = 1, \qquad 2x - y + z = 3.$$
Find the distance between this line and the point $P = (1,2,-1)$.

6. Find the global extrema of $f(x,y) = x^2 + 4y^2$ on the finite region M bounded by the curves $x^2 + (y+1)^2 = 4$, $y = -1$ and $y = x + 1$.

7. Find the global extrema of $f(x,y) = x^2 + y^2 - 6x + 6y$ on the disk of radius 2, centred at the origin.

8. The equation $y^2 + 2xy = 2x - 4x^2$ defines an implicit function $y(x)$. Find and classify its local extrema.

Solutions:

1. First we find stationary points. Partial derivatives:
$$\frac{\partial f}{\partial x} = 6x^2 + 9y^2 + 30x, \qquad \frac{\partial f}{\partial y} = 18xy + 54y.$$
We have to make them equal to zero. We get the system
$$2x^2 + 3y^2 + 10x = 0 \qquad xy + 3y = 0.$$
The second equation looks promising, since we can write it as $y(x+3) = 0$. Thus there are two possibilities:
1) $y = 0$. Then the first equation reads $x^2 + 5x = 0$, which yields $x = 0$ and $x = -5$. This possibility therefore leads to points $(0,0)$ and $(-5,0)$
2) $x = -3$. Then the first equation reads $y^2 = 4$, which yields $y = \pm 2$ and points $(-3,\pm 2)$.
Thus we obtain four stationary points: $(0,0)$, $(-5,0)$, $(-3,2)$, and $(-3,-2)$.
To classify them we need to find second order partial derivatives and form the Hess matrix:
$$\frac{\partial^2 f}{\partial x^2} = 12x + 30, \qquad \frac{\partial^2 f}{\partial x \partial y} = 18y, \qquad \frac{\partial^2 f}{\partial y^2} = 18x + 54$$
$$H = \begin{pmatrix} 12x+30 & 18y \\ 18y & 18x+54 \end{pmatrix}$$
Now the classification.
For $(0,0)$ we get $H = \begin{pmatrix} 30 & 0 \\ 0 & 54 \end{pmatrix}$. Determinants of principal minors (subdeterminants) are $\Delta_1 = a_{11} = 30$ and $\Delta_2 = \det(H) = 30 \cdot 54 = 1620$. Their signs are $\Delta_1 > 0$, $\Delta_2 > 0$, which shows that the point $f(0,0) = 0$ is a local minimum.
For $(-5,0)$ we get $H = \begin{pmatrix} -30 & 0 \\ 0 & -26 \end{pmatrix}$. Subdeterminants are $\Delta_1 = -30$ and $\Delta_2 = 780$. Their signs are $\Delta_1 < 0$, $\Delta_2 > 0$, which shows that the point $f(-5,0) = 125$ is a local maximum.
For $(-3,-2)$ we get $H = \begin{pmatrix} -6 & -36 \\ -36 & 0 \end{pmatrix}$. Subdeterminants are $\Delta_1 = -6$ and $\Delta_2 = -(-36)^2$. Their signs are $\Delta_1 < 0$, $\Delta_2 < 0$, this does not follow pattern for any local extreme. But from $\Delta_2 < 0$ we conclude that the point $f(-3,2) = 81$ is a saddle point.

For $(-3,2)$ we get $H = \begin{pmatrix} -6 & -36 \\ -36 & 0 \end{pmatrix}$. Subdeterminants are $\Delta_1 = -6$ and $\Delta_2 = -36^2$. As above, from $\Delta_2 < 0$ we conclude that the point $f(-3,-2) = 81$ is a saddle point.

Some people prefer a different approach that might be simpler if the derivatives are not too bad, it is also somewhat more organized.
First we evaluate those subdeterminants in general, we obtain $\Delta_1 = 12x + 30$ and
$\Delta_2 = (12x+30)(18x+54)-(18y)^2 = 36(6x^2+23x+45-9y^2)$. Then we substitute the stationary points and reach conclusions:

point:	$(0,0)$	$(-5,0)$	$(-3,2)$	$(-3,-2)$
Δ_1:	$+$	$-$	$-$	$-$
Δ_2:	$+$	$+$	$-$	$-$
conclusion:	loc. min.	loc. max.	saddle	saddle

Then one has to write the answer: $f(0,0) = 0$ is a local minimum, $f(-5,0) = 125$ is a local maximum, $f(-3,2) = f(-3,-2) = 81$ are saddle points.

2. First we find stationary points. Partial derivatives:
$\frac{\partial f}{\partial x} = 3x^2 - 4x - 2y + z$, $\frac{\partial f}{\partial y} = 2y - 2x - z$, $\frac{\partial f}{\partial z} = 2z + x - y + 3$.
We have to solve the system

$$\begin{aligned} 3x^2 - 4x - 2y + z &= 0 \\ 2y - 2x - z &= 0 \\ 2z + x - y + 3 &= 0 \end{aligned}$$

Now noe of the equations has the convenient form of a product, so the method used in the previous problem does not help. Another popular method is elimination.
Since there is x^2 in the first equation, we will try to use the others to get rid of y and z in this first equation and then apply the quadratic rule. We can express $z = 2y - 2x$ from the second equation and put into the first and the third, obtaining $3x^2 - 6x = 0$ and $3y - 3x = -3$. What a piece of luck, the first one already features only x, the third one will also come handy when we express $y = x - 1$.
The equation $3x^2 - 6x = 0$ has two solutions: $x = 0$ and $x = 2$.
If $x = 0$, then $y = -1$ and $z = -2$. If $x = 2$, then $y = 1$ and $z = -2$. Thus we have two stationary points, $(0,-1,-2)$ and $(2,1,-2)$.
Now we use the second derivative test. First we need second partial derivatives arranged into the Hess matrix.

$$H = \begin{pmatrix} 6x-4 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Calculating subdeterminant in general does not sound very appealing (but you can try this approach), we handle each point separately.
For $(0,-1,-2)$ we get

$$H = \begin{pmatrix} -4 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \Longrightarrow \Delta_1 = -4,\ \Delta_2 = \begin{vmatrix} -4 & -2 \\ -2 & 2 \end{vmatrix} = -12,\ \Delta_3 = |H| = -26.$$

Since $\Delta_2 < 0$, at the stationary point $(0,-1,-2)$ there is no local extreme but a saddle point.
Better answer: $f(0,-1,2) = 13$ is a saddle point (we give more information this way).
(Some authors do not use the notion of saddle in cases of more than two variables, they would just say that this point is not a local extreme.)

* It may require some algebraic manipulation to find the solutions: a basic technique is to solve one equation for one of the variables, and then plug the result into the other equation. Another technique is to try to factor one of the equations and then analyze cases.

- Step 3: At *each* critical point, evaluate $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$ and apply the Second Derivatives Test:

If $\boxed{D > 0 \text{ and } f_{xx} > 0\text{: local minimum}}$. If $\boxed{D > 0 \text{ and } f_{xx} < 0\text{: local maximum}}$. If $\boxed{D < 0\text{: saddle point}}$.

- Example: Verify that $f(x, y) = x^2 + y^2$ has only one critical point, a minimum at the origin.
  - First, we have $f_x = 2x$ and $f_y = 2y$. Since they are both defined everywhere, we need only find where they are both zero.
  - Setting both partial derivatives equal to zero yields $x = 0$ and $y = 0$, so the only critical point is $(0, 0)$.
  - To classify the critical points, we compute $f_{xx} = 2$, $f_{xy} = 0$, and $f_{yy} = 2$. Then $D = 2 \cdot 2 - 0^2 = 4$.
  - So, by the classification test, since $D > 0$ and $f_{xx} > 0$ at $(0, 0)$, we see that $\boxed{(0, 0) \text{ is a local minimum}}$.

- Example: For the function $f(x, y) = 3x^2 + 2y^3 - 6xy$, find the critical points and classify them as minima, maxima, or saddle points.
  - First, we have $f_x = 6x - 6y$ and $f_y = 6y^2 - 6x$. Since they are both defined everywhere, we need only find where they are both zero.
  - Next, we can see that f_x is zero only when $y = x$. Then the equation $f_y = 0$ becomes $6x^2 - 6x = 0$, which by factoring we can see has solutions $x = 0$ or $x = 1$. Since $y = x$, we conclude that $\boxed{(0, 0), \text{ and } (1, 1) \text{ are critical points}}$.
  - To classify the critical points, we compute $f_{xx} = 6$, $f_{xy} = -6$, and $f_{yy} = 12y$. Then $D(0, 0) = 6 \cdot 0 - (-6)^2 < 0$ and $D(1, 1) = 6 \cdot 12 - (-6)^2 > 0$.
  - So, by the classification test, $\boxed{(0, 0) \text{ is a saddle point}}$ and $\boxed{(1, 1) \text{ is a local minimum}}$.

- Example: For the function $g(x, y) = x^3 y - 3xy^3 + 8y$, find the critical points and classify them as minima, maxima, or saddle points.
  - First, we have $g_x = 3x^2 y - 3y^3$ and $g_y = x^3 - 9xy^2 + 8$. Since they are both defined everywhere, we need only find where they are both zero.
  - Setting both partial derivatives equal to zero. Since $g_x = 3y(x^2 - y^2) = 3y(x + y)(x - y)$, we see that $g_x = 0$ precisely when $y = 0$ or $y = x$ or $y = -x$.
  - If $y = 0$, then $g_y = 0$ implies $x^3 + 8 = 0$, so that $x = -2$. This yields the point $(x, y) = (-2, 0)$.
  - If $y = x$, then $g_y = 0$ implies $-8x^3 + 8 = 0$, so that $x = 1$. This yields the point $(x, y) = (1, 1)$.
  - If $y = -x$, then $g_y = 0$ implies $-8x^3 + 8 = 0$, so that $x = 1$. This yields the point $(x, y) = (1, -1)$.
  - To summarize, we see that $\boxed{(-2, 0), (1, 1), \text{ and } (1, -1) \text{ are critical points}}$.
  - To classify the critical points, we compute $g_{xx} = 6xy$, $g_{xy} = 3x^2 - 9y^2$, and $g_{yy} = -18xy$.
  - Then $D(-2, 0) = 0 \cdot 0 - (12)^2 < 0$, $D(1, 1) = 6 \cdot (-18) - (-6)^2 < 0$, and $D(1, -1) = (-6) \cdot (18) - (-6)^2 < 0$.
  - So, by the classification test, $\boxed{(-2, 0), (1, 1), \text{ and } (1, -1) \text{ are all saddle points}}$.

- Example: Find the value of the function $h(x, y) = x + 2y^4 - \ln(x^4 y^8)$ at its local minimum, for x and y positive.
  - To solve this problem, we will search for all critical points of $h(x, y)$ that are minima.
  - First, we have $h_x = 1 - \dfrac{4x^3 y^8}{x^4 y^8} = 1 - \dfrac{4}{x}$ and $h_y = 8y^3 - \dfrac{8x^4 y^7}{x^4 y^8} = 8y^3 - \dfrac{8}{y}$. Both partial derivatives are defined everywhere in the given domain.

- We see that $h_x = 0$ only when $x = 4$, and also that $h_y = 0$ is equivalent to $\frac{8}{y}(y^4 - 1) = 0$, which holds for $y = \pm 1$. Since we only want $y > 0$, there is a unique critical point: (4, 1).
- Next, we compute $h_{xx} = \frac{4}{x^2}$, $g_{xy} = 0$, and $g_{yy} = 24y^2 + \frac{8}{y^2}$. Then $D(4,1) = \frac{1}{4} \cdot 32 - 0^2 > 0$.
- Thus, there is a unique critical point, and it is a minimum. Therefore, we conclude that the function has a local minimum at $(4, 1)$, and the minimum value is $h(4, 1) = \boxed{6 - \ln(4^4)}$.

• Example: Find the minimum distance between a point on the plane $x + y + z = 1$ and the point $(2, -1, -2)$.
  - The distance from the point (x, y, z) to $(2, -1, 2)$ is $d = \sqrt{(x-2)^2 + (y+1)^2 + (z+2)^2}$. Since $x+y+z = 1$ on the plane, we can view this as a function of x and y only: $d(x, y) = \sqrt{(x-2)^2 + (y+1)^2 + (3-x-y)^2}$.
  - We could minimize $d(x, y)$ by finding its critical points and searching for a minimum, but it will be much easier to find the minimum value of the squared distance $f(x, y) = d(x, y)^2 = (x-2)^2+(y+1)^2+(3-x-y)^2$.
  - We compute $f_x = 2(x-2) - 2(3-x-y) = 4x + 2y - 10$ and $f_y = 2(y+1) - 2(3-x-y) = 2x + 4y - 4$. Both partial derivatives are defined everywhere, so we need only find where they are both zero.
  - Setting $f_x = 0$ and solving for y yields $y = 5 - 2x$, and then plugging this into $f_y = 0$ yields $2x + 4(5 - 2x) - 4 = 0$, so that $-6x + 16 = 0$. Thus, $x = 8/3$ and then $y = -1/3$.
  - Furthermore, we have $f_{xx} = 4$, $f_{xy} = 2$, and $f_{yy} = 4$, so that $D = f_{xx}f_{yy} - f_{xy}^2 = 12 > 0$. Thus, the point $(x, y) = (8/3, -1/3)$ is a local minimum.
  - Thus, there is a unique critical point, and it is a minimum. We conclude that the distance function has its minimum at $(4, 1)$, so the minimum distance is $d(8/3, -1/3) = \sqrt{(2/3)^2 + (2/3)^2 + (2/3)^2} = \boxed{2/\sqrt{3}}$.

1.4 Optimization of a Function on a Region, Linear Programming

• We now discuss the problem of finding the minimum and maximum values of a function on a region in the plane, rather than the entire plane itself.
  - In general, if the region is not closed (i.e., does not contain its boundary, like the region $x^2 + y^2 < 1$ which does not contain the boundary circle $x^2 + y^2 = 1$) or not bounded (i.e., extends infinitely far away from the origin, like the half-plane $x \geq 0$) then a continuous function may not attain its minimum or maximum values anywhere in the region.
  - In order to ensure that a function does attain its minimum and maximum values at some point inside the region, the region must be both closed and bounded. If the region is not bounded or not closed, we must additionally study what happens to the function as we approach the region's boundary, or what happens as we move far away from the origin.

1.4.1 Optimization on a Region

• A natural first step is to find the critical points of the function. However, if we want to find the absolute minimum or maximum of a function $f(x, y)$ on a closed and bounded region, we must also analyze the function's behavior on the boundary of the region, because the boundary could contain the minimum or maximum.
  - Example: The extreme values of $f(x, y) = x^2 - y^2$ on the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ occur at two of the "corner points": the minimum is -1 occurring at $(0, 1)$, and the maximum $+1$ occurring at $(1, 0)$. We can see that these two points are actually the minimum and maximum on this region without calculus: since squares of real numbers are always nonnegative, on the region in question we have $-1 \leq -y^2 \leq x^2 - y^2 \leq x^2 \leq 1$.

• Unfortunately, unlike the case of a function of one variable where the boundary of an interval $[a, b]$ is very simple (namely, the two values $x = a$ and $x = b$), the boundary of a region in the plane or in higher-dimensional space can be rather complicated.

The characteristic equation reads

$$P_2(\lambda) = (a-\lambda)(b-\lambda) - c^2 = \lambda^2 - 4 = 0\,.$$

Its roots $\lambda_1 = 2$ and $\lambda_2 = -2$ do not vanish and have opposite signs. Therefore, the points $(1, \pm 1)$ are *saddle points* of the function.

Critical point $(0,0)$: The values of the second partial derivatives are

$$a = f''_{xx}(0,0) = -2\,, \quad b = f''_{yy}(0,0) = -2\,, \quad c = f''_{xy}(0,0) = 0\,.$$

The characteristic equation

$$P_2(\lambda) = (a-\lambda)(b-\lambda) = (-2-\lambda)^2 = 0$$

has one negative root of multiplicity 2, that is, $\lambda_1 = \lambda_2 = -2 < 0$. Therefore f has a *local maximum* at $(0,0)$.

Critical point $(2,0)$: The values of the second partial derivatives are

$$a = f''_{xx}(2,0) = 2\,, \quad b = f''_{yy}(2,0) = 2\,, \quad c = f''_{xy}(2,0) = 0\,.$$

The characteristic equation

$$P_2(\lambda) = (a-\lambda)(b-\lambda) = (2-\lambda)^2 = 0$$

has one positive root of multiplicity 2, $\lambda_1 = \lambda_2 = 2 > 0$. Therefore the function has a *local minimum* at $(2,0)$. □

EXAMPLE **25.2**. *Investigate the function $f(x,y) = e^{x^2-y}(5-2x+y)$ for extreme values.*

SOLUTION: The functions is defined on the whole plane and, as the product of an exponential and a polynomial, it has continuous partial derivatives of any order. So its critical points are points where its gradient vanishes, and its local extreme values, if any, can be investigated by the second-derivative test.

Critical points. Using the product rule for partial derivatives,

$$f'_x = e^{x^2-y}\Big(2x(5-2x+y) - 2\Big) = 0 \quad \Rightarrow \quad x(5-2x+y) = 1$$

$$f'_y = e^{x^2-y}\Big((-1)(5-2x+y) + 1\Big) = 0 \quad \Rightarrow \quad 5-2x+y = 1$$

The substitution of the second equation into the first one yields $x = 1$. Then it follows from the second equation that $y = -2$. So the function has just one critical point $(1,-2)$.

Second derivative test. Using the product rule for partial derivatives,

$$f''_{xx} = (f'_x)'_x = e^{x^2-y}\Big[2x\Big(2x(5-2x+y)-2\Big) + 2(5-2x+y) - 4\Big]$$

$$f''_{yy} = (f'_y)'_y = e^{x^2-y}\Big[(-1)\Big((-1)(5-2x+y)+1\Big) - 1\Big]$$

$$f''_{xy} = (f'_y)'_x = e^{x^2-y}\Big[2x\Big((-1)(5-2x+y)+1\Big) + 2\Big]$$

The values of the second partial derivatives at the critical point are

$$a = f''_{xx}(1,-2) = -2e^3\,, \quad b = f''_{yy}(1,-2) = -e^3\,, \quad c = f''_{xy}(1,-2) = 2e^3\,.$$

Therefore $D = ab - c^2 = -2e^6 < 0$. By Corollary **25.1**, the only critical point is a saddle point. The function has no extreme values. □

25.4. Proof of Theorem 25.3. Consider a rotation of the variables (dx, dy):

$$(dx,\ dy) = (dx' \cos\phi - dy' \sin\phi,\ dy' \cos\phi + dx' \sin\phi)$$

Following the proof of Theorem **9.1** (classification of quadric cylinders), the second differential is written in the new variables (dx', dy') as

$$\begin{aligned}
d^2f(\mathbf{r}_0) &= a(dx)^2 + 2cdxdy + b(dy)^2 = a'(dx')^2 + 2c'dx'dy' + b'(dy')^2 \\
a' &= \tfrac{1}{2}\Big(a + b + (a-b)\cos(2\phi) + 2c\sin(2\phi)\Big) \\
b' &= \tfrac{1}{2}\Big(a + b - (a-b)\cos(2\phi) - 2c\sin(2\phi)\Big) \\
2c' &= 2c\cos(2\phi) - (a-b)\sin(2\phi)
\end{aligned}$$

The rotation angle is chosen so that $c' = 0$. Put $A^2 = (a-b)^2 + 4c^2$. If $\cos(2\phi) = (a-b)/A$ and $\sin(2\phi) = 2c/A$, then $c' = 0$. With this choice,

$$a' = \tfrac{1}{2}(a + b + A), \quad b' = \tfrac{1}{2}(a + b - A)$$

Next note that

$$a' + b' = a + b\,, \quad a'b' = \frac{1}{4}((a+b)^2 - A^2) = ab - c^2\,.$$

On the other hand, the roots of the quadratic equation $P_2(\lambda) = 0$ also satisfy the same conditions

$$\lambda_1 + \lambda_2 = a + b\,, \quad \lambda_1\lambda_2 = ab - c^2\,.$$

Thus, $a' = \lambda_1$, $b' = \lambda_2$, and

$$d^2f(\mathbf{r}_0) = \lambda_1(dx')^2 + \lambda_2(dy')^2$$

If λ_1 and λ_2 are strictly positive, then $d^2f(\mathbf{r}_0) > 0$ for all $(dx, dy) \neq (0,0)$ and by Theorem **25.2** the function has a local minimum at $\mathbf{r}_0$. If λ_1 and λ_2 are strictly negative, then $d^2f(\mathbf{r}_0) < 0$ for all $(dx, dy) \neq (0,0)$ and by Theorem **25.2** the function has a local maximum at $\mathbf{r}_0$. If λ_1 and λ_2 do not vanish but have opposite signs, $\lambda_1\lambda_2 < 0$, then in a neighborhood of $\mathbf{r}_0$, the graph of f looks like

$$z = f(\mathbf{r}_0) + \lambda_1(x' - x_0')^2 + \lambda_2(y' - y_0')^2$$

where the coordinates (x', y') are obtained from (x, y) by rotation through the angle ϕ. When λ_1 and λ_2 have different signs, this surface is a hyperbolic paraboloid (a saddle), and f has neither a local minimum nor maximum. Case (4) is easily proved by examples (see Study Problem **25.3**).

For $(-3,2)$ we get $H = \begin{pmatrix} -6 & -36 \\ -36 & 0 \end{pmatrix}$. Subdeterminants are $\Delta_1 = -6$ and $\Delta_2 = -36^2$. As above, from $\Delta_2 < 0$ we conclude that the point $f(-3,-2) = 81$ is a saddle point.

Some people prefer a different approach that might be simpler if the derivatives are not too bad, it is also somewhat more organized.
First we evaluate those subdeterminants in general, we obtain $\Delta_1 = 12x + 30$ and
$\Delta_2 = (12x+30)(18x+54)-(18y)^2 = 36(6x^2+23x+45-9y^2)$. Then we substitute the stationary points and reach conclusions:

point:	$(0,0)$	$(-5,0)$	$(-3,2)$	$(-3,-2)$
Δ_1:	+	−	−	−
Δ_2:	+	+	−	−
conclusion:	loc. min.	loc. max.	saddle	saddle

Then one has to write the answer: $f(0,0) = 0$ is a local minimum, $f(-5,0) = 125$ is a local maximum, $f(-3,2) = f(-3,-2) = 81$ are saddle points.

2. First we find stationary points. Partial derivatives:
$\frac{\partial f}{\partial x} = 3x^2 - 4x - 2y + z$, $\frac{\partial f}{\partial y} = 2y - 2x - z$, $\frac{\partial f}{\partial z} = 2z + x - y + 3$.
We have to solve the system

$$\begin{aligned} 3x^2 - 4x - 2y + z &= 0 \\ 2y - 2x - z &= 0 \\ 2z + x - y + 3 &= 0 \end{aligned}$$

Now noe of the equations has the convenient form of a product, so the method used in the previous problem does not help. Another popular method is elimination.
Since there is x^2 in the first equation, we will try to use the others to get rid of y and z in this first equation and then apply the quadratic rule. We can express $z = 2y - 2x$ from the second equation and put into the first and the third, obtaining $3x^2 - 6x = 0$ and $3y - 3x = -3$. What a piece of luck, the first one already features only x, the third one will also come handy when we express $y = x - 1$.
The equation $3x^2 - 6x = 0$ has two solutions: $x = 0$ and $x = 2$.
If $x = 0$, then $y = -1$ and $z = -2$. If $x = 2$, then $y = 1$ and $z = -2$. Thus we have two stationary points, $(0,-1,-2)$ and $(2,1,-2)$.
Now we use the second derivative test. First we need second partial derivatives arranged into the Hess matrix.

$$H = \begin{pmatrix} 6x-4 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Calculating subdeterminant in general does not sound very appealing (but you can try this approach), we handle each point separately.
For $(0,-1,-2)$ we get

$$H = \begin{pmatrix} -4 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \implies \Delta_1 = -4,\ \Delta_2 = \begin{vmatrix} -4 & -2 \\ -2 & 2 \end{vmatrix} = -12,\ \Delta_3 = |H| = -26.$$

Since $\Delta_2 < 0$, at the stationary point $(0,-1,-2)$ there is no local extreme but a saddle point.
Better answer: $f(0,-1,2) = 13$ is a saddle point (we give more information this way).
(Some authors do not use the notion of saddle in cases of more than two variables, they would just say that this point is not a local extreme.)

For $(2,1,-2)$ we get

$$H = \begin{pmatrix} 8 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \implies \Delta_1 = 8,\ \Delta_2 = \begin{vmatrix} 8 & -2 \\ -2 & 2 \end{vmatrix} = 12,\ \Delta_3 = |H| = 28.$$

Since always $\Delta_i > 0$, we conclude that $f(2,1,-2) = -7$ is a local minimum.
Recall that for a local maximum we need $\Delta_1 < 0$, $\Delta_2 > 0$, and $\Delta_3 < 0$.

3. Since expressing y from the constraint would be messy, this calls for Lagrange multipliers with $g(x,y) = x^2 - 2x + 2y^2 + 4y$. Equations to solve are $\nabla f = \lambda \nabla g$ and $g = 0$, that is,

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} \\ g &= 0 \end{aligned}\right\} \implies \left.\begin{aligned} 2x &= \lambda(2x-2) \\ 4y &= \lambda(4y+4) \\ x^2 - 2x + 2y^2 + 4y &= 0 \end{aligned}\right\} \implies \begin{aligned} x &= \lambda(x-1) \\ y &= \lambda(y+1) \\ x^2 - 2x + 2y^2 + 4y &= 0 \end{aligned}$$

A typical strategy is to eliminate λ from the first two equations in order to obtain some relationship between the variables x, y, this is then used with condition $g = 0$ to find the desired points.
We would like to isolate λ from the first equation. Can we have $x = 1$? The first equation then reads $1 = 0$, which is not true. Thus for sure $x \neq 1$ and we can write $\lambda = \frac{x}{x-1}$. Putting it into the second equation and multiplying out we get $y = -x$. Now this can be put into the constraint, we obtain $3x^2 - 6x = 0$ and two solutions, $x = 0$ and $x = 2$. Thus there are two suspicious points: $(0,0)$ and $(2,-2)$. We substitute them into f: $f(0,0) = 0$, $f(2,-2) = 12$. Comparing values we guess that the former is a local minimum and the latter is a local maximum.
Determining global extrema usually involves some analysis of the situation. We have two local extrema, but we do not know whether they give global extrema. In general, we find global extrema by comparing values at local extrema and also values at "borders" of the set. Thus we need to know more about M, the set determined by the given condition where we look at f.
A frequent trouble arises when the given set is not bounded, since then we have to ask what happens to f when points of M run away to some infinity. Could it happen that x tends to infinity within this set? Since points from M satisfy $2y^2 + 4y = 2x - x^2$, this would force the expression $2y^2 + 4y$ to tend to minus infinity, but that is not possible. Similarly we argue that also y cannot go to infinity and we thus have a bounded set M.
Another source of trouble is if the set M is a curve that has some endpoints, then we would have to check on those. How does M actually look like? In fact, rewriting the condition as

$$(x-1)^2 + 2(y+1)^2 = 3$$

we see that M is an ellipse. This is a close curve without any end, so whatever important happens to values of f on it, it must happen at one of the points we found earlier. Thus we can conclude that $f(0,0) = 0$ is a minimum and $f(2,-2) = 12$ is a maximum of f on the given set.

4. The unknown point $Q = (x,y,z)$ satisfies $x + y - z = 1$, that would be the constraint with $g(x,y,z) = x + y - z$. The function to minimize should be the distance between P and Q, but that would mean a square root. It will be easier to minimize the distance squared, which is equivalent (think about it). Thus we have $f(x,y,z) = \text{dist}(P,Q)^2 = x^2 + (y+3)^2 + (z-2)^2$. We use Lagrange multipliers, the equations $\nabla f = \lambda \nabla g$ and $g = 1$ now give

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} \\ \frac{\partial f}{\partial z} &= \lambda \frac{\partial g}{\partial z} \\ g &= 1 \end{aligned}\right\} \implies \left.\begin{aligned} 2x &= \lambda \cdot 1 \\ 2(y+3) &= \lambda \cdot 1 \\ 2(z-2) &= \lambda \cdot (-1) \\ x + y - z &= 1 \end{aligned}\right\} \implies \begin{aligned} x &= \tfrac{1}{2}\lambda \\ y + 3 &= \tfrac{1}{2}\lambda \\ z - 2 &= -\tfrac{1}{2}\lambda \\ x + y - z &= 1 \end{aligned}$$

- We see that $h_x = 0$ only when $x = 4$, and also that $h_y = 0$ is equivalent to $\frac{8}{y}(y^4 - 1) = 0$, which holds for $y = \pm 1$. Since we only want $y > 0$, there is a unique critical point: $(4, 1)$.
- Next, we compute $h_{xx} = \frac{4}{x^2}$, $g_{xy} = 0$, and $g_{yy} = 24y^2 + \frac{8}{y^2}$. Then $D(4, 1) = \frac{1}{4} \cdot 32 - 0^2 > 0$.
- Thus, there is a unique critical point, and it is a minimum. Therefore, we conclude that the function has a local minimum at $(4, 1)$, and the minimum value is $h(4, 1) = \boxed{6 - \ln(4^4)}$.

- Example: Find the minimum distance between a point on the plane $x + y + z = 1$ and the point $(2, -1, -2)$.
  - The distance from the point (x, y, z) to $(2, -1, 2)$ is $d = \sqrt{(x-2)^2 + (y+1)^2 + (z+2)^2}$. Since $x+y+z = 1$ on the plane, we can view this as a function of x and y only: $d(x, y) = \sqrt{(x-2)^2 + (y+1)^2 + (3-x-y)^2}$.
  - We could minimize $d(x, y)$ by finding its critical points and searching for a minimum, but it will be much easier to find the minimum value of the squared distance $f(x, y) = d(x, y)^2 = (x-2)^2 + (y+1)^2 + (3-x-y)^2$.
  - We compute $f_x = 2(x-2) - 2(3-x-y) = 4x + 2y - 10$ and $f_y = 2(y+1) - 2(3-x-y) = 2x + 4y - 4$. Both partial derivatives are defined everywhere, so we need only find where they are both zero.
  - Setting $f_x = 0$ and solving for y yields $y = 5 - 2x$, and then plugging this into $f_y = 0$ yields $2x + 4(5 - 2x) - 4 = 0$, so that $-6x + 16 = 0$. Thus, $x = 8/3$ and then $y = -1/3$.
  - Furthermore, we have $f_{xx} = 4$, $f_{xy} = 2$, and $f_{yy} = 4$, so that $D = f_{xx}f_{yy} - f_{xy}^2 = 12 > 0$. Thus, the point $(x, y) = (8/3, -1/3)$ is a local minimum.
  - Thus, there is a unique critical point, and it is a minimum. We conclude that the distance function has its minimum at $(4, 1)$, so the minimum distance is $d(8/3, -1/3) = \sqrt{(2/3)^2 + (2/3)^2 + (2/3)^2} = \boxed{2/\sqrt{3}}$.

1.4 Optimization of a Function on a Region, Linear Programming

- We now discuss the problem of finding the minimum and maximum values of a function on a region in the plane, rather than the entire plane itself.
  - In general, if the region is not closed (i.e., does not contain its boundary, like the region $x^2 + y^2 < 1$ which does not contain the boundary circle $x^2 + y^2 = 1$) or not bounded (i.e., extends infinitely far away from the origin, like the half-plane $x \geq 0$) then a continuous function may not attain its minimum or maximum values anywhere in the region.
  - In order to ensure that a function does attain its minimum and maximum values at some point inside the region, the region must be both closed and bounded. If the region is not bounded or not closed, we must additionally study what happens to the function as we approach the region's boundary, or what happens as we move far away from the origin.

1.4.1 Optimization on a Region

- A natural first step is to find the critical points of the function. However, if we want to find the absolute minimum or maximum of a function $f(x, y)$ on a closed and bounded region, we must also analyze the function's behavior on the boundary of the region, because the boundary could contain the minimum or maximum.
  - Example: The extreme values of $f(x, y) = x^2 - y^2$ on the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ occur at two of the "corner points": the minimum is -1 occurring at $(0, 1)$, and the maximum $+1$ occurring at $(1, 0)$. We can see that these two points are actually the minimum and maximum on this region without calculus: since squares of real numbers are always nonnegative, on the region in question we have $-1 \leq -y^2 \leq x^2 - y^2 \leq x^2 \leq 1$.
- Unfortunately, unlike the case of a function of one variable where the boundary of an interval $[a, b]$ is very simple (namely, the two values $x = a$ and $x = b$), the boundary of a region in the plane or in higher-dimensional space can be rather complicated.

Since $\operatorname{grad} P$ is defined everywhere, the only critical points of P are those where $\operatorname{grad} P = \vec{0}$. Thus, solving for q_1, and q_2, we find that

$$q_1 = 699.1 \quad \text{and} \quad q_2 = 896.7.$$

The corresponding prices are

$$p_1 = 390.27 \quad \text{and} \quad p_2 = 320.66.$$

To see whether or not we have found a local maximum, we compute second partial derivatives:

$$\frac{\partial^2 P}{\partial q_1^2} = -0.6, \quad \frac{\partial^2 P}{\partial q_2^2} = -0.4, \quad \frac{\partial^2 P}{\partial q_1 \partial q_2} = -0.2,$$

so,

$$D = \frac{\partial^2 P}{\partial q_1^2}\frac{\partial^2 P}{\partial q_2^2} - \left(\frac{\partial^2 P}{\partial q_1 \partial q_2}\right)^2 = (-0.6)(-0.4) - (-0.2)^2 = 0.2.$$

Therefore we have found a local maximum. The graph of P is an upside-down paraboloid, so $(699.1, 896.7)$ is a global maximum. This point is within the region, so points on the boundary give smaller values of P.

The company should produce 699.1 units of the first item priced at $\$390.27$ per unit, and 896.7 units of the second item priced at $\$320.66$ per unit. The maximum profit $P(699.1, 896.7) \approx \$433{,}000$.

Example 2 A delivery of 480 cubic meters of gravel is to be made to a landfill. The trucker plans to purchase an open-top box in which to transport the gravel in numerous trips. The total cost to the trucker is the cost of the box plus $80 per trip. The box must have height 2 meters, but the trucker can choose the length and width. The cost of the box is $\$100/\text{m}^2$ for the ends, $\$50/\text{m}^2$ for the sides and $\$200/\text{m}^2$ for the bottom. Notice the tradeoff: A smaller box is cheaper to buy but requires more trips. What size box should the trucker buy to minimize the total cost?[4]

Solution We first get an algebraic expression for the trucker's cost. Let the length of the box be x meters and the width be y meters; the height is 2 meters. (See Figure 15.20.)

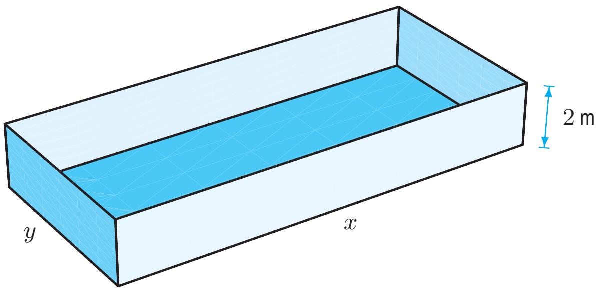

Figure 15.20: The box for transporting gravel

Table 15.2 *Trucker's itemized cost*

Expense	Cost in dollars
Travel: $480/(2xy)$ at \$80/trip	$(240 \cdot 80)/(xy)$
Ends: 2 at $\$100/\text{m}^2 \cdot 2y\ \text{m}^2$	$400y$
Sides: 2 at $\$50/\text{m}^2 \cdot 2x\ \text{m}^2$	$200x$
Bottom: 1 at $\$200/\text{m}^2 \cdot xy\ \text{m}^2$	$200xy$

The volume of the box is $2xy$ m^3, so delivery of 480 m^3 of gravel requires $480/(2xy)$ trips. The number of trips is a whole number; however, we treat it as continuous so that we can optimize using derivatives. The trucker's cost is itemized in Table 15.2. The problem is to minimize

$$\text{Total cost} = \frac{240 \cdot 80}{xy} + 400y + 200x + 200xy = 200\left(\frac{96}{xy} + 2y + x + xy\right).$$

[4] Adapted from Claude McMillan, Jr., *Mathematical Programming*, 2nd ed., p. 156-157 (New York: Wiley, 1978).

The length and width of the box must be positive. Thus, the region is the first quadrant but it does not contain the boundary, $x = 0$ and $y = 0$.

Our problem is to minimize

$$f(x,y) = \frac{96}{xy} + 2y + x + xy.$$

The critical points of this function occur where

$$f_x(x,y) = -\frac{96}{x^2y} + 1 + y = 0$$
$$f_y(x,y) = -\frac{96}{xy^2} + 2 + x = 0.$$

We put the $96/(x^2y)$ and $96/(xy^2)$ terms on the other side of the the equation, divide, and simplify:

$$\frac{96/(x^2y)}{96/(xy^2)} = \frac{1+y}{2+x} \quad \text{so} \quad \frac{y}{x} = \frac{1+y}{2+x} \quad \text{giving} \quad 2y = x.$$

Substituting $x = 2y$ in the equation $f_y(x,y) = 0$ gives

$$-\frac{96}{2y \cdot y^2} + 2 + 2y = 0$$
$$y^4 + y^3 - 24 = 0.$$

The only positive solution to this equation is $y = 2$, so the only critical point in the region is $(4, 2)$.

To check that the critical point is a local minimum, we use the second-derivative test. Since

$$D(4,2) = f_{xx}f_{yy} - (f_{xy})^2 = \frac{192}{4^3 \cdot 2} \cdot \frac{192}{4 \cdot 2^3} - \left(\frac{96}{4^2 \cdot 2^2} + 1\right)^2 = 9 - \frac{25}{4} > 0$$

and $f_{xx}(4,2) > 0$, the point $(4,2)$ is a local minimum. Since the value of f increases without bound as x or y increases without bound and as $x \to 0^+$ and $y \to 0^+$, it can be shown that $(4,2)$ is a global minimum. (See Problem 29.) Thus, the optimal box is 4 meters long and 2 meters wide.

Fitting a Line to Data: Least Squares

Suppose we want to fit the "best" line to some data in the plane. We measure the distance from a line to the data points by adding the squares of the vertical distances from each point to the line. The smaller this sum of squares is, the better the line fits the data. The line with the minimum sum of square distances is called the *least squares line*, or the *regression line*. If the data is nearly linear, the least squares line is a good fit; otherwise it may not be. (See Figure 15.21.)

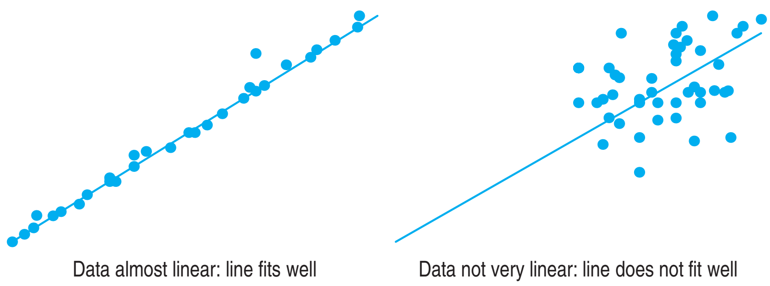

Figure 15.21: Fitting lines to data points

Example 3 Find a least squares line for the following data points: $(1,1)$, $(2,1)$, and $(3,3)$.