17th lesson

https://www2.karlin.mff.cuni.cz/~kuncova/en/teaching.php,
kuncova@karlin.mff.cuni.cz

Theory

Theorem 1 (implicit function)**.** Let $G \subset \mathbb{R}^{n+1}$ be an open set, $F\colon G \to \mathbb{R}$, and $\vec{\tilde{x}} \in \mathbb{R}^n$, $\tilde{y} \in \mathbb{R}$ such that $[\vec{\tilde{x}}, \tilde{y}] \in G$. Suppose that

1. $F \in C^1(G)$,
2. $F(\vec{\tilde{x}}, \tilde{y}) = 0$,
3. $\dfrac{\partial F}{\partial y}(\vec{\tilde{x}}, \tilde{y}) \neq 0$.

Then there exist a neighbourhood $U \subset \mathbb{R}^n$ of the point $\vec{\tilde{x}}$ and a neighbourhood $V \subset \mathbb{R}$ of the point $\tilde{y}$ such that for each $\vec{x} \in U$ there exists a unique $y \in V$ satisfying $F(\vec{x}, y) = 0$. If we denote this y by $\varphi(\vec{x})$, then the resulting function φ is in $C^1(U)$ and

$$\frac{\partial \varphi}{\partial x_j}(\vec{x}) = -\frac{\frac{\partial F}{\partial x_j}(\vec{x}, \varphi(\vec{x}))}{\frac{\partial F}{\partial y}(\vec{x}, \varphi(\vec{x}))} \quad \text{for } \vec{x} \in U,\ j \in \{1, \dots, n\}.$$

Exercises

1. Find the derivative of $y(x)$ given by the equation
   (a) $y^3 + y = x$ at $(-2, -1)$
   (b) $x^3 + y^3 = 6xy$ at $(3, 3)$
   (c) $y + \sin y = x^2 + x$ at $(0, 0)$
   (d) $x^3 + y^3 - 3xy - 3 = 0$ at $(1, 2)$
   (e) $\ln(x + y) = x + y - xy - x^2 - y^2$ at $(0, 0)$
   (f) $x^{2/3} + y^{2/3} = 8$ at $(8, 8)$
   (g) $\sin(xy) + x^2 + y^2 = 1$ at $(0, 1)$
   (h) $\sin(xz) + \sin(yz) = \sin(xy)$ at $(0, 1, \pi)$, find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$
   (i) $x^2 e^y - yze^x = 0$ find $\frac{\partial z}{\partial x}$ at an arbitrary suitable point
2. Find y'' if $\sqrt{x} + \sqrt{y} = 1$ (at an arbitrary suitable point).
3. Find the tangent line at point $(2, 1)$ to the graph given by the equation
$$3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0.$$
4. Let us consider the equation
$$y^3 + 2y = \sin x + 3$$
   Find the slope of the curve at $(0, 1)$. Approximate the value of y when $x = 0.05$. (Find the tangent line and estimate the value at 0.05 at the tangent line.)