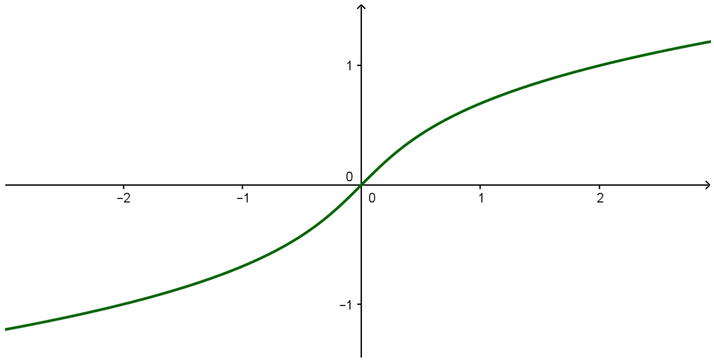
Note it is not expected that you should be able to plot these graphs - they are displayed here for information.



Graph of $y^3 + y = x$.

We can now ask for the slope of the tangent to this curve at a particular point by finding dy/dx.

1a

Example 5 *A curve is given by*

$$y^3 + y = x.$$

(a) Show that the point $(-2, -1)$ *lies on the curve.*

(b) Find dy/dx *by using implicit differentiation.*

(c) Find the slope of the tangent at the point $(-2, -1)$.

Solution.

(a) To show that the point $(-2, -1)$ *lies on the curve all we do is to substitute these values into the equation and see if they satisfy the equation.*
We get on putting $x = -2,\ y = -1$:

$$(-1)^3 + (-1) = -2$$

which is clearly true, hence the point lies on the curve.

(b) *Differentiate both sides of the equation to get:*

$$\begin{aligned}
\frac{d}{dx}(y^3 + y) &= \frac{d}{dx}(x) \Rightarrow \\
3y^2\frac{dy}{dx} + \frac{dy}{dx} &= 1 \Rightarrow \quad \textit{on collecting terms in } dy/dx \\
(3y^2 + 1)\frac{dy}{dx} &= 1 \Rightarrow \\
\frac{dy}{dx} &= \frac{1}{3y^2 + 1}.
\end{aligned}$$

(c) *Putting* $x = -2, \quad y = -1$ *in the expression for* dy/dx *gives the slope of the tangent at the point* $(-2, -1)$ *i.e.*

$$\frac{dy}{dx} = \frac{1}{3 \times (-1)^2 + 1} = \frac{1}{4}.$$

Example 6 *A curve is given by*

$$y + \sin(y) = x^2 + x$$

(a) *Show that the point* $(0, 0)$ *lies on the curve.*

(b) *Find the slope of the tangent at the point* $(0, 0)$.

(c) *Find the equation of the tangent at the point* $(0, 0)$.

Solution.

(a) *To show that the point* $(0, 0)$ *lies on the curve all we do is to substitute these values into the equation and see if they satisfy the equation.*
We get on putting $x = 0, \ y = 0$:

$$0 + \sin(0) = 0^2 + 0$$

which is clearly true, hence the point lies on the curve.

2. Collect all terms with dy/dx on one side of the equation.

$$2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = -2x$$

3. Factor dy/dx: $\quad \frac{dy}{dx}\left(2-3y^2\right) = -2x$

4. Solve for dy/dx: $\quad \frac{dy}{dx} = \frac{2x}{3y^2-2}$ □

Try This 2

Find the derivative $\frac{dy}{dx}$.

a) $y^3 - x - y = 1$ b) $xy^2 = y + 2$

Example 3 Using Implicit Differentiation to find the Slope

1b Find the slope of the graph of $x^3 + y^3 = 6xy$ at the point $(3, 3)$.
The graph is called the **folium of Descartes**, see Figure 1.

Solution First, we determine dy/dx.

$$\begin{aligned}
\frac{d}{dx}\left[x^3+y^3\right] &= 6\frac{d}{dx}[xy] \\
3x^2 + 3y^2\frac{dy}{dx} &= 6\left[x\frac{dy}{dx}+y\right] && \text{Sum and Product Rules} \\
3x^2 + 3y^2\frac{dy}{dx} &= 6x\frac{dy}{dx} + 6y \\
3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} &= 6y - 3x^2 && \text{Collect all } dy/dx \\
\frac{dy}{dx}\left(3y^2-6x\right) &= 6y - 3x^2 && \text{Factor } dy/dx \\
\frac{dy}{dx} &= \frac{6y-3x^2}{3y^2-6x} && \text{Solve for } dy/dx \\
\frac{dy}{dx} &= \frac{2y-x^2}{y^2-2x} && \text{Simplify}
\end{aligned}$$

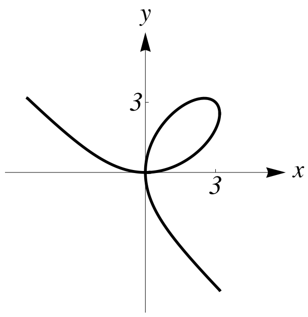


Figure 1
The folium of Descartes at the point $(3, 3)$ has slope $dy/dx = -1$.

Next, substitute the coordinates of $(3, 3)$ into the derivative.
Then the slope of the graph at $(3, 3)$ is

$$\frac{dy}{dx} = \frac{2(3)-3^2}{3^2-2(3)} = -1.$$

□

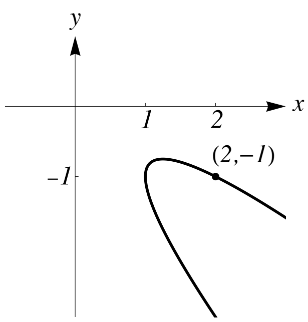


Figure 2
The point $(2, -1)$ on the graph of $y^2 + x^2 = x - 2xy - 1$.

Try This 3

Find the slope of the tangent line to the graph of $y^2 + x^2 = x - 2xy - 1$ at the point $(2, -1)$, see Figure 2.

(b) *Differentiate both sides of the equation to get:*

$$\begin{aligned}
\frac{d}{dx}(y^3 + y) &= \frac{d}{dx}(x) \Rightarrow \\
3y^2\frac{dy}{dx} + \frac{dy}{dx} &= 1 \Rightarrow \quad \textit{on collecting terms in } dy/dx \\
(3y^2 + 1)\frac{dy}{dx} &= 1 \Rightarrow \\
\frac{dy}{dx} &= \frac{1}{3y^2 + 1}.
\end{aligned}$$

(c) *Putting* $x = -2, \quad y = -1$ *in the expression for* dy/dx *gives the slope of the tangent at the point* $(-2, -1)$ *i.e.*

$$\frac{dy}{dx} = \frac{1}{3 \times (-1)^2 + 1} = \frac{1}{4}.$$

1c

Example 6 *A curve is given by*

$$y + \sin(y) = x^2 + x$$

(a) *Show that the point* $(0, 0)$ *lies on the curve.*

(b) *Find the slope of the tangent at the point* $(0, 0)$.

(c) *Find the equation of the tangent at the point* $(0, 0)$.

Solution.

(a) *To show that the point* $(0, 0)$ *lies on the curve all we do is to substitute these values into the equation and see if they satisfy the equation.*
We get on putting $x = 0, \; y = 0$:

$$0 + \sin(0) = 0^2 + 0$$

which is clearly true, hence the point lies on the curve.

(b) *Differentiate both sides of the equation:*

$$\begin{aligned}\frac{d}{dx}(y+\sin(y)) &= \frac{d}{dx}(x^2+x) \Rightarrow \\ \frac{dy}{dx}+\cos(y)\frac{dy}{dx} &= 2x+1 \Rightarrow \text{ on collecting terms in } dy/dx \\ (1+\cos(y))\frac{dy}{dx} &= 2x+1 \Rightarrow \\ \frac{dy}{dx} &= \frac{2x+1}{1+\cos(y)}.\end{aligned}$$

Hence at the point $(0,0)$ *we have the slope of the tangent is*

$$\frac{dy}{dx} = \frac{2\times 0+1}{1+\cos(0)} = \frac{1}{2}.$$

(c) *The equation of the tangent at* $(0,0)$ *is of the form*

$$y = \frac{1}{2}x + c$$

but at $x = 0, \quad y = 0$ *hence* $c = 0$.
So the equation of the tangent is

$$y = \frac{1}{2}x.$$

Note

On differentiating the implicit expression we found:

$$\frac{d(\sin(y))}{dx} = \cos(y)\frac{dy}{dx}.$$

Once again this is because we are using the Chain Rule where $\sin(y)$ *is a function of a function with* y *as the "innermost" function.*

The given equation therefore implicitly defines a function $y = f(x)$ in the neighbourhood of (x_0, y_0). We know that $f(1) = 2$. Further, let us assume f is defined on $\mathscr{O}_{0.11}(1) = (0.89, 1.11)$. What would be the function value of f at $x = 0.9$?

By substituting $x = 0.9$ into the equation $x^3 + y^3 - 3xy - 3 = 0$ one derives

$$y^3 - 2.7y - 2.271 = 0.$$

This equation has to have exactly one solution $y = f(0.9)$ in the neighbourhood of the point $y_0 = 2$. The first approximation of this solution by the Newton's method is

$$y = f(0.9) \doteq 1.963783.$$

■

The theorem below shows us how to differentiate implicit functions. The computed derivatives can be used to decide monotonicity, convexity and concavity of implicit functions, to construct Taylor polynomials to approximate implicit functions, to determine their tangents and differentials.

Theorem 6.1.2 Let $G \subseteq \mathbb{R}^2$ be an open set and let $F \in C^k(G)$ for $k \in \mathbb{N}$. Let $(x_0, y_0) \in G$ be a point such that

$$F(x_0, y_0) = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial y}(x_0, y_0) \neq 0.$$

Then the derivative of an implicit function $y = f(x)$ defined by the equation $F(x, y) = 0$ in a neighbourhood of (x_0, y_0) is given as

$$f'(x) = -\frac{\frac{\partial F}{\partial x}(x, f(x))}{\frac{\partial F}{\partial y}(x, f(x))} \quad \text{for } x \in (x_0 - \delta, x_0 + \delta) \text{ for some } \delta > 0. \tag{6.1}$$

Proof. The proof follows from the rule on how to differentiate composed functions:

The composed function $h(x) = F(x, f(x))$ is identically zero on a neighbourhood of x_0 and thus

$$0 = h'(x) = \frac{\partial F}{\partial x}(x, f(x)) \cdot 1 + \frac{\partial F}{\partial y}(x, f(x)) \cdot \frac{dy}{dx} = \frac{\partial F}{\partial x}(x, f(x)) + \frac{\partial F}{\partial y}(x, f(x)) f'(x)$$

in the same neighbourhood. Because $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$ and because $\frac{\partial F}{\partial y}(x, y)$ is continuous, the derivative $\frac{\partial F}{\partial y}(x, y) \neq 0$ in a neighbourhood of (x_0, y_0). Therefore,

$$f'(x) = -\frac{\frac{\partial F}{\partial x}(x, f(x))}{\frac{\partial F}{\partial y}(x, f(x))} \quad \text{for } x \in (x_0 - \delta, x_0 + \delta) \text{ for some } \delta > 0.$$

■

■ **Example 6.7** Let us differentiate the implicit function $y = f(x)$ given by the equation

$$F(x, y) = x^3 + y^3 - 3xy - 3 = 0$$

in the neighbourhood of $(x_0, y_0) = (1, 2)$.

According to the results in Example 6.6 the function f exists. Let us recall that

$$\frac{\partial F}{\partial y}(x, y) = 3y^2 - 3x \quad \text{and} \quad \frac{\partial F}{\partial y}(1, 2) = 9 \neq 0.$$

Further,

$$\frac{\partial F}{\partial x}(x, y) = 3x^2 - 3y \quad \text{and} \quad \frac{\partial F}{\partial x}(1, 2) = -3.$$

Then

$$f'(1) = -\frac{\frac{\partial F}{\partial x}(1,2)}{\frac{\partial F}{\partial y}(1,2)} = -\frac{-3}{9} = \frac{1}{3}.$$

Moreover,

$$f'(x) = -\frac{\frac{\partial F}{\partial x}(x,f(x))}{\frac{\partial F}{\partial y}(x,f(x))} = \frac{f(x)-x^2}{f(x)^2-x} > 0$$

for $x \in (1-\delta, 1+\delta)$ for $\delta > 0$. This means f is increasing in a neighbourhood of the point $x_0 = 1$. ■

R Higher order derivatives of an implicit function $y = f(x)$ can be derived by differentiating the formula (6.1). For instance:

$$f''(x) = -\frac{\left(\frac{\partial^2 F}{\partial x^2}(x,f(x)) + \frac{\partial^2 F}{\partial y \partial x}(x,f(x))f'(x)\right)\frac{\partial F}{\partial y}(x,f(x)) - \left(\frac{\partial^2 F}{\partial x \partial y}(x,f(x)) + \frac{\partial^2 F}{\partial y^2}(x,f(x))f'(x)\right)\frac{\partial F}{\partial x}(x,f(x))}{\left(\frac{\partial F}{\partial y}(x,f(x))\right)^2}$$

or shortly

$$y'' = -\frac{\left(\frac{\partial^2 F}{\partial x^2}(x,y) + \frac{\partial^2 F}{\partial y \partial x}(x,y)y'\right)\frac{\partial F}{\partial y}(x,y) - \left(\frac{\partial^2 F}{\partial x \partial y}(x,y) + \frac{\partial^2 F}{\partial y^2}(x,y)y'\right)\frac{\partial F}{\partial x}(x,y)}{\left(\frac{\partial F}{\partial y}(x,y)\right)^2}$$

where y is viewed as the function $y = f(x)$.

■ **Example 6.8** From Example 6.6 and 6.7, the equation

$$F(x,y) = x^3 + y^3 - 3xy - 3 = 0$$

defines an implicit function $y = f(x)$ in the neighbourhood of $(x_0, y_0) = (1,2)$ such that $f(1) = 2$ and $f'(1) = \frac{1}{3}$. By differentiating the expression

$$f'(x) = -\frac{\frac{\partial F}{\partial x}(x,f(x))}{\frac{\partial F}{\partial y}(x,f(x))} = \frac{f(x)-x^2}{f(x)^2-x}$$

one derives

$$f''(x) = \frac{(f'(x)-2x)(f(x)^2-x) - (f(x)-x^2)(2f(x)f'(x)-1)}{(f(x)^2-x)^2}$$

and thus

$$f''(1) = \frac{(f'(1)-2)(f(1)^2-1) - (f(1)-1)(2f(1)f'(1)-1)}{(f(1)^2-1)^2} = \frac{(1-6)-(4\frac{1}{3}-1)}{3^2} = \frac{-5-\frac{1}{3}}{3^2} = \frac{-16}{27} < 0$$

which means that f is concave in a neighbourhood of the point $x_0 = 1$.

Note that with the notation $y = f(x)$ one may write

$$y' = \frac{y-x^2}{y^2-x}$$

and by differentiating with respect to x:

$$y'' = \frac{(y'-2x)(y^2-x) - (y-x^2)(2yy'-1)}{(y^2-x)^2}.$$

Consequently one substitutes the values $x = 1$, $y = 2$ and $y' = 1/3$. ■

- **By differentiating the formula for the first derivative:**
The first derivative of the function $f(x)$ in a neighbourhood of 0 defined by the equation $F(x,y)=0$ equals

$$f'(x)=-\frac{\frac{\partial F}{\partial x}(x,f(x))}{\frac{\partial F}{\partial y}(x,f(x))}=-\frac{2e^{2x}+1}{e^{f(x)}+2}.$$

Therefore, since $f(0)=0$, $f'(0)=-1$. Further,

$$f''(x)=[f'(x)]'=-\frac{4e^{2x}(e^{f(x)}+2)-(2e^{2x}+1)e^{f(x)}f'(x)}{(e^{f(x)}+2)^2}$$

and, by evaluating the formula for $x=0$, $f''(0)=-\frac{12+3}{3^2}=-\frac{5}{3}$. Finally,

$$\begin{aligned} f'''(x)=[f''(x)]' &= -\frac{\left(8e^{2x}(e^{f(x)}+2)-(2e^{2x}+1)e^{f(x)}(f'(x))^2-(2e^{2x}+1)e^{f(x)}f''(x)\right)(e^{f(x)}+2)^2}{(e^{f(x)}+2)^4} \\ &+\frac{\left(4e^{2x}(e^{f(x)}+2)-(2e^{2x}+1)e^{f(x)}f'(x)\right)2(e^{f(x)}+2)e^{f(x)}f'(x)}{(e^{f(x)}+2)^4} \end{aligned}$$

and then

$$f'''(0)=-\frac{(24-4+4-3+5)9+(12+3)6}{3^4}=-\frac{36}{9}=-4.$$

- **By differentiating the equation:**
Because the equation $F(x,y)=0$ implicitly defines a function $y=f(x)$ in a neighbourhood of $(0,0)$, we substitute $f(x)$ into y, where x is considered from a neighbourhood of 0:

$$e^{2x}+e^{f(x)}+x+2f(x)-2=0.$$

By differentiating both sides of this equation with respect to x one obtains

$$2e^{2x}+e^{f(x)}f'(x)+1+2f'(x)=0$$

which specifies the first derivative of f in a neighbourhood of 0. Specifically, $f'(0)=-1$. By further differentiation of this equation, the relation specifying the second derivative of f is obtained:

$$4e^{2x}+e^{f(x)}(f'(x))^2+e^{f(x)}f''(x)+2f''(x)=0.$$

Thus, the substitution $x=0$ implies $f''(0)=-\frac{5}{3}$. Lastly, differentiation of the equation specifying $f''(x)$ leads to the equation which specifies $f'''(x)$ for x in a neighbourhood of 0:

$$8e^{2x}+e^{f(x)}(f'(x))^3+e^{f(x)}2f'(x)f''(x)+e^{f(x)}f'(x)f''(x)+e^{f(x)}f'''(x)+2f'''(x)=0.$$

Therefore, by substituting $x=0$, $f'''(0)=-4$.

■

■ **Example 6.11** Show that the equation

$$\ln(x+y)=x+y-xy-x^2-y^2$$

defines an implicit function $y=f(x)$ in a neighbourhood of the point $(0,1)$. Determine the tangent line to the graph of f at the point $(0,1)$.

Let $F(x,y)=\ln(x+y)-x-y+xy+x^2+y^2$. Then, because $F(0,1)=\ln(1)-1+1=0$ and $\frac{\partial F}{\partial y}(0,1)=\left.\left(\frac{1}{x+y}-1+x+2y\right)\right|_{(x,y)=(0,1)}=2\neq 0$, the equation $F(x,y)=0$ implicitly defines a function $y=f(x)$ in a neighbourhood of the point $(0,1)$.

Because

$$f'(0)=-\frac{\frac{\partial F}{\partial x}(0,1)}{\frac{\partial F}{\partial y}(0,1)}=-\frac{\left.\left(\frac{1}{x+y}-1+y+2x\right)\right|_{(x,y)=(0,1)}}{\left.\left(\frac{1}{x+y}-1+x+2y\right)\right|_{(x,y)=(0,1)}}=-1/2,$$

This section has shown how to find the derivatives of implicitly defined functions, whose graphs include a wide variety of interesting and unusual shapes. Implicit differentiation can also be used to further our understanding of "regular" differentiation.

One hole in our current understanding of derivatives is this: what is the derivative of the square root function? That is,

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = ? \tag{3.5.18}$$

We allude to a possible solution, as we can write the square root function as a power function with a rational (or, fractional) power. We are then tempted to apply the Power Rule and obtain

$$\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}. \tag{3.5.19}$$

The trouble with this is that the Power Rule was initially defined only for positive integer powers, $n > 0$. While we did not justify this at the time, generally the Power Rule is proved using something called the Binomial Theorem, which deals only with positive integers. The Quotient Rule allowed us to extend the Power Rule to negative integer powers. Implicit Differentiation allows us to extend the Power Rule to rational powers, as shown below.

Let $y = x^{m/n}$, where m and n are integers with no common factors (so $m = 2$ and $n = 5$ is fine, but $m = 2$ and $n = 4$ is not). We can rewrite this explicit function implicitly as $y^n = x^m$. Now apply implicit differentiation.

$$\begin{aligned} y &= x^{m/n} \\ y^n &= x^m \\ \frac{d}{dx}(y^n) &= \frac{d}{dx}(x^m) \\ n \cdot y^{n-1} \cdot y' &= m \cdot x^{m-1} \\ y' &= \frac{m}{n}\frac{x^{m-1}}{y^{n-1}} \quad (\text{now substitute } x^{m/n} \text{ for } y) \\ &= \frac{m}{n}\frac{x^{m-1}}{(x^{m/n})^{n-1}} \quad (\text{apply lots of algebra}) \\ &= \frac{m}{n}x^{(m-n)/n} \\ &= \frac{m}{n}x^{m/n-1}. \end{aligned}$$

The above derivation is the key to the proof extending the Power Rule to rational powers. Using limits, we can extend this once more to include *all* powers, including irrational (even transcendental!) powers, giving the following theorem.

Theorem 21: Power Rule for Differentiation

Let $f(x) = x^n$, where $n \neq 0$ is a real number. Then f is a differentiable function, and $f'(x) = n \cdot x^{n-1}$.

This theorem allows us to say the derivative of x^π is $\pi x^{\pi-1}$.

We now apply this final version of the Power Rule in the next example, the second investigation of a "famous" curve.

Example 72: Using the Power Rule

Find the slope of $x^{2/3} + y^{2/3} = 8$ at the point $(8, 8)$.

Solution

This is a particularly interesting curve called an *astroid*. It is the shape traced out by a point on the edge of a circle that is rolling around inside of a larger circle, as shown in Figure 2.25.

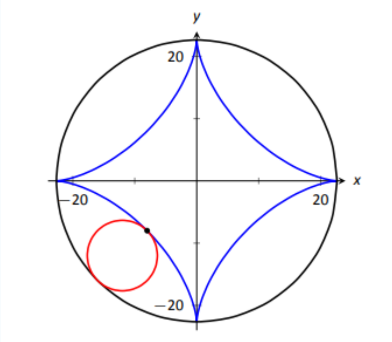


Figure 2.25: An astroid, traced out by a point on the smaller circle as it rolls inside the larger circle.

To find the slope of the astroid at the point $(8,8)$, we take the derivative implicitly.

$$\begin{aligned}\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y' &= 0\\ \frac{2}{3}y^{-1/3}y' &= -\frac{2}{3}x^{-1/3}\\ y' &= -\frac{x^{-1/3}}{y^{-1/3}}\\ y' &= -\frac{y^{1/3}}{x^{1/3}} = -\sqrt[3]{\frac{y}{x}}.\end{aligned}$$

Plugging in $x=8$ and $y=8$, we get a slope of -1. The astroid, with its tangent line at $(8,8)$, is shown in Figure 2.26.

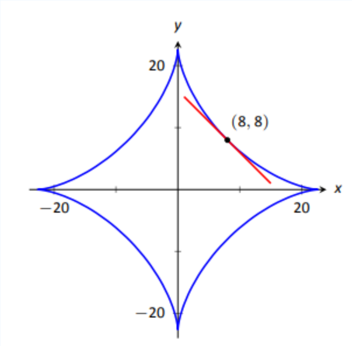


Figure 2.26: An astroid with a tangent line.

Implicit Differentiation and the Second Derivative

We can use implicit differentiation to find higher order derivatives. In theory, this is simple: first find $\frac{dy}{dx}$, then take its derivative with respect to x. In practice, it is not hard, but it often requires a bit of algebra. We demonstrate this in an example.

Example 73: Finding the second derivative

Given $x^2+y^2=1$, find $\frac{d^2y}{dx^2}=y''$.

Solution

We found that $y'=\frac{dy}{dx}=-x/y$ in Example 71. To find y'', we apply implicit differentiation to y'.

two equations, so this is no problem, we eliminate. We get

$$\frac{\partial^2 f}{\partial x\partial y}=2\sqrt{s}\frac{\partial f}{\partial t}+2\sqrt{s^3}\frac{\partial^2 f}{\partial s\partial t}+t2\sqrt{s}\frac{\partial^2 f}{\partial t^2}$$

Dividing by 2 we get exactly the same answer as before.
Note that we were lucky that in the equation for $\frac{\partial f}{\partial t}$, only one partial derivative remained on the right. In general we can expect both equations in part 1) to feature derivatives by x and y, like the first equation does. The we would need to differentiate both equations by both s and t, obtaining four equations with four unknown partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x\partial y}$, and $\frac{\partial^2 f}{\partial y\partial x}$, the system then would have to be solved. So indeed, this procedure is longer, but as we remarked before, more general.
Bonus: We will find $\frac{\partial^2 f}{\partial y\partial x}$. We start with

$$\frac{\partial f}{\partial x}=2x\frac{\partial f}{\partial s}+2xy\frac{\partial f}{\partial t}.$$

and then calculate (similarly as above)

$$\begin{aligned}\frac{\partial^2 f}{\partial y\partial x}&=\frac{\partial}{\partial y}\Big(2x\frac{\partial f}{\partial s}+2xy\frac{\partial f}{\partial t}\Big)=2x\frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial s}\Big)+2x\frac{\partial f}{\partial t}+2xy\frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial t}\Big)\\&=2x\Big[\frac{\partial^2 f}{\partial s^2}\frac{\partial s}{\partial y}+\frac{\partial^2 f}{\partial t\partial s}\frac{\partial t}{\partial y}\Big]+2x\frac{\partial f}{\partial t}+2xy\Big[\frac{\partial^2 f}{\partial s\partial t}\frac{\partial s}{\partial y}+\frac{\partial^2 f}{\partial t^2}\frac{\partial t}{\partial y}\Big]\\&=2x^3\frac{\partial^2 f}{\partial t\partial s}+2x\frac{\partial f}{\partial t}+2x^3y\frac{\partial^2 f}{\partial t^2}.\end{aligned}$$

Thus

$$\frac{1}{2}\frac{\partial^2 f}{\partial y\partial x}=\sqrt{s}\frac{\partial f}{\partial t}+\big(\sqrt{s}\big)^3\frac{\partial^2 f}{\partial t\partial s}+t\sqrt{s}\frac{\partial^2 f}{\partial t^2}.$$

If f is “nice”, the two mixed derivatives should be the same.

10. We have $F(x,y)=\sin(xy)+x^2+y^2$. We check that $F(0,1)=1$, so the question makes sense.
a) $\frac{\partial F}{\partial y}(0,1)=x\cos(xy)+2y\big|_{x=0,y=1}=2$. Since $\frac{\partial F}{\partial y}(0,1)\neq 0$, by the Implicit Function Theorem there is a function $y(x)$ defined on some neighborhood of $x=0$ such that $y(0)=1$ and $F(x,y(x))=0$.
b) We can use the Implicit Function Theorem to find $y'(x)$, but it is easier just to differentiate the given equation, keeping in mind that now y is a function of x. We get

$$\begin{gathered}\sin(xy)+x^2+y^2=1\\ [\sin(xy(x))]'+[x^2]'+[y(x)^2]'=[1]'\\ \cos(xy)\cdot[y+xy']+2x+2y\cdot y'=0. \qquad (\star)\end{gathered}$$

Substituting $(0,1)$ into it we get $\cos(0)\cdot[1+0\cdot y'(0)]+0+2\cdot y'(0)=0$, solving for $y'(0)$ we get $y'(0)=-\frac{1}{2}$.
We have the point $(0,1)$ and the slope $y'(0)=-\frac{1}{2}$, consequently the equation of the tangent line is $y=-\frac{1}{2}(x-0)+1$, that is, $x+2y=2$.
c) There are two possibilities. One is to take another derivative of the equation $(\star)$:

$$-\sin(xy)[y+xy']\cdot[y+xy']+\cos(xy)[y'+y'+xy'']+2+2y'y'+2yy''=0.$$

Substituting $x=0$, $y=1$, and $y'=-\frac{1}{2}$, then solving for $y''(0)$ we get $y''(0)=-\frac{3}{4}$.
Alternative solution: We can solve $(\star)$ in general for y':

$$y'(x)=-\frac{2x+\cos(xy)y}{2y+\cos(xy)x}$$

and then take the derivative:

$$y''(x) = -\frac{[2x+\cos(xy)y]'\cdot[2y+\cos(xy)x] - [2x+\cos(xy)y]\cdot[2y+\cos(xy)x]'}{[2y+\cos(xy)x]^2}$$

$$= -\frac{2-\sin(xy)(y+xy')y+\cos(xy)y'}{2y+\cos(xy)x} + \frac{(2x+\cos(xy)y)(2y'-\sin(xy)(y+xy')x+\cos(xy))}{[2y+\cos(xy)x]^2}.$$

Then we put in $x=0$, $y=1$, $y'=-\frac{1}{2}$ and it is done.
Obviously the first solution is preferable.

11. The equation can be written as $F(x,y,z)=0$, where

1h

$$F(x,y,z)=\sin(xz)+\sin(yz)-\sin(xy).$$

a) For the function $z(x,y)$ to exist, we need $\frac{\partial F}{\partial z}(0,1,\pi)\neq 0$. Here $\frac{\partial F}{\partial z}=x\cos(xz)+y\cos(yz)$, so $\frac{\partial F}{\partial z}(0,1,\pi)=-1\neq 0$, and the Implicit Function Theorem does the rest.
b) The given equation defines a level surface of F, so the normal vector can be found using gradient of F. We have $\nabla F=\big(-y\cos(xy)+z\cos(yz),-x\cos(xy)+z\cos(yz),x\cos(xz)+y\cos(yz)\big)$, so $\vec{n}=\nabla F(0,1,\pi)=(\pi-1,-\pi,-1)$.
The tangent plane is given by $(\pi-1)x-\pi(y-1)-(z-\pi)=0$, that is,

$$(\pi-1)x-\pi y-z+2\pi=0.$$

Alternative solution: We will treat it as the question of finding the tangent plane to the graph of $z=z(x,y)$. The normal vector is then given as $(z_x,z_y,-1)$. To find the partial derivatives, we differentiate the given equation by x and by y, remembering that now $z=z(x,y)$:

$$\cos(xz)[z+xz_x]+\cos(yz)yz_x=\cos(xy)y$$
$$\cos(xz)xz_y+\cos(yz)[z+yz_y]=\cos(xy)x$$

We substitute in $(0,1,\pi)$ and get $z_x(0,1)=\pi-1$, $z_y(0,1)=-\pi$, thus $\vec{n}=(\pi-1,-\pi,-1)$ as we had before.
Note that we used the handy shortcut $z_x=\frac{\partial z}{\partial x}$, $z_y=\frac{\partial z}{\partial y}$ to simplify the writing.
c) To find the second partial derivative, it is easiest to use the alternative solution of b). There we differentiated the original equation by x, so we now differentiate that result by y to get the desired z_{xy}. Again, we have to remember that $z=z(x,y)$, but now also $z_x=z_x(x,y)$.

$$\big[\cos(xz)[z+xz_x]+\cos(yz)yz_x\big]_y=\big[\cos(xy)y\big]_y \implies$$
$$-\sin(xz)xz_y[z+xz_x]+\cos(xz)[z_y+xz_{xy}]-\sin(yz)[z+yz_y]yz_x+\cos(yz)z_x+\cos(yz)yz_{xy}$$
$$=-\sin(xy)xy+\cos(xy).$$

We substitute in the point $(0,1,\pi)$ and also $z_x(0,1)=\pi-1$, $z_y(0,1)=-\pi$ and solve the resulting equation to obtain $z_{xy}(0,1)=2-2\pi$.
Of course, we can also take the equation we obtained in b) by differentiating the given equation with respect to y and then differentiate it by x, obtaining the same answer.

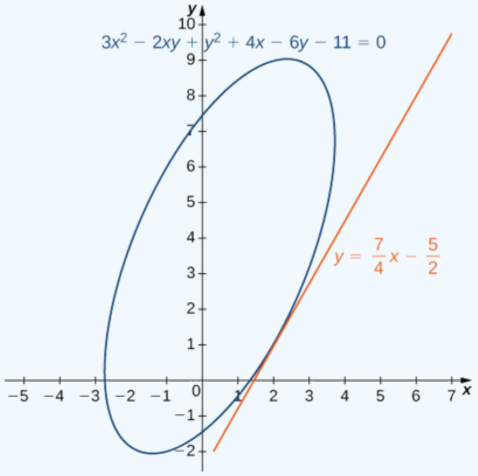


Figure 5: Graph of the rotated ellipse defined by $3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$.

b. We have $f(x, y, z) = x^2 e^y - yze^x$. Therefore,

$$\frac{\partial f}{\partial x} = 2xe^y - yze^x$$

$$\frac{\partial f}{\partial y} = x^2 e^y - ze^x$$

$$\frac{\partial f}{\partial z} = -ye^x$$

Using Equation 7,

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial y}\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}$$

$$= -\frac{2xe^y - yze^x}{-ye^x}\text{and} = -\frac{x^2e^y - ze^x}{-ye^x}$$

$$= \frac{2xe^y - yze^x}{ye^x} = \frac{x^2e^y - ze^x}{ye^x}$$

Exercise 5

Find dy/dx if y is defined implicitly as a function of x by the equation $x^2 + xy - y^2 + 7x - 3y - 26 = 0$. What is the equation of the tangent line to the graph of this curve at point $(3, -2)$?

Hint

Calculate $\partial f/dx$ and $\partial f/dy$, then use Equation 6.

Solution

$$\frac{dy}{dx} = \frac{2x + y + 7}{2y - x + 3}\bigg|_{(3,-2)} = \frac{2(3) + (-2) + 7}{2(-2) - (3) + 3} = -\frac{11}{4}$$

Equation of the tangent line: $y = -\frac{11}{4}x + \frac{25}{4}$

Key Concepts

- The chain rule for functions of more than one variable involves the partial derivatives with respect to all the independent variables.

2

$$\frac{d}{dx}(\sqrt{x}+\sqrt{y}) = \frac{d1}{dx}$$
$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$, we bring the terms with $\frac{dy}{dx}$ to the left and all other terms to the right:

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

multiplying both sides by $2\sqrt{y}$, we get

$$\frac{dy}{dx} = -\frac{2\sqrt{y}}{2\sqrt{x}} = -\frac{\sqrt{y}}{\sqrt{x}}.$$

To calculate $y'' = \frac{d}{dx}(\frac{dy}{dx})$, we have

$$y'' = -\frac{d}{dx}\left(\frac{\sqrt{y}}{\sqrt{x}}\right) = -\left[\frac{\sqrt{x}\frac{d(\sqrt{y})}{dx} - \sqrt{y}\frac{d(\sqrt{x})}{dx}}{(\sqrt{x})^2}\right] = -\left[\frac{\sqrt{x}[\frac{1}{2\sqrt{y}}\frac{dy}{dx}] - \sqrt{y}[\frac{1}{2\sqrt{x}}]}{x}\right]$$

From above, we know that $\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$. Substituting that into the expression for y'', we get

$$y'' = -\left[\frac{\sqrt{x}[\frac{1}{2\sqrt{y}}[-\frac{\sqrt{y}}{\sqrt{x}}]] - \sqrt{y}[\frac{1}{2\sqrt{x}}]}{x}\right].$$

After cancellation and factoring $-1/2$ out of each term, we get

$$y'' = \frac{\frac{1}{2}[1 - \frac{\sqrt{y}}{\sqrt{x}}]}{x}.$$

Example Find $\frac{dy}{dx}$ by implicit differentiation if $y\sin(x^2) = x\sin(y^2)$.
(Please attempt to solve this before looking at the solution on the next page)

Let's now return to the problem that we started before the previous theorem. Using Note and the function $f(x,y)=x^2+3y^2+4y-4$, we obtain

$$\frac{\partial f}{\partial x}=2x$$

$$\frac{\partial f}{\partial y}=6y+4.$$

Then Equation 6 gives

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{2x}{6y+4}=-\frac{x}{3y+2}, \tag{8}$$

which is the same result obtained by the earlier use of implicit differentiation.

③

Example 5: Implicit Differentiation by Partial Derivatives

a. Calculate dy/dx if y is defined implicitly as a function of x via the equation $3x^2-2xy+y^2+4x-6y-11=0$. What is the equation of the tangent line to the graph of this curve at point $(2,1)$?
b. Calculate $\partial z/\partial x$ and $\partial z/\partial y$, given $x^2e^y-yze^x=0$.

Solution

a. Set $f(x,y)=3x^2-2xy+y^2+4x-6y-11=0$, then calculate f_x and f_y: $f_x=6x-2y+4$ $f_y=-2x+2y-6$.

The derivative is given by

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=\frac{6x-2y+4}{-2x+2y-6}=\frac{3x-y+2}{x-y+3}.$$

The slope of the tangent line at point $(2,1)$ is given by

$$\frac{dy}{dx}\Big|_{(x,y)=(2,1)}=\frac{3(2)-1+2}{2-1+3}=\frac{7}{4}$$

To find the equation of the tangent line, we use the point-slope form (Figure 5):

$$y-y_0=m(x-x_0)$$

$$y-1=\frac{7}{4}(x-2)$$

$$y=\frac{7}{4}x-\frac{7}{2}+1$$

$$y=\frac{7}{4}x-\frac{5}{2}.$$

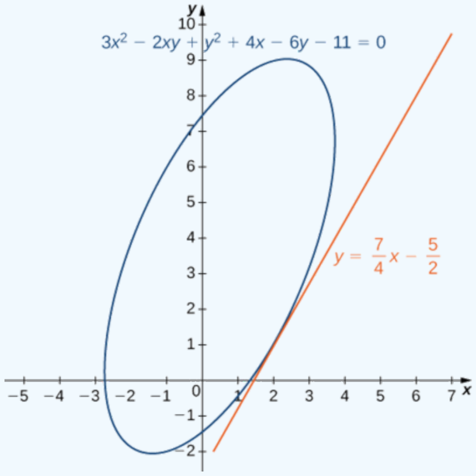


Figure 5: Graph of the rotated ellipse defined by $3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$.

b. We have $f(x, y, z) = x^2 e^y - yze^x$. Therefore,

$$\frac{\partial f}{\partial x} = 2xe^y - yze^x$$

$$\frac{\partial f}{\partial y} = x^2 e^y - ze^x$$

$$\frac{\partial f}{\partial z} = -ye^x$$

Using Equation 7,

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial y}\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}$$

$$= -\frac{2xe^y - yze^x}{-ye^x} \text{and} = -\frac{x^2 e^y - ze^x}{-ye^x}$$

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Exercise 5

Find dy/dx if y is defined implicitly as a function of x by the equation $x^2 + xy - y^2 + 7x - 3y - 26 = 0$. What is the equation of the tangent line to the graph of this curve at point $(3, -2)$?

Hint

Calculate $\partial f/dx$ and $\partial f/dy$, then use Equation 6.

Solution

$$\frac{dy}{dx} = \frac{2x + y + 7}{2y - x + 3}\bigg|_{(3,-2)} = \frac{2(3) + (-2) + 7}{2(-2) - (3) + 3} = -\frac{11}{4}$$

Equation of the tangent line: $y = -\frac{11}{4}x + \frac{25}{4}$

Key Concepts

- The chain rule for functions of more than one variable involves the partial derivatives with respect to all the independent variables.

Example 7 *A curve is given by the equation*

$$y^3 + 2y = \sin(x) + 3.$$

Find the slope of the curve at the point $(0, 1)$.
Also approximate the value of y *when* $x = 0.05$.
Solution.

In this example we use y' *as shorthand for* dy/dx.
Note that $(0, 1)$ *lies on the curve as we can see that* $x = 0, \quad y = 1$ *satisfies the equation.*
Differentiating both sides of the equation with respect to x *gives:*

$$\begin{aligned} 3y^2y' + 2y' &= \cos(x) \Rightarrow \\ (3y^2 + 2)y' &= \cos(x) \Rightarrow \\ y' &= \frac{\cos(x)}{3y^2 + 2}. \end{aligned}$$

At $x = 0, \quad y = 1$:

$$y' = \frac{1}{3 + 2} = \frac{1}{5}.$$

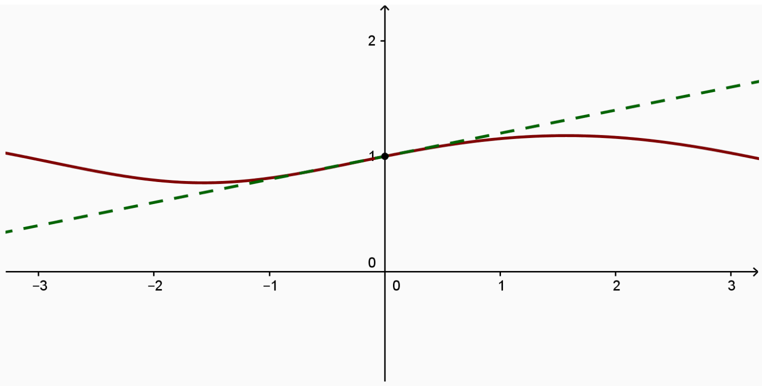
This is the slope of the curve at $(0, 1)$.
To approximate the value of y *at* $x = 0.05$ *we use the equation of the tangent at* $(0, 1)$ *which is of the form*

$$y = \frac{1}{5}x + c$$

and since $y = 1$ *when* $x = 0$ *we have* $c = 1$. *Hence the equation of the tangent is*

$$y = \frac{1}{5}x + 1.$$

The approximation for y *at* $x = 0.05$ *is then* $y = 0.05/5 + 1 = 1.01$.
The true value is $y = 1.009936398$ *and the error is* 0.000064 *to 6 decimal places.*
The graph together with the tangent at $(0, 1)$ *is Figure 2.*



Graph of $y^3 + 2y = \sin(x) + 3$ *with tangent at* $(0, 1)$.

Exercise 2

A curve is given by the equation

$$y^3 + y^2 = x^3 + 1.$$

Find the slope of the curve at the point $(-1, -1)$.
Also approximate the value of y when $x = -0.95$.

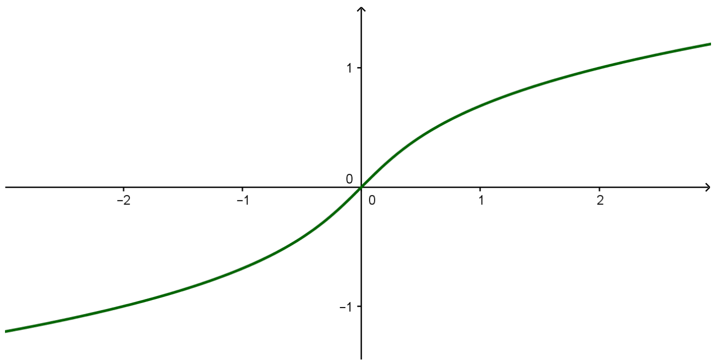
Solutions to exercise 2

Once again we use y' as shorthand for dy/dx.

Note that $(-1, -1)$ lies on the curve as we can see that $x = -1, \;\; y = -1$ satisfies the equation.
Differentiating both sides of the equation with respect to x gives:

$$\begin{aligned} 3y^2y' + 2yy' &= 3x^2 \Rightarrow \\ (3y^2 + 2y)y' &= 3x^2 \Rightarrow \\ y' &= \frac{3x^2}{3y^2 + 2y}. \end{aligned}$$

Note it is not expected that you should be able to plot these graphs - they are displayed here for information.



Graph of $y^3 + y = x$.

We can now ask for the slope of the tangent to this curve at a particular point by finding dy/dx.

1a

Example 5 *A curve is given by*

$$y^3 + y = x.$$

(a) Show that the point $(-2, -1)$ *lies on the curve.*

(b) Find dy/dx *by using implicit differentiation.*

(c) Find the slope of the tangent at the point $(-2, -1)$.

Solution.

(a) To show that the point $(-2, -1)$ *lies on the curve all we do is to substitute these values into the equation and see if they satisfy the equation.*
We get on putting $x = -2,\ y = -1:$

$$(-1)^3 + (-1) = -2$$

which is clearly true, hence the point lies on the curve.