18th lesson

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Theory

Theorem 1 (Lagrange multiplier theorem)**.** Let $G \subset \mathbb{R}^2$ be an open set, $f, g \in C^1(G)$, $M = \{[x, y] \in G;\ g(x, y) = 0\}$ and let $[\tilde{x}, \tilde{y}] \in M$ be a point of local extremum of f with respect to M. Then at least one of the following conditions holds:

1. $\operatorname{grad} g(\tilde{x}, \tilde{y}) = \vec{o}$,
2. there exists $\lambda \in \mathbb{R}$ satisfying

$$\frac{\partial f}{\partial x}(\tilde{x}, \tilde{y}) + \lambda \frac{\partial g}{\partial x}(\tilde{x}, \tilde{y}) = 0,$$
$$\frac{\partial f}{\partial y}(\tilde{x}, \tilde{y}) + \lambda \frac{\partial g}{\partial y}(\tilde{x}, \tilde{y}) = 0.$$

Theorem 2 (Lagrange multipliers theorem)**.** Let $m, n \in \mathbb{N}$, $m < n$, $G \subset \mathbb{R}^n$ an open set, $f, g_1, \dots, g_m \in C^1(G)$,

$$M = \{\vec{z} \in G;\ g_1(\vec{z}) = 0, g_2(\vec{z}) = 0, \dots, g_m(\vec{z}) = 0\}$$

and let $\vec{\tilde{z}} \in M$ be a point of local extremum of f with respect to the set M. Then at least one of the following conditions holds:

1. the vectors
$$\operatorname{grad} g_1(\vec{\tilde{z}}), \operatorname{grad} g_2(\vec{\tilde{z}}), \dots, \operatorname{grad} g_m(\vec{\tilde{z}})$$
are linearly dependent,
2. there exist numbers $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$ satisfying
$$\operatorname{grad} f(\vec{\tilde{z}}) + \lambda_1 \operatorname{grad} g_1(\vec{\tilde{z}}) + \lambda_2 \operatorname{grad} g_2(\vec{\tilde{z}}) + \dots + \lambda_m \operatorname{grad} g_m(\vec{\tilde{z}}) = \vec{o}.$$

Remarks 3.
- For $m = 1$: One vector is linearly dependent if it is the zero vector.
- For $m = 2$: Two vectors are linearly dependent if one of them is a multiple of the other one.

Exercises

1. Find suspect points of a function subject to the given constraint
   (a) $f(x,y) = 2x + 3y,\ x^2 + 4y^2 = 100$
   (b) $f(x,y) = x^2 + 4y^2 - 2x + 8y,\ x + 2y = 7$ (find local minimum)
   (c) $f(x,y) = x^2 - 10x - y^2,\ x^2 + 4y^2 = 16$
   (d) $f(x,y) = x^2 + 2y^2,\ x^2 - 2x + 2y^2 + 4y = 0$
2. Find suspect points of a function subject to the given constraint
   (a) $f(x,y,z) = x^2 + y^2 + z^2,\ x + y - 2z - 6 = 0$
   (b) $f(x,y,z) = x + 2y + 2z,\ x^2 + y^2 + z^2 = 9$
   (c) $f(x,y,z) = y^2 - 10z,\ x^2 + y^2 + z^2 = 36$
   (d) $f(x,y,z) = x^2 + y^2 + z^2,\ x^4 + y^4 + z^4 = 1$
3. Find suspect points of a function subject to the given constraints
   (a) $f(x,y,z) = x^2 + 2y^2 + z^2,\ x + 2y + 3z = 1,\ x - 2y + z = 5$
   (b) $f(x,y,z) = 4y - 2z,\ x^2 + y^2 = 1,\ 2x - y - z = 2$
   (c) $f(x,y,z) = 3x - y - 3z,\ x + y - z = 0,\ x^2 + 2z^2 = 1$
   (d) $f(x,y,z) = 3x^2 + y,\ x^2 + z^2 = 9,\ 4x - 3y = 9$
4. Find a point, that is closest to $P = (0, -3, 2)$ and lies on the plane $x + y - z = 1$.
5. Find the rectangle (sides are parallel to the coordinates) which is inscribed into ellipse $x^2 + 2y^2 = 1$ and has the possible greatest area.

6. The farmer has 100 m of fencing and wants to make a place for sheep next to the river - it means, that he has to fence only three sides of the rectangular place. Of course, he wants to have maximum dimension for the sheep.

   How to use the Lagrange multipliers?

   (a) $f(x, y) = xy$, $g(x, y) = 2x + y - 100$
   (b) $f(x, y) = 2x + 2y - 100$, $g(x, y) = xy$
   (c) $f(x, y) = xy$, $g(x, y) = x + y - 100$
   (d) $f(x, y) = x + y$, $g(x, y) = xy - 100$

   (Inspiration: `https://www.cpp.edu/conceptests/question-library/mat214.shtml#Partial%20Derivatives`)

Source 1: https://www.cbr.com/shaun-the-sheep-best-worst-episodes-imdb/