- Since we are interested in positive r, we can do a little bit more checking to conclude that the can's volume is indeed maximized at the critical point, so the radius is $r = 5\,\text{cm}$, the height is $h = 10\,\text{cm}$, and the resulting volume is $V = 250\pi\text{cm}^3$.

- Using the technique of Lagrange multipliers, however, we can perform a constrained optimization without having to solve the constraint equations. This technique is especially useful when the constraints are difficult or impossible to solve explicitly.

- Method (Lagrange multipliers, 1 constraint): To find the extreme values of $f(x,y,z)$ subject to a constraint $g(x,y,z) = c$, define the Lagrange function $L(x,y,z,\lambda) = f(x,y,z) - \lambda \cdot [g(x,y,z) - c]$. Then any extreme value of $f(x,y,z)$ subject to the constraint $g(x,y,z) = c$ must occur at a critical point of $L(x,y,z,\lambda)$. In other words, it is sufficient to solve the system of four variables x,y,z,λ given by $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$, $g(x,y,z) = c$, and then search among the resulting triples (x,y,z) to find the minimum and maximum.
  - If we have two variables, we would instead solve the system $f_x = \lambda g_x$, $f_y = \lambda g_y$, $g(x,y) = c$.
  - Remark: The value λ is called a Lagrange multiplier.
  - Here is the intuitive idea behind the method:
    * Imagine we are walking around the level set $g(x,y,z) = c$, and consider what the contours of $f(x,y,z)$ are doing as we move around.
    * In general the contours of f and g will be different, and they will cross one another.
    * But if we are at a point where f is maximized, then if we walk around nearby that maximum, we will see only contours of f with a smaller value than the maximum.
    * Thus, at that maximum, the contour $g(x,y,z) = c$ is tangent to the contour of f.
    * This information can, in turn, be reinterpreted as saying that the vector $\nabla f = \langle f_x, f_y, f_z \rangle$ is parallel to the vector $\nabla g = \langle g_x, g_y, g_z \rangle$, or in other words, there exists a scalar λ for which $\nabla f = \lambda \nabla g$. This yields the explicit conditions $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$ given above.

- For completeness we also mention that there is an analogous procedure for a problem with two constraints:

- Method (Lagrange Multipliers, 2 constraints): To find the extreme values of $f(x,y,z)$ subject to a pair of constraints $g(x,y,z) = c$ and $h(x,y,z) = d$, define the Lagrange function $L(x,y,z,\lambda,\mu) = f(x,y,z) - \lambda \cdot [g(x,y,z) - c] - \mu \cdot [h(x,y,z) - d]$. Then any extreme value of $f(x,y,z)$ subject to the constraint constraints $g(x,y,z) = c$ and $h(x,y,z) = d$ must occur at a critical point of $L(x,y,z,\lambda,\mu)$.
  - The method also works with more than three variables, and has a natural generalization to more than two constraints. (It is fairly rare to encounter systems with more than two constraints.)

- Example: Find the maximum and minimum values of $f(x,y) = 2x+3y$ subject to the constraint $x^2+4y^2 = 100$.
  - We have $g = x^2 + 4y^2$, and we compute $f_x = 2$, $g_x = 2x$, $f_y = 3$, and $g_y = 8y$.
  - Thus we have the system $2 = 2x\lambda$, $3 = 8y\lambda$, and $x^2 + 4y^2 = 100$.
  - Solving the first two equations gives $x = \dfrac{1}{\lambda}$ and $y = \dfrac{3}{8\lambda}$. Then plugging in to the third equation yields $\left(\dfrac{1}{\lambda}\right)^2 + 4\left(\dfrac{3}{8\lambda}\right)^2 = 100$, so that $\dfrac{1}{\lambda^2} + \dfrac{9}{16\lambda^2} = 100$. Multiplying both sides by $16\lambda^2$ yields $25 = 100(16\lambda^2)$, so that $\lambda^2 = \dfrac{1}{64}$, hence $\lambda = \pm\dfrac{1}{8}$.
  - Thus, we obtain the two points $(x,y) = (8,3)$ and $(-8,-3)$.
  - Since $f(8,3) = 25$ and $f(-8,-3) = -25$, the maximum is $\boxed{f(8,3) = 25}$ and the minimum is $\boxed{f(-8,-3) = -25}$.

- Example: Find the maximum and minimum values of $f(x,y,z) = x + 2y + 2z$ subject to the constraint $x^2 + y^2 + z^2 = 9$.
  - We have $g = x^2 + y^2 + z^2$, and also $f_x = 1$, $g_x = 2x$, $f_y = 2$, $g_y = 2y$, $f_z = 2$, $g_z = 2z$.
  - Thus we have the system $1 = 2x\lambda$, $2 = 2y\lambda$, $2 = 2z\lambda$, and $x^2 + y^2 + z^2 = 9$.

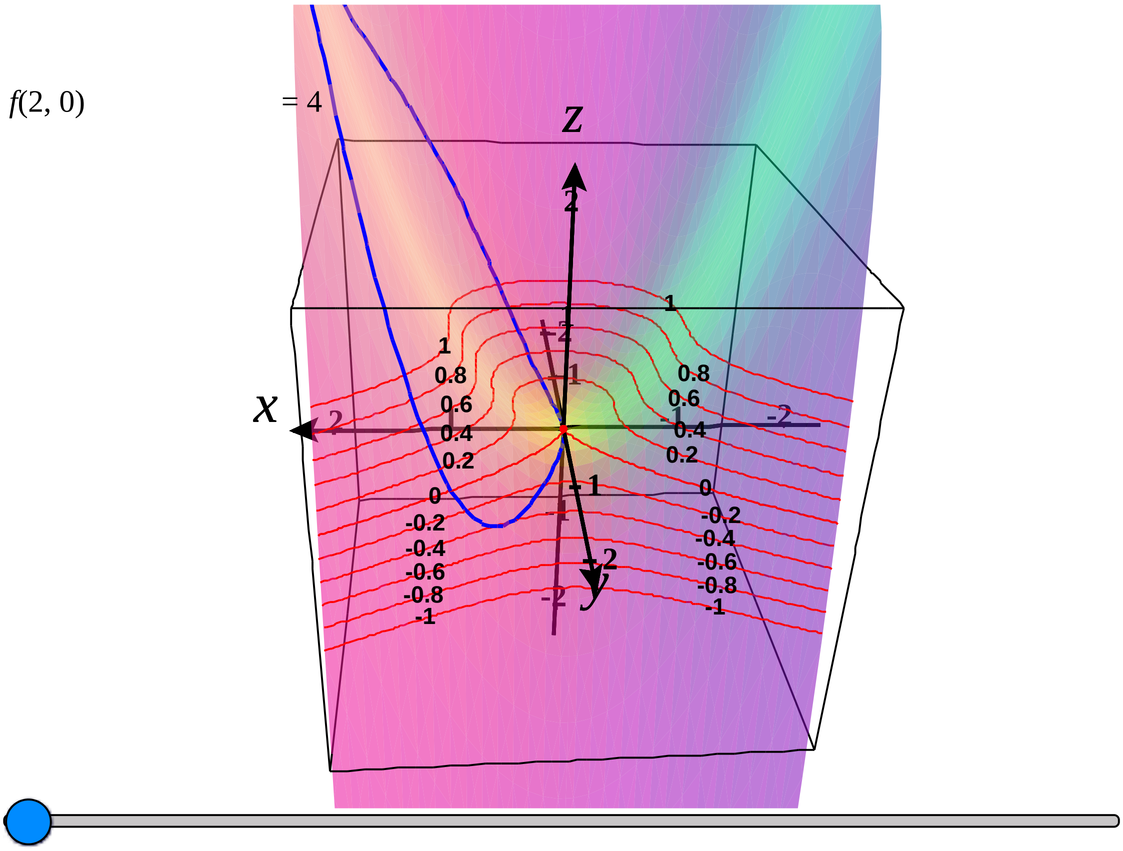


Figure **13.10.3***: Graph of* $f(x,y)=x^2-y^3$ *along with the constraint* $(x-1)^2+y^2=1$ *. Note that there is no relative extremum at* $(0,0)$*, although this point will satisfy the Lagrange Multiplier equation with* $\lambda=0$.

Example 13.10.1: Using Lagrange Multipliers

Use the method of Lagrange multipliers to find the minimum value of $f(x,y)=x^2+4y^2-2x+8y$ subject to the constraint $x+2y=7$.

Solution

Let's follow the problem-solving strategy:

1. The objective function is $f(x,y)=x^2+4y^2-2x+8y$. The constraint function is equal to the left-hand side of the constraint equation when only a constant is on the right-hand side. So here $g(x,y)=x+2y$. The problem asks us to solve for the minimum value of f, subject to the constraint (Figure 13.10.4).

$\nabla g = (1,2) \neq (0,0)$

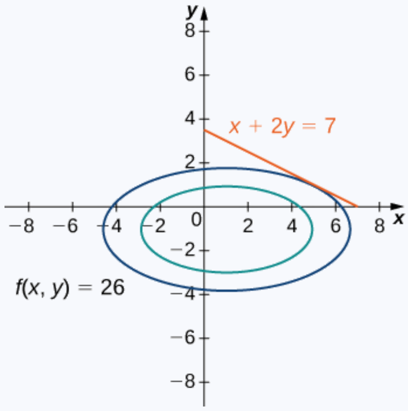


Figure 13.10.4: Graph of level curves of the function $f(x,y) = x^2 + 4y^2 - 2x + 8y$ corresponding to $c = 10$ and 26. The red graph is the constraint function.

2. We then must calculate the gradients of both f and g:

$$\vec{\nabla} f(x,y) = (2x-2)\,\hat{\mathbf{i}} + (8y+8)\,\hat{\mathbf{j}} \tag{13.10.4}$$
$$\vec{\nabla} g(x,y) = \hat{\mathbf{i}} + 2\hat{\mathbf{j}}.$$

The equation $\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$ becomes

$$(2x-2)\,\hat{\mathbf{i}} + (8y+8)\,\hat{\mathbf{j}} = \lambda\left(\hat{\mathbf{i}} + 2\hat{\mathbf{j}}\right), \tag{13.10.5}$$

which can be rewritten as

$$(2x-2)\,\hat{\mathbf{i}} + (8y+8)\,\hat{\mathbf{j}} = \lambda\hat{\mathbf{i}} + 2\lambda\hat{\mathbf{j}}. \tag{13.10.6}$$

Next, we set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$2x - 2 = \lambda \tag{13.10.7}$$
$$8y + 8 = 2\lambda. \tag{13.10.8}$$

The equation $g(x,y) = k$ becomes $x + 2y = 7$. Therefore, the system of equations that needs to be solved is

$$2x - 2 = \lambda \tag{13.10.9}$$
$$8y + 8 = 2\lambda \tag{13.10.10}$$
$$x + 2y = 7. \tag{13.10.11}$$

3. This is a linear system of three equations in three variables. We start by solving the second equation for λ and substituting it into the first equation. This gives $\lambda = 4y + 4$, so substituting this into the first equation gives

$$2x - 2 = 4y + 4.$$

Solving this equation for x gives $x = 2y + 3$. We then substitute this into the third equation:

$$\begin{aligned}(2y+3) + 2y &= 7\\ 4y &= 4\\ y &= 1.\end{aligned}$$

Since $x = 2y + 3$, this gives $x = 5$.

4. Next, we evaluate $f(x,y) = x^2 + 4y^2 - 2x + 8y$ at the point $(5,1)$,

$$f(5,1) = 5^2 + 4(1)^2 - 2(5) + 8(1) = 27. \tag{13.10.12}$$

To ensure this corresponds to a minimum value on the constraint function, let's try some other points on the constraint from either side of the point $(5,1)$, such as the intercepts of $g(x,y) = 0$, Which are $(7,0)$ and $(0,3.5)$.

We get $f(7,0)=35>27$ and $f(0,3.5)=77>27$.

So it appears that f has a relative minimum of 27 at $(5,1)$, subject to the given constraint.

Exercise 13.10.1

Use the method of Lagrange multipliers to find the maximum value of

$$f(x,y)=9x^2+36xy-4y^2-18x-8y$$

subject to the constraint $3x+4y=32$.

Let's now return to the problem posed at the beginning of the section.

Example 13.10.2: Golf Balls and Lagrange Multipliers

The golf ball manufacturer, Pro-T, has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising y, according to the function

$$z=f(x,y)=48x+96y-x^2-2xy-9y^2,$$

where z is measured in thousands of dollars. The budgetary constraint function relating the cost of the production of thousands golf balls and advertising units is given by $20x+4y=216$. Find the values of x and y that maximize profit, and find the maximum profit.

Solution:

Again, we follow the problem-solving strategy:

1. The objective function is $f(x,y)=48x+96y-x^2-2xy-9y^2$. To determine the constraint function, we divide both sides by 4, which gives $5x+y=54$. The constraint function is equal to the left-hand side, so $g(x,y)=5x+y$. The problem asks us to solve for the maximum value of f, subject to this constraint.
2. So, we calculate the gradients of both f and g:

$$\begin{aligned}\vec{\nabla} f(x,y) &=(48-2x-2y)\hat{\mathbf{i}}+(96-2x-18y)\hat{\mathbf{j}}\\ \vec{\nabla} g(x,y) &=5\hat{\mathbf{i}}+\hat{\mathbf{j}}.\end{aligned}$$

   The equation $\vec{\nabla} f(x,y)=\lambda\vec{\nabla} g(x,y)$ becomes

$$(48-2x-2y)\hat{i}+(96-2x-18y)\hat{\mathbf{j}}=\lambda(5\hat{\mathbf{i}}+\hat{\mathbf{j}}),$$

   which can be rewritten as

$$(48-2x-2y)\hat{\mathbf{i}}+(96-2x-18y)\hat{\mathbf{j}}=\lambda 5\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}.$$

   We then set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$\begin{aligned}48-2x-2y &=5\lambda\\ 96-2x-18y &=\lambda.\end{aligned}$$

   The equation $g(x,y)=k$ becomes $5x+y=54$. Therefore, the system of equations that needs to be solved is

$$\begin{aligned}48-2x-2y &=5\lambda\\ 96-2x-18y &=\lambda\\ 5x+y &=54.\end{aligned}$$

3. We use the left-hand side of the second equation to replace λ in the first equation:

to state the method of Lagrange multipliers using a new piece of notation. The *gradient* of a function of two variables $f(x, y)$ is the (two component) vector

$$\nabla f(x, y) = \langle f_x(x, y)\,,\ f_y(x, y)\rangle$$

Theorem 20 (Lagrange Multipliers).

Let $f(x, y)$ and $g(x, y)$ have continuous first partial derivatives in a region of $\mathbb{R}^2$ that contains the curve C given by the equation $g(x, y) = 0$. Assume that $\nabla g(x, y) \neq \mathbf{0}$ there. If f, restricted to the curve C, has a local extreme value at the point (a, b) on C, then there is a real number λ (called a *Lagrange multiplier*) such that

$$\nabla f(a, b) = \lambda \nabla g(a, b) \qquad \text{i.e.} \qquad f_x(a, b) = \lambda g_x(a, b) \quad f_y(a, b) = \lambda g_y(a, b)$$

So to find the maximum and minimum values of $f(x, y)$ on a curve $g(x, y) = 0$, assuming that both the objective function $f(x, y)$ and constraint function $g(x, y)$ have continuous first partial derivatives and that $\nabla g(x, y) \neq \mathbf{0}$, you

1. build up a list of candidate points (x, y) by finding all solutions to the equations

$$f_x(x, y) = \lambda g_x(x, y) \qquad f_y(x, y) = \lambda g_y(x, y) \qquad g(x, y) = 0$$

2. and then you evaluate $f(x, y)$ at each (x, y) on the list of candidates. The biggest of these candidate values is the absolute maximum and the smallest of these candidate values is the absolute minimum.

Example 21

1c

Find the maximum and minimum of $x^2 - 10x - y^2$ on the ellipse $x^2 + 4y^2 = 16$.

Solution. For this problem the objective function is $f(x, y) = x^2 - 10x - y^2$ and the constraint function is $g(x, y) = x^2 + 4y^2 - 16$. The first order derivatives of these functions are

$$f_x = 2x - 10 \qquad f_y = -2y \qquad g_x = 2x \qquad g_y = 8y$$

So, according to the method of Lagrange multipliers, we need to find all solutions to

$$2x - 10 = \lambda(2x) \iff (\lambda - 1)x = -5 \tag{7a}$$
$$-2y = \lambda(8y) \iff (4\lambda + 1)y = 0 \tag{7b}$$
$$0 = x^2 + 4y^2 - 16 \tag{7c}$$

From (7b), we see that we must have either $\lambda = -1/4$ or $y = 0$.

- If $\lambda = -1/4$, (7a) gives $-\frac{5}{4}x = -5$, i.e. $x = 4$, and then (7c) gives $y = 0$.

$\nabla g = (2x, 8y) \to (0,0) \notin M$

- If $y = 0$, then (7c) gives $x = \pm 4$.

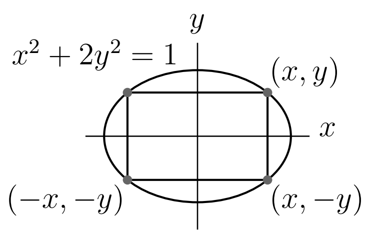
So we have the following table of candidates.

point	$(4, 0)$	$(-4, 0)$
value of f	-24	56
	min	max

Example 21

Example 22

Find the rectangle of largest area (with sides parallel to the coordinates axes) that can be inscribed in the ellipse $x^2 + 2y^2 = 1$.



Solution. Call the coordinates of the upper right corner of the rectangle (x, y), as in the figure above. The four corners of the rectangle are $(\pm x, \pm y)$ so the rectangle has width $2x$ and height $2y$ and the objective function is $f(x, y) = 4xy$. The constraint function for this problem is $g(x, y) = x^2 + 2y^2 - 1$. The first order derivatives of these functions are

$$f_x = 4y \qquad f_y = 4x \qquad g_x = 2x \qquad g_y = 4y$$

So, according to the method of Lagrange multipliers, we need to find all solutions to

$$4y = \lambda(2x) \qquad \iff \qquad y = \frac{1}{2}\lambda x \tag{8a}$$

$$4x = \lambda(4y) \qquad \implies \qquad x = \lambda y = \frac{1}{2}\lambda^2 x \qquad \implies \qquad x\left(1 - \frac{\lambda^2}{2}\right) = 0 \tag{8b}$$

$$0 = x^2 + 2y^2 - 1 \tag{8c}$$

So (8b) is satisfied if either $x = 0$ or $\lambda = \sqrt{2}$ or $\lambda = -\sqrt{2}$.

- If $x = 0$, then (8a) gives $y = 0$ too. But $(0, 0)$ violates the constraint.
- If $\lambda = \sqrt{2}$, then (8a) gives $x = \sqrt{2}y$ and then (8c) gives $2y^2 + 2y^2 = 1$ so that $y = \pm 1/2$ and $x = \pm 1/\sqrt{2}$.
- If $\lambda = -\sqrt{2}$, then (8a) gives $x = -\sqrt{2}y$ and then (8c) gives $2y^2 + 2y^2 = 1$ so that $y = \pm 1/2$ and $x = \mp 1/\sqrt{2}$.

For $(2,1,-2)$ we get

$$H = \begin{pmatrix} 8 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \implies \Delta_1 = 8,\ \Delta_2 = \begin{vmatrix} 8 & -2 \\ -2 & 2 \end{vmatrix} = 12,\ \Delta_3 = |H| = 28.$$

Since always $\Delta_i > 0$, we conclude that $f(2,1,-2) = -7$ is a local minimum.
Recall that for a local maximum we need $\Delta_1 < 0$, $\Delta_2 > 0$, and $\Delta_3 < 0$.

1d $\nabla g = (2x-2, 4y+4) \to (1,-1) \notin M$

3. Since expressing y from the constraint would be messy, this calls for Lagrange multipliers with $g(x,y) = x^2 - 2x + 2y^2 + 4y$. Equations to solve are $\nabla f = \lambda \nabla g$ and $g = 0$, that is,

$$\left.\begin{array}{l} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ g = 0 \end{array}\right\} \implies \left.\begin{array}{l} 2x = \lambda(2x-2) \\ 4y = \lambda(4y+4) \\ x^2 - 2x + 2y^2 + 4y = 0 \end{array}\right\} \implies \begin{array}{l} x = \lambda(x-1) \\ y = \lambda(y+1) \\ x^2 - 2x + 2y^2 + 4y = 0 \end{array}$$

A typical strategy is to eliminate λ from the first two equations in order to obtain some relationship between the variables x, y, this is then used with condition $g = 0$ to find the desired points.
We would like to isolate λ from the first equation. Can we have $x = 1$? The first equation then reads $1 = 0$, which is not true. Thus for sure $x \neq 1$ and we can write $\lambda = \frac{x}{x-1}$. Putting it into the second equation and multiplying out we get $y = -x$. Now this can be put into the constraint, we obtain $3x^2 - 6x = 0$ and two solutions, $x = 0$ and $x = 2$. Thus there are two suspicious points: $(0,0)$ and $(2,-2)$. We substitute them into f: $f(0,0) = 0$, $f(2,-2) = 12$. Comparing values we guess that the former is a local minimum and the latter is a local maximum.
Determining global extrema usually involves some analysis of the situation. We have two local extrema, but we do not know whether they give global extrema. In general, we find global extrema by comparing values at local extrema and also values at "borders" of the set. Thus we need to know more about M, the set determined by the given condition where we look at f.
A frequent trouble arises when the given set is not bounded, since then we have to ask what happens to f when points of M run away to some infinity. Could it happen that x tends to infinity within this set? Since points from M satisfy $2y^2 + 4y = 2x - x^2$, this would force the expression $2y^2 + 4y$ to tend to minus infinity, but that is not possible. Similarly we argue that also y cannot go to infinity and we thus have a bounded set M.
Another source of trouble is if the set M is a curve that has some endpoints, then we would have to check on those. How does M actually look like? In fact, rewriting the condition as

$$(x-1)^2 + 2(y+1)^2 = 3$$

we see that M is an ellipse. This is a close curve without any end, so whatever important happens to values of f on it, it must happen at one of the points we found earlier. Thus we can conclude that $f(0,0) = 0$ is a minimum and $f(2,-2) = 12$ is a maximum of f on the given set.

4. The unknown point $Q = (x,y,z)$ satisfies $x + y - z = 1$, that would be the constraint with $g(x,y,z) = x + y - z$. The function to minimize should be the distance between P and Q, but that would mean a square root. It will be easier to minimize the distance squared, which is equivalent (think about it). Thus we have $f(x,y,z) = \text{dist}(P,Q)^2 = x^2 + (y+3)^2 + (z-2)^2$. We use Lagrange multipliers, the equations $\nabla f = \lambda \nabla g$ and $g = 1$ now give

$$\left.\begin{array}{l} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ \frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z} \\ g = 1 \end{array}\right\} \implies \left.\begin{array}{l} 2x = \lambda \cdot 1 \\ 2(y+3) = \lambda \cdot 1 \\ 2(z-2) = \lambda \cdot (-1) \\ x + y - z = 1 \end{array}\right\} \implies \begin{array}{l} x = \frac{1}{2}\lambda \\ y + 3 = \frac{1}{2}\lambda \\ z - 2 = -\frac{1}{2}\lambda \\ x + y - z = 1 \end{array}$$

The answer is that the method of Lagrange multipliers is a general method that is effective in solving a wide variety of problems. It may not always be possible to express one variable in terms of the others (recall our discussion on implicit functions). Furthermore, the method of Lagrangians is very useful in more general or abstract problems involving an arbitrary number of independent variables and/or constraints. For example, in a future course or courses in Physics (e.g., thermal physics, statistical mechanics), you should see a derivation of the famous "Boltzman distribution" of the energies of atoms in an ideal gas using Lagrange multipliers.

2a

Example: Let us return to the optimization problem with constraints discusssed earlier: Find the point P on the plane $x + y - 2z = 6$ that lies closest to the origin. Recall that we sought to minimize the square of the distance:

$$\text{Minimize} \quad f(x, y, z) = x^2 + y^2 + z^2$$
$$\text{subject to} \quad x + y - 2z - 6 = 0.$$

$\nabla g = (1, 1, -2) \neq (0,0,0)$

Solution: The Lagrangian function associated with this problem is

$$\begin{aligned} L(x, y, z, \lambda) &= f(x, y, z) + \lambda F(x, y, z) \\ &= x^2 + y^2 + z^2 + \lambda(x + y - 2z - 6). \end{aligned} \tag{11}$$

We must find the critical points of L in terms of the *four* variables x, y, z and λ:

$$\begin{aligned} \frac{\partial L}{\partial x} &= 2x + \lambda = 0 \\ \frac{\partial L}{\partial y} &= 2y + \lambda = 0 \\ \frac{\partial L}{\partial z} &= 2z - 2\lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= x + y - 2z - 6 = 0. \end{aligned} \tag{12}$$

Note that the final equation simply correponds to the constraint applied to the problem. Clever, eh? There are often several ways to solve problems involving Lagrangians and Lagrangian multipliers. The most important point to remember is that *one method does not often work for all problems.* In this case, we can find the critical point rather easily as follows. We use the first three equations to express x, y and z in terms of λ:

$$x = -\frac{\lambda}{2}, \quad y = -\frac{\lambda}{2}, \quad z = \lambda. \tag{13}$$

We now substitute these results into the fourth equation:

$$-\frac{\lambda}{2} - \frac{\lambda}{2} - 2\lambda - 6 = 0 \quad => \quad 3\lambda = -6, \tag{14}$$

which implies that $\lambda = -2$. From the above three equations, we have determined x, y and z:

$$x = 1, \quad y = 1, \quad z = -2. \tag{15}$$

Therefore the desired point is $(1, 1, -2)$, which is in agreement with the result obtained in the previous lecture.

We continue with another illustrative example.

Example:

$$\text{Maximize/minimize} \quad f(x, y, z) = xyz \quad \text{on the ellipse} \quad x^2 + 2y^2 + 3z^2 = 1. \tag{16}$$

The ellipse represents the constraint in this problem. We first express this constraint in the form $F(x, y, z) = 0$, i.e.,

$$F(x, y, z) = x^2 + 2y^2 + 3z^2 = 1 = 0. \tag{17}$$

The Lagrangian associated with this problem is then

$$L(x, y, z, \lambda) = xyz + \lambda(x^2 + 2y^2 + 3z^2 - 1). \tag{18}$$

The critical points of the Lagrangian must satisfy the following equations

$$\begin{aligned} \frac{\partial L}{\partial x} &= yz + 2\lambda x = 0 \qquad \text{(a)} \\ \frac{\partial L}{\partial y} &= xz + 4\lambda y = 0 \qquad \text{(b)} \\ \frac{\partial L}{\partial z} &= xy + 6\lambda z = 0 \qquad \text{(c)} \end{aligned} \tag{19}$$

The final condition $\frac{\partial L}{\partial \lambda} = 0$ yields the constraint.

Once again, we're faced with the problem of solving this system of equations, which is now *nonlinear*. Here is a "trick" that works because of the symmetry of the problem. (It won't always

- Since we are interested in positive r, we can do a little bit more checking to conclude that the can's volume is indeed maximized at the critical point, so the radius is $r = 5\,\text{cm}$, the height is $h = 10\,\text{cm}$, and the resulting volume is $V = 250\pi\text{cm}^3$.

- Using the technique of Lagrange multipliers, however, we can perform a constrained optimization without having to solve the constraint equations. This technique is especially useful when the constraints are difficult or impossible to solve explicitly.

- Method (Lagrange multipliers, 1 constraint): To find the extreme values of $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, define the Lagrange function $L(x, y, z, \lambda) = f(x, y, z) - \lambda \cdot [g(x, y, z) - c]$. Then any extreme value of $f(x, y, z)$ subject to the constraint $g(x, y, z) = c$ must occur at a critical point of $L(x, y, z, \lambda)$. In other words, it is sufficient to solve the system of four variables x, y, z, λ given by $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$, $g(x, y, z) = c$, and then search among the resulting triples (x, y, z) to find the minimum and maximum.
  - If we have two variables, we would instead solve the system $f_x = \lambda g_x$, $f_y = \lambda g_y$, $g(x, y) = c$.
  - Remark: The value λ is called a Lagrange multiplier.
  - Here is the intuitive idea behind the method:
    - Imagine we are walking around the level set $g(x, y, z) = c$, and consider what the contours of $f(x, y, z)$ are doing as we move around.
    - In general the contours of f and g will be different, and they will cross one another.
    - But if we are at a point where f is maximized, then if we walk around nearby that maximum, we will see only contours of f with a smaller value than the maximum.
    - Thus, at that maximum, the contour $g(x, y, z) = c$ is tangent to the contour of f.
    - This information can, in turn, be reinterpreted as saying that the vector $\nabla f = \langle f_x, f_y, f_z \rangle$ is parallel to the vector $\nabla g = \langle g_x, g_y, g_z \rangle$, or in other words, there exists a scalar λ for which $\nabla f = \lambda \nabla g$. This yields the explicit conditions $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$ given above.

- For completeness we also mention that there is an analogous procedure for a problem with two constraints:

- Method (Lagrange Multipliers, 2 constraints): To find the extreme values of $f(x, y, z)$ subject to a pair of constraints $g(x, y, z) = c$ and $h(x, y, z) = d$, define the Lagrange function $L(x, y, z, \lambda, \mu) = f(x, y, z) - \lambda \cdot [g(x, y, z) - c] - \mu \cdot [h(x, y, z) - d]$. Then any extreme value of $f(x, y, z)$ subject to the constraint constraints $g(x, y, z) = c$ and $h(x, y, z) = d$ must occur at a critical point of $L(x, y, z, \lambda, \mu)$.
  - The method also works with more than three variables, and has a natural generalization to more than two constraints. (It is fairly rare to encounter systems with more than two constraints.)

- Example: Find the maximum and minimum values of $f(x, y) = 2x + 3y$ subject to the constraint $x^2 + 4y^2 = 100$.
  - We have $g = x^2 + 4y^2$, and we compute $f_x = 2$, $g_x = 2x$, $f_y = 3$, and $g_y = 8y$.
  - Thus we have the system $2 = 2x\lambda$, $3 = 8y\lambda$, and $x^2 + 4y^2 = 100$.
  - Solving the first two equations gives $x = \dfrac{1}{\lambda}$ and $y = \dfrac{3}{8\lambda}$. Then plugging in to the third equation yields $\left(\dfrac{1}{\lambda}\right)^2 + 4\left(\dfrac{3}{8\lambda}\right)^2 = 100$, so that $\dfrac{1}{\lambda^2} + \dfrac{9}{16\lambda^2} = 100$. Multiplying both sides by $16\lambda^2$ yields $25 = 100(16\lambda^2)$, so that $\lambda^2 = \dfrac{1}{64}$, hence $\lambda = \pm\dfrac{1}{8}$.
  - Thus, we obtain the two points $(x, y) = (8, 3)$ and $(-8, -3)$.
  - Since $f(8, 3) = 25$ and $f(-8, -3) = -25$, the maximum is $\boxed{f(8, 3) = 25}$ and the minimum is $\boxed{f(-8, -3) = -25}$.

2b

- Example: Find the maximum and minimum values of $f(x, y, z) = x + 2y + 2z$ subject to the constraint $x^2 + y^2 + z^2 = 9$.
  - We have $g = x^2 + y^2 + z^2$, and also $f_x = 1$, $g_x = 2x$, $f_y = 2$, $g_y = 2y$, $f_z = 2$, $g_z = 2z$.
  - Thus we have the system $1 = 2x\lambda$, $2 = 2y\lambda$, $2 = 2z\lambda$, and $x^2 + y^2 + z^2 = 9$.

$\nabla g = (2x, 2y, 2z) \to (0,0,0) \notin M$

- Solving the first three equations gives $x = \frac{1}{2\lambda}$, $y = \frac{1}{\lambda}$, $z = \frac{1}{\lambda}$; plugging in to the last equation yields $\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 = 9$, so $\frac{9}{4\lambda^2} = 9$, so that $\lambda = \pm\frac{1}{2}$.
- This gives the two points $(x, y, z) = (1, 2, 2)$ and $(-1, -2, -2)$.
- Since $f(1, 2, 2) = 9$ and $f(-1, -2, -2) = -9$, the maximum is $\boxed{f(1, 2, 2) = 9}$ and the minimum is $\boxed{f(-1, -2, -2) = -9}$.

- Example: Maximize the volume $V = \pi r^2 h$ of a cylindrical can given that its surface area $SA = 2\pi r^2 + 2\pi rh$ is $150\pi\,\text{cm}^2$.
  - We have $f(r, h) = \pi r^2 h$ and $g(r, h) = 2\pi r^2 + 2\pi rh$, so $f_r = 2\pi rh$, $g_r = 4\pi r + 2\pi h$, $f_h = \pi r^2$, $g_h = 2\pi r$.
  - Thus we have the system $2\pi rh = (4\pi r + 2\pi h)\lambda$, $\pi r^2 = (2\pi r)\lambda$, and $2\pi r^2 + 2\pi rh = 150\pi$.
  - We clearly cannot have $r = 0$ since that contradicts the third equation, so we can assume $r \neq 0$.
  - Cancelling r from the second equation and then solving for λ yields $\lambda = \frac{r}{2}$. Plugging into the first equation (and cancelling the πs) yields $2rh = (4r + 2h) \cdot \frac{r}{2}$, so dividing by r yields $2h = 2r + h$, so that $h = 2r$.
  - Finally, plugging in $h = 2r$ to the third equation (after cancelling the πs) yields $2r^2 + 4r^2 = 150$, so that $r^2 = 25$ and thus $r = \pm 5$.
  - The two candidate points are $(r, h) = (5, 10)$ and $(-5, -10)$; since we only want positive values we are left only with $(5, 10)$, which by the physical setup of the problem must be the maximum.
  - Therefore, the maximum volume occurs with $r = 5$cm and $h = 10$cm, and is $\boxed{f(5, 10) = 250\pi \text{cm}^3}$.

- Example: An assembly line involving f full-time workers, p part-time workers, and r robots has a total production level of $T(f, p, r) = 80 f^{0.7} p^{0.2} r^{0.1}$ gizmos per day. Each full-time worker's compensation totals \$200 per day, each part-time worker's compensation totals \$80 per day, and each robot's maintenance costs total \$40 per day. If the daily operating budget is \$4000, how many of each type of worker, and how many robots, should be employed to maximize total daily production?
  - We wish to maximize the function $T(f, p, r) = 80 f^{0.7} p^{0.2} r^{0.1}$ subject to the constraint $200f + 80p + 40r = 4000$, so that $g(f, p, r) = 200f + 80p + 40r$.
  - We may maximize the function T directly, but the resulting calculation is somewhat unpleasant. It is easier to maximize the logarithm of T, namely $\ln(T) = \ln(80) + 0.7\ln(f) + 0.2\ln(p) + 0.1\ln(r)$, instead.
  - Taking the partial derivatives then yields the system $\frac{0.7}{f} = 200\lambda$, $\frac{0.2}{p} = 80\lambda$, $\frac{0.1}{r} = 40\lambda$, and $200f + 80p + 40r = 4000$.
  - The first three equations yield $f = \frac{7}{2000\lambda}$, $p = \frac{1}{400\lambda}$, and $r = \frac{1}{400\lambda}$.
  - Plugging these expressions into the last equation then yields $200 \cdot \frac{7}{2000\lambda} + 80 \cdot \frac{1}{400\lambda} + 40 \cdot \frac{1}{400\lambda} = 4000$, which simplifies to $\frac{1}{\lambda} = 4000$ and thus $\lambda = \frac{1}{4000}$. This yields a unique candidate triple $(f, p, r) = (14, 10, 10)$, which by the setup of the problem must be a maximum.
  - We conclude that the maximum production occurs with $\boxed{\text{14 full-time workers, 10 part-time workers, and 10 robots}}$, and is $T(14, 10, 10) \approx 1012.46$ gizmos per day.
  - Remark: If we tried to maximize T directly, the system of equations would be $56 f^{-0.3} p^{0.2} r^{0.1} = 200\lambda$, $16 f^{0.7} p^{-0.8} r^{0.1} = 80\lambda$, $8 f^{0.7} p^{0.2} r^{-0.9} = 40\lambda$, $200f + 80p + 40r = 4000$. This system is not nearly as easy to solve as the one above; one approach is to divide the second and third equations by the first one, then solve for two of f, p, r in terms of the other one, and finally plug in to the last equation.

Well, you're at the end of my handout. Hope it was helpful.

2c

$\nabla g = (2x, 2y, 2z) \rightarrow (0,0,0) \notin M$

Step 1
Before proceeding with the problem let's note because our constraint is the sum of three terms that are squared (and hence positive) the largest possible range of *x* is $-6 \le x \le 6$ (the largest values would occur if $y = 0$ and $z = 0$). Likewise, we'd get the same ranges for both *y* and *z*.

Note that, at this point, we don't know if *x*, *y* or *z* will actually be the largest possible value. At this point we are simply acknowledging what they are. What this allows us to say is that whatever our answers will be they must occur in these bounded ranges and hence by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

This step is an important (and often overlooked) step in these problems. It always helps to know that absolute extrema exist prior to actually trying to find them!

Step 2
The first actual step in the solution process is then to write down the system of equations we'll need to solve for this problem.

$$\begin{aligned} 0 &= 2x\lambda \\ 2y &= 2y\lambda \\ -10 &= 2z\lambda \\ x^2 + y^2 + z^2 &= 36 \end{aligned}$$

Step 3
For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be "easy" and which will be "hard" until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you've ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren't really as "bad" as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

For this system let's start with the third equation and note that because the left side is -10, or more importantly can never by zero, we can see that we must therefore have $z \ne 0$ and $\lambda \ne 0$. The fact that λ can't be zero is really important for this problem.

Step 4
Okay, because we now know that $\lambda \ne 0$ we can see that the only way for the first equation to be true is to have $x = 0$.

Therefore, no matter what else is going on with *y* and *z* in this problem we must always have $x = 0$ and we'll need to keep that in mind.

Step 5

Next, let's take a look at the second equation. A quick rewrite of this equation gives,

$$2y - 2y\lambda = 2y(1-\lambda) = 0 \qquad \to \qquad y = 0 \;\text{ or }\; \lambda = 1$$

Step 6
We now have two possibilities from Step 4. Either $y = 0$ or $\lambda = 1$. We'll need to go through both of these possibilities and see what we get.

Let's start by assuming that $y = 0$ and recall from Step 3 that we also know that $x = 0$. In this case we can plug these values into the constraint to get,

$$z^2 = 36 \qquad \to \qquad z = \pm 6$$

Therefore, from this part we get two points that are potential absolute extrema,

$$(0,0,-6) \qquad\qquad (0,0,6)$$

Step 7
Next, let's assume that $\lambda = 1$. If we head back to the third equation we can see that we now have,

$$-10 = 2z \qquad \to \qquad z = -5$$

So, under this assumption we must have $z = -5$ and recalling once more from Step 3 that we have $x = 0$ we can now plug these into the constraint to get,

$$y^2 + 25 = 36 \qquad \to \qquad y^2 = 11 \qquad \to \qquad y = \pm\sqrt{11}$$

So, this part gives us two more points that are potential absolute extrema,

$$\left(0,-\sqrt{11},-5\right) \qquad\qquad \left(0,\sqrt{11},-5\right)$$

Step 8
In total, it looks like we have four points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points. Here are those function evaluations.

$$f\left(0,-\sqrt{11},-5\right) = 61 \qquad f\left(0,\sqrt{11},-5\right) = 61 \qquad f(0,0,-6) = 60 \qquad f(0,0,6) = -60$$

The absolute maximum is then 61 which occurs at $\left(0,-\sqrt{11},-5\right)$ and $\left(0,\sqrt{11},-5\right)$. The absolute minimum is -60 which occurs at $(0,0,6)$. Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

1. $3 = 2\lambda x$
2. $1 = 2\lambda y$
3. $x^2 + y^2 = 10$

Now we want to solve for each variable. At this point, you should take a moment and try to cleverly think of a way to solve for one of the three. Let's plug in equations (1) and (2) into (3). This allows us to solve for λ.

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 10 \implies \lambda = \pm\frac{1}{2}$$

Now, we plug λ back into our original equations and get $x = \pm 3$ and $y = \pm 1$. We get the following extreme points

$$(3, 1), (-3, -1)$$

We can classify them by simply finding their values when plugging into $f(x, y)$.

- $f(3, 1) = 9 + 1 = 10$
- $f(-3, -1) = -9 - 1 = -10$

So the maximum happens at $\boxed{(3, 1)}$ and the minimum happens at $\boxed{(-3, -1).}$

Example 5.8.1.2 *Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint* $x^4 + y^4 + z^4 = 1$.

$$f(x, y, z) = x^2 + y^2 + z^2$$

- $\nabla f = \langle 2x, 2y, 2z \rangle$
- $\nabla g = \langle 4x^3, 4y^3, 4z^3 \rangle$

This gives us the following equation

$$\langle 2x, 2y, 2z \rangle = \lambda \left\langle 4x^3, 4y^3, 4z^3 \right\rangle$$

Therefore, we have the following equations:

1. $2x = 4\lambda x^3$
2. $2y = 4\lambda y^3$
3. $2z = 4\lambda z^3$

4. $x^4 + y^4 + z^4 = 1$

If x, y, z are nonzero, then we can consider
Therefore, we have the following equations:

1. $1 = 2\lambda x^2$
2. $1 = 2\lambda y^2$
3. $1 = 2\lambda z^2$
4. $x^4 + y^4 + z^4 = 1$

Remember, we can only make this simplification if all the variables are nonzero! In this form, we can plug in (1), (2), and (3) into (4). This gives us

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1$$

From this, we can solve for λ to get

$$\lambda = \pm\frac{\sqrt{3}}{2}$$

Now, we plug λ back into our original equations and get $\pm\frac{1}{\sqrt[4]{3}}$ for each variable.

Regardless of the sign, we see that

$$f\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\frac{1}{\sqrt[4]{3}}, \pm\frac{1}{\sqrt[4]{3}}\right) = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \boxed{\sqrt{3}}$$

Now, what if one of the variables is zero? How can we deal with that? We can assume one of the variables is zero and see what happens. That means when we plug into equation (4), we only get two nonzero terms.

If x is zero, then

$$(0)^2 + \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1 \implies \lambda = \pm\frac{1}{\sqrt{2}}$$

Notice that if y or z were chosen to be zero instead of x, we'd still conclude that $\lambda = \pm\frac{1}{\sqrt{2}}$. That's why we can just consider one of the variables and think of it as considering all three possibilities. In this case, we get one variable to be zero and the remaining nonzero variables as $\pm\frac{1}{\sqrt[4]{2}}$. Therefore, we get the critical points

$$\left(0, \pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}\right), \left(\pm\frac{1}{\sqrt[4]{2}}, 0, \pm\frac{1}{\sqrt[4]{2}}\right), \left(\pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}, 0\right)$$

For either of these points, we get

$$f(\text{ critical point }) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Still, we haven't considered *all* possible values. What if two variables were zero? Then, when we plug into equation (4), we get

$$(0)^2 + (0)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1 \implies \lambda = \pm\frac{1}{2}$$

For the variable that is not zero, we'd get the value ±1. Therefore, we have the critical points

$$(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$$

For either of these points, we get

$$f(\text{ critical point }) = 1$$

It's not possible that all three variables are zero. Otherwise equation (4) would be false. Therefore, are maximums are obtained at the points

- $\left(\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$
- $\left(-\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$
- $\left(\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$
- $\left(\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$
- $\left(-\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$
- $\left(-\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$
- $\left(\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$
- $\left(-\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$

The minimums are obtained at the points

- $(1, 0, 0)$
- $(-1, 0, 0)$
- $(0, 1, 0)$
- $(0, -1, 0)$
- $(0, 0, 1)$
- $(0, 0, -1)$

This example shows how complicated these problems can get, especially with an added dimension. We could have easily missed the minimum values if we weren't careful.

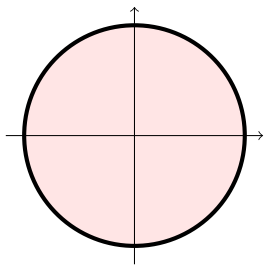
Example 5.8.1.3 *Use Lagrange multipliers to find the absolute maximum and absolute minimum of*

$$f(x, y) = xy$$

over the region $D = \{(x, y) \mid x^2 + y^2 \leq 8\}$.

As before, we will find the critical points of f over D. Then, we'll restrict f to the boundary of D and find all extreme values. It is in this second step that we will use Lagrange multipliers.

The region D is a circle of radius $2\sqrt{2}$.



- $f_x(x, y) = y$
- $f_y(x, y) = x$

We therefore have a critical point at $(0, 0)$ and $f(0, 0) = 0$.

Now let us consider the boundary. We will use Lagrange multipliers and let the constraint be $x^2 + y^2 = 9$. We begin with $\nabla f = \lambda \nabla g$.

$$\langle y, x \rangle = \lambda \langle 2x, 2y \rangle$$

This gives us the following equations:

Before we outline the solution of this problem, let us step back for a moment and consider its geometrical interpretation. The level sets of $f(x, y, z)$ are concentric ellipsoids centered at the origin $(0, 0, 0)$. As we move outward, the values associated with these level sets increase.

The two constraints represent equations of planes. The fact that they must be satisfied simultaneously means that the planes are intersecting – in this case, their intersection produces a line in $\mathbf{R}^3$. As one moves along this line, closer and closer to the origin, the value of f on the line will decrease as it intersects level sets associated with lower and lower values of f. If the line actually were to go through the origin (which is not the case, since $(0, 0, 0)$ does not satisfy any of the two equations), the value of f evaluated on the line would go to zero.) At some point, a minimal value of f will be attained, and the values will begin to increase.

3a

Solution: The two constraints can be written in the form

$$\begin{aligned} F(x, y, z) &= x + 2y + 3z - 1 = 0 \\ G(x, y, z) &= x - 2y + z - 5 = 0. \end{aligned} \tag{56}$$

The associated Lagrangian function then has the form

$$L(x, y, z, \lambda, \mu) = x^2 + 2y^2 + z^2 + \lambda(x + 2y + 3z - 1) + \mu(x - 2y + z - 5). \tag{57}$$

The conditions for a critical point become:

$$\begin{aligned} \frac{\partial L}{\partial x} = 0 \quad &: \quad 2x + \lambda + \mu = 0 \\ \frac{\partial L}{\partial y} = 0 \quad &: \quad 4y + 2\lambda - 2\mu = 0 \\ \frac{\partial L}{\partial z} = 0 \quad &: \quad 2z + 3\lambda + \mu = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \quad &: \quad x + 2y + 3z - 1 = 0 \\ \frac{\partial L}{\partial \mu} = 0 \quad &: \quad x - 2y + z - 5 = 0. \end{aligned} \tag{58}$$

We search for x, y, z, λ and μ that simultaneously satisfy these equations.

As mentioned in class, there are usually several ways to solve these equations. And a method that solves one problem will not necessarily apply to another. To solve this problem, one method is to use

$\nabla g_1 = (1, 2, 3)$
$\nabla g_2 = (1, -2, 1)$ } Independent

the first three equations from above to express x, y and z in terms of λ and μ:

$$x = -\frac{1}{2}(\lambda + \mu), \quad y = -\frac{1}{4}(2\lambda - 2\mu), \quad z = -\frac{1}{2}(3\lambda + \mu). \tag{59}$$

Now substitute these results into the final two equations, which represent the constraints:

$$\begin{aligned} -6\lambda - \mu &= 1 \\ -\lambda - 2\mu &= 5. \end{aligned} \tag{60}$$

The solution of this simultaneous linear system is given by

$$\lambda = \frac{3}{11}, \qquad \mu = -\frac{29}{11}. \tag{61}$$

From these values, we compute x, y and z to be

$$x = \frac{13}{11}, \quad y = -\frac{16}{11}, \quad z = \frac{10}{11}. \tag{62}$$

This is the only critical point for this problem. At this point, $f(x, y, z) = \frac{71}{11}$. This must correspond to a global minimum since $f(x, y, z)$ can assume arbitrary large values by letting x, y and z become arbitrarily large while they satisfy the two constraints.

An important application of Lagrange multipliers to Physics – the Boltzman distribution of Statistical Mechanics

The discussion in this section is intended to be brief. We outline the application of method of Lagrangian multipliers a fundamental problem in Statistical Mechanics: finding the most probable distribution of energies assumed by a system of atoms or molecules. From now on, we simply refer to these particles as molecules.

Consider a system of N independent, identical and distinguishable atoms or molecules, for example, a container of oxygen gas. By "distinguishable," we mean that we can index each molecule uniquely and keep track of it. We assume that each molecule can exist in one of n states, $1, 2, \cdots, n$, with respective energies $E_1, E_2, \cdots, E_n$. (Examples of these energies: the electronic energies that can be assumed by an atom, the vibrational energies of a diatomic molecule.) We'll also let N_k denote the "occupation number" of the kth state, i.e. the number of molecules in that state.

$Dg_1 = (2, -1, -1)$
$Dg_2 = (2x, 2y, 0)$ } $x = y = 0 \notin M$

So, in this case we get two Lagrange Multipliers. Also, note that the first equation really is three equations as we saw in the previous examples. Let's see an example of this kind of optimization problem.

3b

Example 5 Find the maximum and minimum of $f(x,y,z) = 4y - 2z$ subject to the constraints $2x - y - z = 2$ and $x^2 + y^2 = 1$.

Solution
Verifying that we will have a minimum and maximum value here is a little trickier. Clearly, because of the second constraint we've got to have $-1 \le x, y \le 1$. With this in mind there must also be a set of limits on *z* in order to make sure that the first constraint is met. If one really wanted to determine that range you could find the minimum and maximum values of $2x - y$ subject to $x^2 + y^2 = 1$ and you could then use this to determine the minimum and maximum values of *z*. We won't do that here. The point is only to acknowledge that once again the possible solutions must lie in a closed and bounded region and so minimum and maximum values must exist by the Extreme Value Theorem.

Here is the system of equations that we need to solve.

$$0 = 2\lambda + 2\mu x \qquad (f_x = \lambda g_x + \mu h_x) \qquad (14)$$
$$4 = -\lambda + 2\mu y \qquad (f_y = \lambda g_y + \mu h_y) \qquad (15)$$
$$-2 = -\lambda \qquad (f_z = \lambda g_z + \mu h_z) \qquad (16)$$
$$2x - y - z = 2 \qquad (17)$$
$$x^2 + y^2 = 1 \qquad (18)$$

First, let's notice that from equation (16) we get $\lambda = 2$. Plugging this into equation (14) and equation (15) and solving for *x* and *y* respectively gives,

$$0 = 4 + 2\mu x \qquad \Rightarrow \qquad x = -\frac{2}{\mu}$$
$$4 = -2 + 2\mu y \qquad \Rightarrow \qquad y = \frac{3}{\mu}$$

Now, plug these into equation (18).

$$\frac{4}{\mu^2} + \frac{9}{\mu^2} = \frac{13}{\mu^2} = 1 \qquad \Rightarrow \qquad \mu = \pm\sqrt{13}$$

So, we have two cases to look at here. First, let's see what we get when $\mu = \sqrt{13}$. In this case we know that,

$$x = -\frac{2}{\sqrt{13}} \qquad y = \frac{3}{\sqrt{13}}$$

Plugging these into equation (17) gives,

$$-\frac{4}{\sqrt{13}} - \frac{3}{\sqrt{13}} - z = 2 \qquad \Rightarrow \qquad z = -2 - \frac{7}{\sqrt{13}}$$

So, we've got one solution.

Let's now see what we get if we take $\mu = -\sqrt{13}$. Here we have,

$$x = \frac{2}{\sqrt{13}} \qquad\qquad y = -\frac{3}{\sqrt{13}}$$

Plugging these into equation (17) gives,

$$\frac{4}{\sqrt{13}} + \frac{3}{\sqrt{13}} - z = 2 \qquad \Rightarrow \qquad z = -2 + \frac{7}{\sqrt{13}}$$

and there's a second solution.

Now all that we need to is check the two solutions in the function to see which is the maximum and which is the minimum.

$$f\left(-\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, -2 - \frac{7}{\sqrt{13}}\right) = 4 + \frac{26}{\sqrt{13}} = 11.2111$$

$$f\left(\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}, -2 + \frac{7}{\sqrt{13}}\right) = 4 - \frac{26}{\sqrt{13}} = -3.2111$$

So, we have a maximum at $\left(-\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, -2 - \frac{7}{\sqrt{13}}\right)$ and a minimum at $\left(\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}, -2 + \frac{7}{\sqrt{13}}\right)$.

2. Solve this system of equations to get (possibly multiple) solutions for x, y, z, λ, and μ.
3. For each solution (x, y, z, λ, μ), find $f(x, y, z)$ and compare the values you get. The largest value corresponds to maximums, the smallest value corresponds to minimums.

5.8.2 Examples

3C

Example 5.8.2.1 *Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraints $x + y - z = 0$ and $x^2 + 2z^2 = 1$.*

$$f(x, y, z) = 3x - y - 3z$$

As you'll see, the technique is basically the same. It only requires that we look at more equations.

- $\nabla f = \langle 3, -1, -3 \rangle$
- $\nabla g = \langle 1, 1, -1 \rangle$
- $\nabla h = \langle 2x, 0, 4z \rangle$

These combine to

$$\langle 3, -1, -3 \rangle = \lambda \langle 1, 1, -1 \rangle + \mu \langle 2x, 0, 4z \rangle$$

Therefore, we have the following equations:

1. $3 = \lambda + 2\mu x$
2. $-1 = \lambda$
3. $-3 = -\lambda + 4\mu z$
4. $x + y - z = 0$
5. $x^2 + 2z^2 = 1$

Already, we that $\lambda = -1$ and

$$-3 = 1 + 4\mu z \implies -\frac{1}{\mu} = z$$

$$3 = -1 + 2\mu x \implies \frac{2}{\mu} = x$$

$\nabla g_1 = (1, 1, -1)$
$\nabla g_2 = (2x, 0, 4z)$ } $x = z = 0 \notin M$

We can combine these two facts to get $x = -2z$. Let's use our fifth equation to solve.

$$1 = 4z^2 + 2z^2 \implies 6z^2 = 1 \implies z = \pm\frac{1}{\sqrt{6}}, \quad x = \mp\frac{2}{\sqrt{6}}$$

Plugging in either x or z to solve for μ will give you $\mu = \mp\sqrt{6}$. That means we know

$$\lambda = -1, \ \mu = \mp\sqrt{6}, \ x = \mp\frac{2}{\sqrt{6}} \ \text{ and } \ z = \pm\frac{1}{\sqrt{6}}$$

We now use equation 4 to find y.

$$\mp\frac{2}{\sqrt{6}} + y \mp \frac{1}{\sqrt{6}} = 0 \implies y = \pm\frac{3}{\sqrt{6}}$$

Therefore, my two points are

$$\left(\frac{2}{\sqrt{6}}, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) \text{ and } \left(-\frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

- $f\left(\frac{2}{\sqrt{6}}, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) = \frac{6}{\sqrt{6}} + \frac{3}{\sqrt{6}} + \frac{3}{\sqrt{6}} = 2\sqrt{6}$ (This is a max)
- $f\left(-\frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = -\frac{6}{\sqrt{6}} - \frac{3}{\sqrt{6}} - \frac{3}{\sqrt{6}} = -2\sqrt{6}$ (This is a min)

For all these problems, it's important to take your time when simplifying. When we do algebra, we often want to just "plug and chug" through the problem. After all, that has worked pretty well in the past. Here, however, you'll quickly find that you will go in circles and never solve for anything. Just take your time and think about each step you take. It often helps to write out all the equations before trying to plug in anything.

The absolute maximum is then $\frac{2}{3}$ which occurs at $\left(2,-\frac{1}{3},-1\right)$ and $\left(2,\frac{1}{3},1\right)$. The absolute minimum is $-\frac{2}{3}$ which occurs at $\left(2,-\frac{1}{3},1\right)$ and $\left(2,\frac{1}{3},-1\right)$. Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

Before leaving this problem we should note that some of the solution processes for the systems that arise with Lagrange multipliers can be quite involved. It can be easy to get lost in the details of the solution process and forget to go back and take care of one or more possibilities. You need to always be very careful and before finishing a problem go back and make sure that you've dealt with all the possible solution paths in the problem.

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5. Find the maximum and minimum values of $f\left(x,y,z\right)=3x^2+y$ subject to the constraints $4x-3y=9$ and $x^2+z^2=9$.

Step 1
Before proceeding with the problem let's note that the second constraint is the sum of two terms that are squared (and hence positive). Therefore, the largest possible range of *x* is $-3\le x\le 3$ (the largest values would occur if $z=0$). We'll get a similar range for *z*.

Now, the first constraint is not the sum of two (or more) positive numbers. However, we've already established that *x* is restricted to $-3\le x\le 3$ and this will give $-7\le y\le 1$ as the largest possible range of *y*'s. Note that we can easily get this range by acknowledging that the first constraint is just a line and so the extreme values of *y* will correspond to the extreme values of *x*.

So, because we now know that our answers must occur in these bounded ranges by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

This step is an important (and often overlooked) step in these problems. It always helps to know that absolute extrema exist prior to actually trying to find them!

Step 2
The first step here is to write down the system of equations we'll need to solve for this problem.

$$\begin{aligned} 6x &= 4\lambda + 2x\mu \\ 1 &= -3\lambda \\ 0 &= 2z\mu \\ 4x-3y &= 9 \\ x^2+z^2 &= 9 \end{aligned}$$

Step 3
For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be "easy" and which will be "hard" until you start the solution process.

$\nabla g_1 = (2x, 0, 2z)$
$\nabla g_2 = (4, -3, 0)$
$\}$ $x = z = 0 \notin M$

Do not be afraid of these systems. They are probably unlike anything you've ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren't really as "bad" as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

With this system we get a "freebie" to start off with. Notice that from the second equation we quickly can see that $\lambda = -\frac{1}{3}$ regardless of any of the values of the other variables in the system.

Step 4

Next, from the third equation we can see that we have either $z = 0$ or $\mu = 0$, So, we have 2 possibilities to look at. Let's take a look at $z = 0$ first.

In this case we can go straight to the second constraint to get,

$$x^2 = 9 \qquad \rightarrow \qquad x = \pm 3$$

We can in turn plug each of these possibilities into the first constraint to get values for *y*.

$$\begin{aligned} x = -3 \quad &: \quad -12 - 3y = 9 \qquad \rightarrow \qquad y = -7 \\ x = 3 \quad &: \quad 12 - 3y = 9 \qquad \rightarrow \qquad y = 1 \end{aligned}$$

Okay, from this step we have two possible absolute extrema.

$$\left(-3,-7,0\right) \qquad \left(3,1,0\right)$$

Step 5

Now let's go back and take a look at what happens if $\mu = 0$. If we plug this into the first equation in our system (and recalling that we also know that $\lambda = -\frac{1}{3}$) we get,

$$6x = -\tfrac{4}{3} \qquad \rightarrow \qquad x = -\tfrac{2}{9}$$

We can plug this into each of our constraints to get values of *y* (from the first constraint) and *z* (form the second constraint). Here is that work,

$$\begin{aligned} 4\left(-\tfrac{2}{9}\right) - 3y = 9 \qquad &\rightarrow \qquad y = -\tfrac{89}{27} \\ \left(-\tfrac{2}{9}\right)^2 + z^2 = 9 \qquad &\rightarrow \qquad z = \pm\tfrac{5\sqrt{29}}{9} \end{aligned}$$

This leads to two more potential absolute extrema.

$$\left(-\tfrac{2}{9},-\tfrac{89}{27},-\tfrac{5\sqrt{29}}{9}\right) \qquad \left(-\tfrac{2}{9},-\tfrac{89}{27},\tfrac{5\sqrt{29}}{9}\right)$$

Step 6
In total, it looks like we have four points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points. Here are those function evaluations.

$$f\left(-3,-7,0\right)=20 \qquad f\left(3,1,0\right)=28 \qquad f\left(-\tfrac{2}{9},-\tfrac{89}{27},-\tfrac{5\sqrt{29}}{9}\right)=-\tfrac{85}{27} \qquad f\left(-\tfrac{2}{9},-\tfrac{89}{27},\tfrac{5\sqrt{29}}{9}\right)=-\tfrac{85}{27}$$

The absolute maximum is then 28 which occurs at $\left(3,1,0\right)$. The absolute minimum is $-\tfrac{85}{27}$ which occurs at $\left(-\tfrac{2}{9},-\tfrac{89}{27},-\tfrac{5\sqrt{29}}{9}\right)$ and $\left(-\tfrac{2}{9},-\tfrac{89}{27},\tfrac{5\sqrt{29}}{9}\right)$. Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

Before leaving this problem we should note that, in this case, the value of the absolute extrema (as opposed to the location) did not actually depend on the value of *z* in any way as the function we were optimizing in this problem did not depend on *z*. This will happen sometimes and we shouldn't get too worried about it when it does.

Note however that we still need the values of *z* for the location of the absolute extrema. We need the values of *z* for the location because the points that give the absolute extrema are also required to satisfy the constraint and the second constraint in our problem does involve *z*'s!

For $(2,1,-2)$ we get

$$H = \begin{pmatrix} 8 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \implies \Delta_1 = 8,\ \Delta_2 = \begin{vmatrix} 8 & -2 \\ -2 & 2 \end{vmatrix} = 12,\ \Delta_3 = |H| = 28.$$

Since always $\Delta_i > 0$, we conclude that $f(2,1,-2) = -7$ is a local minimum.
Recall that for a local maximum we need $\Delta_1 < 0$, $\Delta_2 > 0$, and $\Delta_3 < 0$.

3. Since expressing y from the constraint would be messy, this calls for Lagrange multipliers with $g(x,y) = x^2 - 2x + 2y^2 + 4y$. Equations to solve are $\nabla f = \lambda \nabla g$ and $g = 0$, that is,

$$\left.\begin{aligned} \tfrac{\partial f}{\partial x} &= \lambda \tfrac{\partial g}{\partial x} \\ \tfrac{\partial f}{\partial y} &= \lambda \tfrac{\partial g}{\partial y} \\ g &= 0 \end{aligned}\right\} \implies \left.\begin{aligned} 2x &= \lambda(2x-2) \\ 4y &= \lambda(4y+4) \\ x^2 - 2x + 2y^2 + 4y &= 0 \end{aligned}\right\} \implies \begin{aligned} x &= \lambda(x-1) \\ y &= \lambda(y+1) \\ x^2 - 2x + 2y^2 + 4y &= 0 \end{aligned}$$

A typical strategy is to eliminate λ from the first two equations in order to obtain some relationship between the variables x, y, this is then used with condition $g = 0$ to find the desired points.
We would like to isolate λ from the first equation. Can we have $x = 1$? The first equation then reads $1 = 0$, which is not true. Thus for sure $x \neq 1$ and we can write $\lambda = \frac{x}{x-1}$. Putting it into the second equation and multiplying out we get $y = -x$. Now this can be put into the constraint, we obtain $3x^2 - 6x = 0$ and two solutions, $x = 0$ and $x = 2$. Thus there are two suspicious points: $(0,0)$ and $(2,-2)$. We substitute them into f: $f(0,0) = 0$, $f(2,-2) = 12$. Comparing values we guess that the former is a local minimum and the latter is a local maximum.
Determining global extrema usually involves some analysis of the situation. We have two local extrema, but we do not know whether they give global extrema. In general, we find global extrema by comparing values at local extrema and also values at "borders" of the set. Thus we need to know more about M, the set determined by the given condition where we look at f.
A frequent trouble arises when the given set is not bounded, since then we have to ask what happens to f when points of M run away to some infinity. Could it happen that x tends to infinity within this set? Since points from M satisfy $2y^2 + 4y = 2x - x^2$, this would force the expression $2y^2 + 4y$ to tend to minus infinity, but that is not possible. Similarly we argue that also y cannot go to infinity and we thus have a bounded set M.
Another source of trouble is if the set M is a curve that has some endpoints, then we would have to check on those. How does M actually look like? In fact, rewriting the condition as

$$(x-1)^2 + 2(y+1)^2 = 3$$

we see that M is an ellipse. This is a close curve without any end, so whatever important happens to values of f on it, it must happen at one of the points we found earlier. Thus we can conclude that $f(0,0) = 0$ is a minimum and $f(2,-2) = 12$ is a maximum of f on the given set.

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4. The unknown point $Q = (x,y,z)$ satisfies $x + y - z = 1$, that would be the constraint with $g(x,y,z) = x + y - z$. The function to minimize should be the distance between P and Q, but that would mean a square root. It will be easier to minimize the distance squared, which is equivalent (think about it). Thus we have $f(x,y,z) = \text{dist}(P,Q)^2 = x^2 + (y+3)^2 + (z-2)^2$.
We use Lagrange multipliers, the equations $\nabla f = \lambda \nabla g$ and $g = 1$ now give

$$\left.\begin{aligned} \tfrac{\partial f}{\partial x} &= \lambda \tfrac{\partial g}{\partial x} \\ \tfrac{\partial f}{\partial y} &= \lambda \tfrac{\partial g}{\partial y} \\ \tfrac{\partial f}{\partial z} &= \lambda \tfrac{\partial g}{\partial z} \\ g &= 1 \end{aligned}\right\} \implies \left.\begin{aligned} 2x &= \lambda \cdot 1 \\ 2(y+3) &= \lambda \cdot 1 \\ 2(z-2) &= \lambda \cdot (-1) \\ x + y - z &= 1 \end{aligned}\right\} \implies \begin{aligned} x &= \tfrac{1}{2}\lambda \\ y + 3 &= \tfrac{1}{2}\lambda \\ z - 2 &= -\tfrac{1}{2}\lambda \\ x + y - z &= 1 \end{aligned}$$

$\nabla g = (1,1,-1) \neq (0,0,0)$

Again, we start by eliminating λ from the first three equations, for instance by substituting for $\frac{1}{2}\lambda$ from the first equation into the next two. Then

$$\left.\begin{array}{l} y+3=x \\ 2-z=x \\ x+y-z=1 \end{array}\right\} \implies \left.\begin{array}{l} y=x-3 \\ z=2-x \\ x+y-z=1 \end{array}\right\} \implies x+(x-3)+(2-x)=1 \implies x=2.$$

We easily find the other unknowns and obtain a suspicious point $Q=(2,-1,0)$. Is the function f and hence the distance really minimal, not for instance maximal at Q? We try another point from the plane. Say, the point $R=(0,1,0)$ has $\text{dist}(R,P)=\sqrt{4^2+2^2}=\sqrt{20}$. On the other hand, the distance from Q to P is $\text{dist}(Q,P)=\sqrt{2^2+2^2+2^2}=\sqrt{12}$, so it looks like the desired minimum.

We can also argue that it is possible to go to infinity within the given plane, we can easily let $x\to\infty$ and the other coordinates adjust, then also the distance goes to infinity (this is obvious when we imagine the situation) and thus the value we found cannot be the maximum.

Alternative: The first three equations offer the possibility of easily expressing all variables using λ (say, $z=2-\frac{1}{2}\lambda$). When we do it and substitute into the constrain, we get an equation with one unknown λ, namely $\frac{3}{2}\lambda=6$. For this we have $\lambda=4$ and we now have exactly the same $x=2$ etc. as before.

Note: Instead of Lagrange multipliers one could use the constraint to get $x=1-y+z$ and substitute into f, obtaining $F(y,z)=(1-y+z)^2+(y+3)^2+(z-2)^2$. We find its local extrema:

$$\left.\begin{array}{l} \frac{\partial F}{\partial y}=0 \\ \frac{\partial F}{\partial z}=0 \end{array}\right\} \implies \left.\begin{array}{r} -2(1-y+z)+2(y+3)=0 \\ 2(1-y+z)+2(z-2)=0 \end{array}\right\} \implies y=-1,\ z=0.$$

5. The distance between a point and a line is given as the distance between the given point and the closest point of the line, so we have to find that point.

We have two constraints, one given by $g(x,y,z)=x+y+z=1$, the other by $h(x,y,z)=2x-y+z=3$. We want a point $Q=(x,y,z)$ satisfying these constraints such that its distance from P is minimal, we will minimize the distance squared $f(x,y,z)=(x-1)^2+(y-2)^2+(z+1)^2$. Now there will be two Lagrange multipliers, we call them λ and μ (it is easier to write g,h and λ,μ rather then g_1,g_2 and λ_1,λ_2 as in the theorem). The equations are $\nabla f=\lambda\nabla g+\mu\nabla h$, $g=1$ and $h=3$, that is,

$$\left.\begin{array}{l} \frac{\partial f}{\partial x}=\lambda\frac{\partial g}{\partial x}+\mu\frac{\partial h}{\partial x} \\ \frac{\partial f}{\partial y}=\lambda\frac{\partial g}{\partial y}+\mu\frac{\partial h}{\partial y} \\ \frac{\partial f}{\partial z}=\lambda\frac{\partial g}{\partial z}+\mu\frac{\partial h}{\partial z} \\ g=1 \\ h=3 \end{array}\right\} \implies \left.\begin{array}{l} 2(x-1)=\lambda\cdot 1+\mu\cdot 2 \\ 2(y-2)=\lambda\cdot 1+\mu\cdot(-1) \\ 2(z+1)=\lambda\cdot 1+\mu\cdot 1 \\ x+y+z=1 \\ 2x-y+z=3 \end{array}\right\} \implies \begin{array}{l} 2(x-1)=\lambda+2\mu \\ 2(y-2)=\lambda-\mu \\ 2(z+1)=\lambda+\mu \\ x+y+z=1 \\ 2x-y+z=3 \end{array}$$

We will again try to eliminate the multipliers from the first three equations. For instance, we add the second and the third equation, get $\lambda=y+z-1$, putting it back into the third equation we get $\mu=z-y+3$. Substituting λ,μ into the first equation we get $2x+y-3z=6$. This is typical, we had 3 equations with 5 unknowns, so after using up two equations we end up with one and only three unknowns.

Now we also take into account the two constraints, so we get

$$\left.\begin{array}{l} 2x+y-3z=6 \\ x+y+z=1 \\ 2x-y+z=3 \end{array}\right\} \implies x=2,\ y=0,\ z=-1.$$

We then calculate the distance from $Q=(2,0,-1)$ to P: $\text{dist}(P,Q)=\sqrt{5}$. Just to make sure

- If $y = 0$, then (7c) gives $x = \pm 4$.

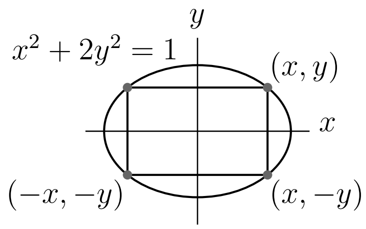
So we have the following table of candidates.

point	$(4, 0)$	$(-4, 0)$
value of f	-24	56
	min	max

Example 21

Example 22

Find the rectangle of largest area (with sides parallel to the coordinates axes) that can be inscribed in the ellipse $x^2 + 2y^2 = 1$.



Solution. Call the coordinates of the upper right corner of the rectangle (x, y), as in the figure above. The four corners of the rectangle are $(\pm x, \pm y)$ so the rectangle has width $2x$ and height $2y$ and the objective function is $f(x, y) = 4xy$. The constraint function for this problem is $g(x, y) = x^2 + 2y^2 - 1$. The first order derivatives of these functions are

$$f_x = 4y \qquad f_y = 4x \qquad g_x = 2x \qquad g_y = 4y$$

So, according to the method of Lagrange multipliers, we need to find all solutions to

$$4y = \lambda(2x) \iff y = \frac{1}{2}\lambda x \tag{8a}$$

$$4x = \lambda(4y) \implies x = \lambda y = \frac{1}{2}\lambda^2 x \implies x\Big(1 - \frac{\lambda^2}{2}\Big) = 0 \tag{8b}$$

$$0 = x^2 + 2y^2 - 1 \tag{8c}$$

So (8b) is satisfied if either $x = 0$ or $\lambda = \sqrt{2}$ or $\lambda = -\sqrt{2}$.

- If $x = 0$, then (8a) gives $y = 0$ too. But $(0, 0)$ violates the constraint.
- If $\lambda = \sqrt{2}$, then (8a) gives $x = \sqrt{2}y$ and then (8c) gives $2y^2 + 2y^2 = 1$ so that $y = \pm 1/2$ and $x = \pm 1/\sqrt{2}$.
- If $\lambda = -\sqrt{2}$, then (8a) gives $x = -\sqrt{2}y$ and then (8c) gives $2y^2 + 2y^2 = 1$ so that $y = \pm 1/2$ and $x = \mp 1/\sqrt{2}$.

$\nabla g = (2x, 4y) \to x = y = 0 \notin M$

The rectangle of largest area has the vertex $\left(1/\sqrt{2}, 1/2\right)$ in the first quadrant.

Example 22

Example 23

Find the ends of the major and minor axes of the ellipse $3x^2 - 2xy + 3y^2 = 4$. They are the points on the ellipse that are farthest from and nearest to the origin.

Solution. Let (x, y) be a point on $3x^2 - 2xy + 3y^2 = 4$. This point is at the end of a major axis when it maximizes its distance from the centre, $(0, 0)$ of the ellipse. It is at the end of a minor axis when it minimizes its distance from $(0, 0)$. So we wish to maximize and minimize the distance $\sqrt{x^2 + y^2}$ subject to the constraint $g(x, y) = 3x^2 - 2xy + 3y^2 - 4 = 0$. Now maximizing/minmizing $\sqrt{x^2 + y^2}$ is equivalent to maximizing/minmizing $\left(\sqrt{x^2 + y^2}\right)^2 = x^2 + y^2$. So we are free to choose the objective function $f(x, y) = x^2 + y^2$, which we will do, because it makes the derivatives cleaner. Since

$$f_x(x, y) = 2x \qquad f_y(x, y) = 2y \qquad g_x(x, y) = 6x - 2y \qquad g_y(x, y) = -2x + 6y$$

we need to find all solutions to

$$2x = \lambda(6x - 2y) \qquad \iff \qquad (1 - 3\lambda)x + \lambda y = 0 \tag{9a}$$

$$2y = \lambda(-2x + 6y) \qquad \iff \qquad \lambda x + (1 - 3\lambda)y = 0 \tag{9b}$$

$$0 = 3x^2 - 2xy + 3y^2 - 4 \tag{9c}$$

To start, let's concentrate on the first two equations. Pretend, for a couple of minutes, that we already know the value of λ and are trying to find x and y. Note that λ cannot be zero because if it is, (9a) forces $x = 0$ and (9b) forces $y = 0$ and $(0, 0)$ is not on the ellipse. So we may divide by λ and (9a) gives $y = -\frac{1-3\lambda}{\lambda}x$. Subbing this into (9b) gives $\lambda x - \frac{(1-3\lambda)^2}{\lambda}x = 0$. Again, x cannot be zero, since then $y = -\frac{1-3\lambda}{\lambda}x$ would give $y = 0$ and $(0, 0)$ is still not on the ellipse. So we may divide $\lambda x - \frac{(1-3\lambda)^2}{\lambda}x = 0$ by x, giving

$$\lambda - \frac{(1 - 3\lambda)^2}{\lambda} = 0 \iff (1 - 3\lambda)^2 - \lambda^2 = 0 \iff 8\lambda^2 - 6\lambda + 1 = (2\lambda - 1)(4\lambda - 1) = 0$$

We now know that λ must be either $\frac{1}{2}$ or $\frac{1}{4}$. Subbing these into either (9a) or (9b) gives

$$\lambda = \frac{1}{2} \implies -\frac{1}{2}x + \frac{1}{2}y = 0 \implies x = y \overset{(9c)}{\implies} 3x^2 - 2x^2 + 3x^2 = 4 \implies x = \pm 1$$

$$\lambda = \frac{1}{4} \implies \frac{1}{4}x + \frac{1}{4}y = 0 \implies x = -y \overset{(9c)}{\implies} 3x^2 + 2x^2 + 3x^2 = 4 \implies x = \pm\frac{1}{\sqrt{2}}$$

Here " $\overset{(9c)}{\implies}$ " indicates that we have just used (9c). The ends of the minor axes are $\pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. The ends of the major axes are $\pm(1, 1)$.

Example 23

6. The farmer has 100 m of fencing and wants to make a place for sheep next to the river - it means, that he has to fence only three sides of the rectangular place. Of course, he wants to have maximum dimension for the sheep.

   How to use the Lagrange multipliers?

   (a) $f(x, y) = xy$, $g(x, y) = 2x + y - 100$
   (b) $f(x, y) = 2x + 2y - 100$, $g(x, y) = xy$
   (c) $f(x, y) = xy$, $g(x, y) = x + y - 100$
   (d) $f(x, y) = x + y$, $g(x, y) = xy - 100$

   (Inspiration: `https://www.cpp.edu/conceptests/question-library/mat214.shtml#Partial%20Derivatives`)

Source 1: https://www.cbr.com/shaun-the-sheep-best-worst-episodes-imdb/