19th lesson

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Theory

Theorem 1 (Necessary condition of the existence of local extremum)**.** Let $n \in \mathbb{N}$, $G \subset \mathbb{R}^n$ be an open set, $a \in G$ and $i \in \{1, \cdots, n\}$. Let us suppose that a function $f : G \to \mathbb{R}$ has a local extremum at a point a. Then the partial derivative $\frac{\partial f}{\partial x}_i(a)$ does not exist or $\frac{\partial f}{\partial x}_i(a) = 0$.

Theorem 2 (attaining extrema)**.** Let $M \subset \mathbb{R}^n$ be a non-empty compact set and $f\colon M \to \mathbb{R}$ a function continuous on M. Then f attains its maximum and minimum on M.

Theorem 3 (characterisation of compact subsets of $\mathbb{R}^n$)**.** The set $M \subset \mathbb{R}^n$ is compact if and only if M is bounded and closed.

Algorithm

1. Check, if f is continuous and M is bounded and closed (=compact). Sketch M.
2. Find critical points at the interior of M = points with zero gradients inside M.
3. Find critical points at the boundary (Lagrange multipliers or expressing y or x).
4. Add problematic points (like square corners).
5. Compare values at all points. Find global maximum and minimum.

Hints

Remarks 4. Segment between points $A = [a_1, a_2]$ and $B = [b_1, b_2]$ can be parametrized as

$$\begin{aligned} x &= a_1 + t(b_1 - a_1), \\ y &= a_2 + t(b_2 - a_2), \qquad t \in [0, 1] \end{aligned}$$

Exercises

1. Find the global extrema of a function f on a set M

   (a) $f(x, y) = x^2 - xy + y$, $M = \{[x, y] \in \mathbb{R}^2 : 0 \le x \le 2, 0 \le y \le 3\}$
   (b) $f(x, y) = 4x^2 + 10y^2$, $M = \{[x, y] \in \mathbb{R}^2 : x^2 + y^2 \le 4\}$
   (c) $f(x, y) = xy - 3x$, $M = \{[x, y] \in \mathbb{R}^2 : x^2 \le y \le 9\}$
   (d) $f(x, y) = x^3 + 6xy - y^3$, M is triangle with vertices $(0, 0)$, $(4, 0)$, $(0, -4)$
   (e) $f(x, y) = x^2 + 4y^2$, M is bounded by the curves $x^2 + (y + 1)^2 = 4$, $y = x + 1$ and $y \ge 0$

(f) $f(x,y) = 4x^2 + y^2 - 4xy$, M is bounded by the curves $y = x^2$ and $y = 4$
(Here you will find a lot of candidate points.)

(g) $f(x,y) = 2x^2 + y - 3xy$, M is bounded by the curves $y = 1 - x$, $y = 1 + x$, $y = -1 - x$ and $y = -1 + x$

(h) $f(x,y) = x^2 - 8x + y^2 + 7$, $M = \{[x,y] \in \mathbb{R}^2 : x^2 + y^2 \leq 1, y \geq 0\}$

2. A sweet factory has a production function dependent on the input of Sugar, Toffee and Candyfloss:
$$f(s,t,c) = 50c^{2/5}t^{1/5}s^{1/5}.$$
The budget is 24 000 CZK and the factory can buy Candyfloss at 80 CZK, Toffee at 12 CZK and Sugar at 10 CZK per kg. How to maximize the production?
(Inspired by: Calculus: Single and Multivariable, Deborah Hughes-Hallett and col.)

Source 1: `https://wiki.openttd.org/File/en/Manual/Base%20Set/Industries/SweetFactory.png`