- ◦ Ultimately, one needs to find a parametrization $(x(t), y(t))$ of the boundary of the region, or some other description. (This may require breaking the boundary into several pieces, depending on the shape of the region.)
  - ◦ Then, by plugging the parametrization of the boundary curve into the function, we obtain a function $f(x(t), y(t))$ of the single variable t, which we can then analyze to determine the behavior of the function on the boundary.

- To find the absolute minimum and maximum values of a function on a given closed and bounded region R, follow these steps:
  - ◦ Step 1: Find all of the critical points of f that lie inside the region R.
  - ◦ Step 2: Parametrize the boundary of the region R (separating into several components if necessary) as $x = x(t)$ and $y = y(t)$, then plug in the parametrization to obtain a function of t, $f(x(t), y(t))$. Then search for "boundary-critical" points, where the t-derivative $\frac{d}{dt}$ of $f(x(t), y(t))$ is zero. Also include endpoints, if the boundary components have them.
    - ∗ A line segment from (a, b) to (c, d) can be parametrized by $x(t) = a + t(c - a)$, $y(t) = b + t(d - b)$, for $0 \le t \le 1$.
    - ∗ A curve of the form $y = g(x)$ can be parametrized by $x(t) = t$, $y(t) = g(t)$.
  - ◦ Step 3: Plug the full list of critical and boundary-critical points into f, and find the largest and smallest values.

1a

- Example: Find the absolute maximum and minimum of $f(x, y) = x^2 - xy + y$ on the rectangle $0 \le x \le 2$, $0 \le y \le 3$.
  - ◦ First, we find the critical points: since $f_x = 2x - y$ and $f_y = -x + 1$, there is a single critical point $\boxed{(1, 2)}$.
  - ◦ Next, we analyze the boundary of the region, which has 4 components:
    - ∗ Component #1, a line segment from $(0, 0)$ to $(2, 0)$: This component is parametrized by $x = 2t$, $y = 0$ for $0 \le t \le 1$. On this component we have $f(2t, 0) = 4t^2$, which has a critical point at $t = 0$ corresponding to $(x, y) = \boxed{(0, 0)}$. We also have boundary points $\boxed{(0, 0),\ (2, 0)}$.
    - ∗ Component #2, a line segment from $(2, 0)$ to $(2, 3)$: This component is parametrized by $x = 2$, $y = 3t$ for $0 \le t \le 1$. On this component we have $f(2, 3t) = 4 - 3t$, which has no critical point. We get only the boundary points $\boxed{(2, 0),\ (2, 3)}$.
    - ∗ Component #3, a line segment from $(0, 3)$ to $(2, 3)$: This component is parametrized by $x = 2t$, $y = 3$ for $0 \le t \le 1$. On this component we have $f(2t, 3) = 4t^2 - 6t + 3$, which has derivative $8t - 6$ hence has a critical point at $t = 3/4$ corresponding to $(x, y) = \boxed{(3/2, 3)}$. We also have boundary points $\boxed{(0, 3),\ (2, 3)}$.
    - ∗ Component #4, a line segment from $(0, 0)$ to $(0, 3)$: This component is parametrized by $x = 0$, $y = 3t$ for $0 \le t \le 1$. On this component we have $f(0, 3t) = 3$, which has no critical point. We get only the boundary points $\boxed{(0, 0),\ (0, 3)}$.
  - ◦ Our full list of points to analyze is $(1, 2)$, $(0, 0)$, $(2, 0)$, $(2, 3)$, $(3/2, 3)$, $(0, 3)$. We have $f(1, 2) = 1$, $f(0, 0) = 0$, $f(2, 0) = 4$, $f(2, 3) = 1$, $f(3/2, 3) = 3/4$, and $f(0, 3) = 3$. The $\boxed{\text{maximum is } 4}$ and the $\boxed{\text{minimum is } 0}$.

- Example: Find the absolute minimum and maximum of $f(x, y) = x^3 + 6xy - y^3$ on the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, -4)$.
  - ◦ First, we find the critical points: we have $f_x = 3x^2 + 6y$ and $f_y = -3y^2 + 6x$. Solving $f_y = 0$ yields $x = y^2/2$ and then plugging into $f_x = 0$ gives $y^4/4 + 2y = 0$ so that $y(y^3 + 8) = 0$: thus, we see that $\boxed{(0, 0) \text{ and } (2, -2) \text{ are critical points}}$.
  - ◦ Next, we analyze the boundary of the region. Here, the boundary has 3 components.

It turns out that we really need to do the same thing here if we want to know that we've found all the locations of the absolute extrema. The method of Lagrange multipliers will find the absolute extrema, it just might not find all the locations of them as the method does not take the end points of variables ranges into account (note that we might luck into some of these points but we can't guarantee that).

So, after going through the Lagrange Multiplier method we should then ask what happens at the end points of our variable ranges. For the example that means looking at what happens if $x = 0$, $y = 0$, $z = 0$, $x = 1$, $y = 1$, and $z = 1$. In the first three cases we get the points listed above that do happen to also give the absolute minimum. For the later three cases we can see that if one of the variables are 1 the other two must be zero (to meet the constraint) and those were actually found in the example. Sometimes that will happen and sometimes it won't.

In the case of this example the end points of each of the variable ranges gave absolute extrema but there is no reason to expect that to happen every time. In Example 2 above, for example, the end points of the ranges for the variables do not give absolute extrema (we'll let you verify this).

The moral of this is that if we want to know that we have every location of the absolute extrema for a particular problem we should also check the end points of any variable ranges that we might have. If all we are interested in is the value of the absolute extrema then there is no reason to do this.

Okay, it's time to move on to a slightly different topic. To this point we've only looked at constraints that were equations. We can also have constraints that are inequalities. The process for these types of problems is nearly identical to what we've been doing in this section to this point. The main difference between the two types of problems is that we will also need to find all the critical points that satisfy the inequality in the constraint and check these in the function when we check the values we found using Lagrange Multipliers.

Let's work an example to see how these kinds of problems work.

Example 4 Find the maximum and minimum values of $f(x,y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \le 4$.

Solution

Note that the constraint here is the inequality for the disk. Because this is a closed and bounded region the Extreme Value Theorem tells us that a minimum and maximum value must exist.

The first step is to find all the critical points that are in the disk (*i.e.* satisfy the constraint). This is easy enough to do for this problem. Here are the two first order partial derivatives.

$$
\begin{aligned}
f_x &= 8x \quad &\Rightarrow \quad 8x &= 0 \quad &\Rightarrow \quad x &= 0 \\
f_y &= 20y \quad &\Rightarrow \quad 20y &= 0 \quad &\Rightarrow \quad y &= 0
\end{aligned}
$$

So, the only critical point is $(0,0)$ and it does satisfy the inequality.

At this point we proceed with Lagrange Multipliers and we treat the constraint as an equality instead of the inequality. We only need to deal with the inequality when finding the critical points.

So, here is the system of equations that we need to solve.

$$8x = 2\lambda x$$
$$20y = 2\lambda y$$
$$x^2 + y^2 = 4$$

From the first equation we get,

$$2x(4-\lambda) = 0 \qquad \Rightarrow \qquad x = 0 \quad \text{or} \quad \lambda = 4$$

If we have $x = 0$ then the constraint gives us $y = \pm 2$.

If we have $\lambda = 4$ the second equation gives us,

$$20y = 8y \qquad \Rightarrow \qquad y = 0$$

The constraint then tells us that $x = \pm 2$.

If we'd performed a similar analysis on the second equation we would arrive at the same points.

So, Lagrange Multipliers gives us four points to check : $(0,2)$, $(0,-2)$, $(2,0)$, and $(-2,0)$.

To find the maximum and minimum we need to simply plug these four points along with the critical point in the function.

$$f(0,0) = 0 \qquad \text{Minimum}$$
$$f(2,0) = f(-2,0) = 16$$
$$f(0,2) = f(0,-2) = 40 \qquad \text{Maximum}$$

In this case, the minimum was interior to the disk and the maximum was on the boundary of the disk.

The final topic that we need to discuss in this section is what to do if we have more than one constraint. We will look only at two constraints, but we can naturally extend the work here to more than two constraints.

We want to optimize $f(x,y,z)$ subject to the constraints $g(x,y,z) = c$ and $h(x,y,z) = k$. The system that we need to solve in this case is,

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$$
$$g(x,y,z) = c$$
$$h(x,y,z) = k$$

    * Component #1, joining $(0,0)$ to $(4,0)$: This component is parametrized by $x = t$, $y = 0$ for $0 \le t \le 4$. On this component we have $f(t,0) = t^3$, which has a critical point only at $t = 0$, which corresponds to $(x,y) = \boxed{(0,0)}$. Also add the boundary point $\boxed{(4,0)}$.
    * Component #2, joining $(0,-4)$ to $(4,0)$: This component is parametrized by $x = t$, $y = t - 4$ for $0 \le t \le 4$. On this component we have $f(t,t-4) = 18t^2 - 72t + 64$, which has a critical point for $t = 2$, corresponding to $(x,y) = \boxed{(2,-2)}$. Also add the boundary points $\boxed{(4,0)}$ and $\boxed{(0,-4)}$.
    * Component #3, joining $(0,0)$ to $(0,-4)$: This component is parametrized by $x = 0$, $y = -t$ for $0 \le t \le 4$. On this component we have $f(0,t) = t^3$, which has a critical point for $t = 0$, corresponding to $(x,y) = \boxed{(0,0)}$. Also add the boundary point $\boxed{(0,-4)}$.
  - Our full list of points to analyze is $(0,0)$, $(4,0)$, $(0,-4)$, and $(2,-2)$. We compute $f(0,0) = 0$, $f(4,0) = 64$, $f(0,-4) = 64$, $f(2,-2) = -8$, and so we see that $\boxed{\text{maximum is 64}}$ and the $\boxed{\text{minimum is -8}}$.

1c

- Example: Find the absolute maximum and minimum of $f(x,y) = xy - 3x$ on the region with $x^2 \le y \le 9$.
  - First, we find the critical points: since $f_x = y - 3$ and $f_y = x$, there is a single critical point $\boxed{(0,3)}$.
  - Next, we analyze the boundary of the region, which (as a quick sketch reveals) has 2 components.
    * Component #1, a line segment from $(-3,9)$ to $(3,9)$: This component is parametrized by $x = t$, $y = 9$ for $-3 \le t \le 3$. On this component we have $f(t,9) = 6t$, which has no critical point. We only have boundary points $\boxed{(-3,9),\ (3,9)}$.
    * Component #2, a parabolic arc parametrized by $x = t$, $y = t^2$ for $-3 \le t \le 3$. On this component we have $f(t,t^2) = t^3 - 3t$, which has critical points at $t = \pm 1$, corresponding to $(x,y) = \boxed{(-1,1),\ (1,1)}$. The boundary points $(-3,9)$, $(3,9)$ are already listed above.
  - Our full list of points to analyze is $(0,3)$, $(-3,9)$, $(3,9)$, $(-1,1)$, $(1,1)$. We have $f(0,3) = 0$, $f(-3,9) = -18$, $f(3,9) = 18$, $f(-1,1) = 2$, and $f(1,1) = -2$. The $\boxed{\text{maximum is 18}}$ and the $\boxed{\text{minimum is -18}}$.

1.4.2 Linear Programming

- A particular special class of optimization problems involves searching for the minimum or maximum value of a linear function subject to various linear constraints (i.e., on a region defined by linear inequalities such as $3x + 2y \le 8$ or $x + z \ge 0$): these are known as linear programming problems[1].
- We can use the same methods for optimization of a function on a region to solve linear programming problems. However, linear programming problems have some convenient features that make it easier to identify potential minima and maxima, which we will illustrate with an example:
- Example: Find the minimum and maximum values of the function $f(x,y) = 2 + 3x + 5y$ subject to the constraints $x \ge 0$, $y \ge 0$, $x + y \le 10$, $x + 2y \le 15$.
  - First we search for critical points of f: since $f_x = 3$ and $f_y = 5$, there are no critical points.
  - Next, we need to determine the structure of the region, which is shown below:

[1] Although there are numerous computational algorithms (such as the simplex method) that exist for solving large-scale linear programming problems, the word "programming" in "linear programming" does not refer to computer programs. Instead, it comes from the United States military usage of the word "program" in reference to training and logistics schedules, whose optimization was among the first applied examples of linear programming.

- Ultimately, one needs to find a parametrization $(x(t), y(t))$ of the boundary of the region, or some other description. (This may require breaking the boundary into several pieces, depending on the shape of the region.)
- Then, by plugging the parametrization of the boundary curve into the function, we obtain a function $f(x(t), y(t))$ of the single variable t, which we can then analyze to determine the behavior of the function on the boundary.

- To find the absolute minimum and maximum values of a function on a given closed and bounded region R, follow these steps:
  - Step 1: Find all of the critical points of f that lie inside the region R.
  - Step 2: Parametrize the boundary of the region R (separating into several components if necessary) as $x = x(t)$ and $y = y(t)$, then plug in the parametrization to obtain a function of t, $f(x(t), y(t))$. Then search for "boundary-critical" points, where the t-derivative $\frac{d}{dt}$ of $f(x(t), y(t))$ is zero. Also include endpoints, if the boundary components have them.
    - A line segment from (a, b) to (c, d) can be parametrized by $x(t) = a + t(c - a)$, $y(t) = b + t(d - b)$, for $0 \le t \le 1$.
    - A curve of the form $y = g(x)$ can be parametrized by $x(t) = t$, $y(t) = g(t)$.
  - Step 3: Plug the full list of critical and boundary-critical points into f, and find the largest and smallest values.

- Example: Find the absolute maximum and minimum of $f(x, y) = x^2 - xy + y$ on the rectangle $0 \le x \le 2$, $0 \le y \le 3$.
  - First, we find the critical points: since $f_x = 2x - y$ and $f_y = -x + 1$, there is a single critical point $\boxed{(1, 2)}$.
  - Next, we analyze the boundary of the region, which has 4 components:
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    - Component #4, a line segment from $(0, 0)$ to $(0, 3)$: This component is parametrized by $x = 0$, $y = 3t$ for $0 \le t \le 1$. On this component we have $f(0, 3t) = 3$, which has no critical point. We get only the boundary points $\boxed{(0, 0),\ (0, 3)}$.
  - Our full list of points to analyze is $(1, 2)$, $(0, 0)$, $(2, 0)$, $(2, 3)$, $(3/2, 3)$, $(0, 3)$. We have $f(1, 2) = 1$, $f(0, 0) = 0$, $f(2, 0) = 4$, $f(2, 3) = 1$, $f(3/2, 3) = 3/4$, and $f(0, 3) = 3$. The $\boxed{\text{maximum is } 4}$ and the $\boxed{\text{minimum is } 0}$.

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* Component #3, joining $(0,0)$ to $(0,-4)$: This component is parametrized by $x = 0$, $y = -t$ for $0 \le t \le 4$. On this component we have $f(0,t) = t^3$, which has a critical point for $t = 0$, corresponding to $(x,y) = \boxed{(0,0)}$. Also add the boundary point $\boxed{(0,-4)}$.

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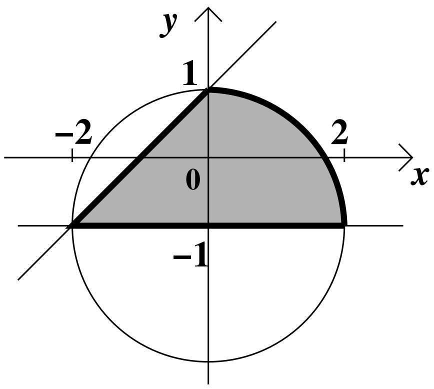
that we have the minimum, we pick another point from the line. For instance, $R = (0, -1, 2)$ satisfies both given equations and $\text{dist}(P, R) = \sqrt{19}$.

Note: If the question did not ask for Lagrange multipliers, we could have also used the given conditions to conclude that, say, $y = \frac{x}{2} + 1$, $z = 2 - \frac{3}{2}x$, substitute into f and then minimize it, obtaining $x = 2$.

6. First we sketch the set M. There are two straight lines and a circle, they split the plane into several parts. However, only one of them is both finite and bounded by all three of these curves.



We have a bounded closed set with non-empty interior and a boundary (border) consisting of three parts. Theory guarantees that global extrema happen either inside, then they have to be local extrema, or on the boundary. This determines the classical algorithm, we will gather all possible candidates and at the end compare values.
1) First we check on interior of M, where extrema happen at points of local extrema. We need not classify them, it is enough to check on all candidates, that is, on stationary points.

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 0 \\ \frac{\partial f}{\partial y} &= 0 \end{aligned}\right\} \implies \left.\begin{aligned} 2x &= 0 \\ 8y &= 0 \end{aligned}\right\} \implies x = y = 0.$$

Since $(0, 0) \in M$, we get the first candidate: $f(0, 0) = 0$.

2) Now we need to check on the boundary. There are three parts and each of them has the same form, it is a curve that is cut off at ends. For each part the procedure is again the same, global extrema can be attained either at endpoints or at points that give local extrema with respect to the considered curve. This guides our steps.
2a) We start with the quarter-circle. It is a curve determined by $g(x, y) = x^2 + (y + 1)^2 = 4$ and by conditions $0 \le x \le 2$, $-1 \le y \le 1$. It has two endpoints, so we have to check on them: $f(2, -1) = 8$ and $f(0, 1) = 4$ are candidates.
Then we need to find local extrema of f with respect to the constraint $g(x, y) = 4$, this should be possible using Lagrange multipliers. The equations $\nabla f = \lambda \nabla g$ and $g = 4$ read

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial g} \\ g &= 4 \end{aligned}\right\} \implies \begin{aligned} 2x &= \lambda 2x \\ 8y &= \lambda 2(y + 1) \\ x^2 + (y + 1)^2 &= 4 \end{aligned}$$

The first equation invites us to cancel, but we have to be careful. Could it happen that $x = 0$? Then the third equation gives $y = 1$ and the second one $\lambda = 2$. We are getting a candidate $(0, 1)$, but we already checked on it, so it is on the list.
To find other points we can assume that $x \neq 0$, then the first equation givess $\lambda = 1$, putting it into the second one we get $y = \frac{1}{3}$ and the third equation yields $x = \frac{2}{3}\sqrt{5}$. We have another candidate, $f\left(\frac{2}{3}\sqrt{5}, \frac{1}{3}\right) = \frac{8}{3}$.

By the way, comparing values at the three points we can guess that $f\big(\frac{2}{3}\sqrt{5},\frac{1}{3}\big)=\frac{8}{3}$ is a local minimum of f with respect to the arc we investigate here.
2b) The next part to explore is the oblique segment given by $y=x+1$, $-2\le x\le 0$. It has two endpoints, $(0,1)$ we already listed and the other is $f(-2,-1)=8$, another candidate for extrema on M.
To see what happens in the middle of this curve we can again use Lagrange multipliers, this time applied to $g(x,y)=y-x=1$:

$$\left.\begin{aligned}2x&=-\lambda\\8y&=\lambda\\y-x&=1\end{aligned}\right\}\implies\left.\begin{aligned}4y&=-x\\y-x&=1\end{aligned}\right\}\implies 5y=1\implies y=\tfrac{1}{5},$$

then $x=-\frac{4}{5}$. We have another candidate, $f\big(-\frac{4}{5},\frac{1}{5}\big)=\frac{4}{5}$ (which seems to be a local minimum of f with respect to that oblique line).
Since the condition is so simple, one might be tempted to simply substitute the expression $y=x+1$ into f, obtaining

$$\varphi(x)=f(x,x+1)=x^2+4(x+1)^2=5x^2+8x+4.$$

The one can use the usual tools of one-variable calculus to find candidates for extrema of this function over the interval $[-2,0]$, arriving at the same results as above.
2c) Finally we check on the horizontal segment, which is given by $y=-1$, $-2\le x\le 2$. Again, candidates come as endpoints, but we already included them above, and local extrema from the middle. Here the simplest way is to substitute, we are interested in extrema of $\varphi(x)=f(x,-1)=x^2+4$ over $-2\le x\le 2$. Endpoints are already done, local extrema are given by the condition $\varphi'(x)=0$, which gives $x=0$. We have another candidate to consider, $f(0,-1)=4$.

Now we put it all together. Candidates are $f(0,0)=0$, $f(2,-1)=8$, $f(0,1)=4$, $f(-2,-1)=8$, $f\big(\frac{2}{3}\sqrt{5},\frac{1}{3}\big)=\frac{8}{3}$, $f\big(-\frac{4}{5},\frac{1}{5}\big)=\frac{4}{5}$, and $f(0,-1)=4$. Comparing values we arrive at the answer: Global maximum of f over M is $f(-2,-1)=f(2,-1)=8$, global minimum is $f(0,0)=0$.
Just out of curiosity, global minimum with respect to the whole border is $f\big(-\frac{4}{5},\frac{1}{5}\big)=\frac{4}{5}$.

Remark: If we wanted to express M using set notation, we could start with the disc given by the first condition and intersect it with two half-planes given by the lines:

$$M=\{(x,y)\in\mathbb{R}^2;\ x^2+(y+1)^2\le 4\ \text{ and }\ y\ge -1\ \text{ and }\ y\le x+1\}.$$

7. The set M can be described as $M=\{(x,y)\in\mathbb{R}^2;\ x^2+y^2\le 4\}$. To find the global extrema we first look at what is happening inside, that is, we check on stationary points of f:

$$\left.\begin{aligned}\tfrac{\partial f}{\partial x}&=0\\\tfrac{\partial f}{\partial y}&=0\end{aligned}\right\}\implies\left.\begin{aligned}2x-6&=0\\2y+6&=0\end{aligned}\right\}\implies x=3,\ y=-3.$$

However, the point $(3,-3)$ is not in M and therefore we disregard it.
Next we have to check on the boundary, that is, we have to find extrema of f given the condition $x^2+y^2=4$. This calls for Lagrange multipliers:

$$\left.\begin{aligned}\tfrac{\partial f}{\partial x}&=\lambda\tfrac{\partial g}{\partial x}\\\tfrac{\partial f}{\partial y}&=\lambda\tfrac{\partial g}{\partial y}\\g&=4\end{aligned}\right\}\implies\left.\begin{aligned}2x-6&=\lambda\cdot 2x\\2y+6&=\lambda\cdot 2y\\x^2+y^2&=4\end{aligned}\right\}\implies\begin{aligned}x-3&=\lambda x\\y+3&=\lambda y\\x^2+y^2&=4\end{aligned}$$

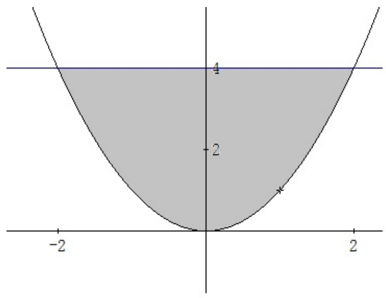
Now we would like to eliminate λ from the first two equations. Could it happen that $x=0$? Then the first equation would read $-3=0$, not possible; similarly we have $y\ne 0$. Thus we can express λ from the first equation, substitute into the second and eventually obtain $y=-x$. Putting this into the constraint equation we get $x=\pm\sqrt{2}$. Thus there are two candidates, $(\sqrt{2},-\sqrt{2})$ and $(-\sqrt{2},\sqrt{2})$. We put them into f: $f(\sqrt{2},-\sqrt{2})=4-12\sqrt{2}$, $f(-\sqrt{2},\sqrt{2})=4+12\sqrt{2}$, so it seems that the first is the minimum, the second is the maximum.

Example 3 *Find the absolute extrema of the function*

$$z = f(x, y) = 4x^2 + y^2 - 4yx$$

in the plane region bounded by the curves $y = x^2$ *and* $y = 4$.

Solution: *The region* D *represents the intersection between a parabola and a line:*



Step 1: Determine the critical points of z in D.

The system given by the nullity of the first partial derivatives of z *is*

$$\left.\begin{aligned} \frac{\partial f}{\partial x}(x, y) = 8x - 4y = 0 \\ \frac{\partial f}{\partial y}(x, y) = 2y - 4x = 0 \end{aligned}\right\} \Rightarrow y = 2x$$

So, every point of the form $(x, 2x)$, *with* $x \in \mathbb{R}$ *is a critical point of* $f(x, y)$. *But only the points with* $x \in [0, 2]$ *are in* D.

Step 2: Analize the boundary points of D.

In this case we have a combination of a polygonal and a curve defined by an equation. We have to consider the vertices of the figure, that is, the points $(-2, 4)$ *and* $(2, 4)$.

Moreover, we have to consider the constraint extrema of z *at each curve too. For the first curve* $y = x^2$, *one has:*

$$g(x) = f(x, x^2) = 4x^2 + x^4 - 4x^3 \Rightarrow g'(x) = 8x + 4x^3 - 12x^2 = 4x(2 + x^2 - 3x)$$

Then, $g'(x) = 0$ *implies* $x = 0$, $x = 2$ *or* $x = 1$. *From* $y = x^2$, *the points* $(0, 0)$, $(2, 4)$ *and* $(1, 1)$ *are points to be considered as possible extrema.*

For the other line, $y = 4$, one has

$$g(x) = f(x,4) = 4x^2 + 16 - 16x \Rightarrow g'(x) = 8x - 16 = 0 \Rightarrow x = 2 \Rightarrow (2,4).$$

Step 3: Choose the maximum and minimum values.

Compare the value of the function at each point obtained in Steps 1 and 2:

Critical point (x,y)	$f(x,y) = 4x^2 + y^2 - 4yx$
$(x, 2x)$ *with* $x \in [0,2]$	0
$(-2,4)$	64
$(1,1)$	1

Note that the point $(2,4)$ is included in the poins $(x,2x)$ for $x = 2$.

Then, the absolute maximum of $f(x,y)$ is 64 and occurs at the point $(-2,4)$. And the absolute minimum of $f(x,y)$ is 0 and occurs at all the points of the form $(x,2x)$ with $x \in [0,2]$.

4. Closing

We have studied that a function $f(x,y)$ that is continuous in a compact region D reaches a maximum and a minimum values in this region. These extrema values are obtained between the critical points of the function that are inside D and the points of the boundary of D. The method to find these absolute extrema consists on the obtaining of all the points that can be possible extrema and the comparison between the value of the function on them to finally choose the maximum and minimum value of $f(x,y)$ and the points where they are reached.

The examples presented illustrate the method in compact regions with different characteristics: defined by a function ($g(x,y) = 0$), defined by a polygonal and defined by a combination of both. Depending on the case we will have to consider the critical points of the function in the region D, the vertices of the region and the constrained extrema of $f(x,y)$ in the boundary of the region.

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Step 1: Obtain the critical points of $f(x, y)$ and select those that are in D.

Step 2: Obtain the constrained extrema of $f(x, y)$ under the condition given by the boundary of the region D.

- If the boundary is given by $g(x, y) = 0$, then we apply the Lagrange multipliers method.
- If the boundary is a polygonal, then we choose the vertex and the critical points at each line.

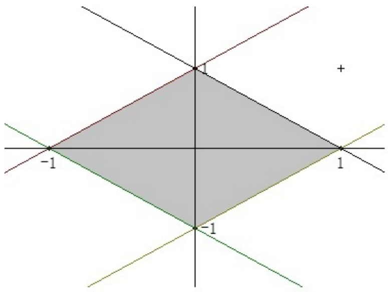
Step 3: Choose the maximum and the minimum between all the points obtained in Steps 1 and 2.

Example 1 *Find the absolute extrema of the function*

$$z = f(x, y) = 2x^2 + y - 3xy$$

in the plane region D bounded by the lines $y = 1 - x$, $y = 1 + x$, $y = -1 - x$ and $y = -1 + x$.

Solution: *The region D represents a quadrilateral*



Step 1: Determine the critical points of z in D.

Critical points of a function $f(x, y)$ are those points where the first partial derivatives are zero or do not exist. In this case, we have that the system giving the nullity of the first partial derivatives of z is

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x}(x, y) = 4x - 3y = 0 \\ \dfrac{\partial f}{\partial y}(x, y) = 1 - 3x = 0 \end{array}\right\} \Rightarrow x = \frac{1}{3} \quad \text{and} \quad y = \frac{4x}{3} = \frac{4}{9}.$$

So, the point $\left(\frac{1}{3}, \frac{4}{9}\right)$ *is a critical point of* z*. Clearly, this point is inside the region* D*.*

Step 2: Analize the boundary points of D.

The vertices of the figure are the points $(1,0)$, $(0,1)$, $(-1,0)$ *and* $(0,-1)$*. So, they must be considered as possible maxima or minima.*

Moreover, one has to consider also the constraint extrema of z *at each line. For the first line* $y = 1 - x$*, one has:*

$$g(x) = f(x, 1-x) = 2x^2 + 1 - x - 3x(1-x) = 5x^2 - 4x + 1 \Rightarrow g'(x) = 10x - 4$$

Then, $g'(x) = 0$ *implies* $x = \frac{2}{5}$ *and* $y = 1 - x = \frac{3}{5}$*. That is, the point* $\left(\frac{2}{5}, \frac{3}{5}\right)$ *is a point to be considered as a possible extrema. / For the other lines, we make a similar reasoning. For* $y = 1 + x$*, one has*

$$g(x) = f(x, 1+x) = -x^2 - 2x + 1 \Rightarrow g'(x) = -2x - 2 = 0 \Rightarrow x = -1 \Rightarrow (-1, 0)$$

For $y = -1 - x$*, it results*

$$g(x) = f(x, -1-x) = 5x^2 + 2x - 1 \Rightarrow g'(x) = 10x + 2 = 0 \Rightarrow x = -\frac{1}{5} \Rightarrow$$

$$\Rightarrow \left(-\frac{1}{5}, -\frac{4}{5}\right)$$

Finally, for $y = -1 + x$*, the point is obtained by*

$$g(x) = f(x, -1+x) = -x^2 + 4x - 1 \Rightarrow g'(x) = -2x + 4 = 0 \Rightarrow x = 2 \Rightarrow (2, 1).$$

Step 3: Choose the maximum and minimum values.

Now, we have to compare the value of the function at each point obtained in Steps 1 and 2:

Critical point (x, y)	$f(x, y) = 2x^2 + y - 3xy$
$\left(\frac{1}{3}, \frac{4}{9}\right)$	$\frac{2}{9}$
$(1, 0)$	2
$(0, 1)$	1
$(-1, 0)$	2
$(0, -1)$	-1
$\left(\frac{2}{5}, \frac{3}{5}\right)$	$\frac{1}{5}$
$\left(-\frac{1}{5}, -\frac{4}{5}\right)$	$-\frac{30}{25}$
$(2, 1)$	3

Then, the absolute maximum of $f(x, y)$ *is* 3 *and occurs at the point* $(2, 1)$. *And the absolute minimum of* $f(x, y)$ *is* $-\frac{30}{25}$ *and occurs at the point* $\left(-\frac{1}{5}, -\frac{4}{5}\right)$.

Example 2 *Find the absolute extrema of the function*

$$f(x, y) = x^2 + 3y^2$$

in the circle $D = \{(x, y) \in \mathbb{R}^2 \ / \ (x - 1)^2 + y^2 \leq 4\}$.

Solution: *The region* D *represents a circle centered at the point* $(1, 0)$ *and with radius* 2.

Step 1: Determine the critical points of z in D.

The system given by the nullity of the first partial derivatives of z *is*

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x}(x, y) = 2x = 0 \\ \dfrac{\partial f}{\partial y}(x, y) = 6y = 0 \end{array}\right\} \Rightarrow x = 0 \quad \text{and} \quad y = 0$$

So, the point $(0, 0)$ *is a critical point of* z. *Clearly, this point is in* D.

Step 2: Analize the boundary points of D.

In this case we do not have vertices. Then, we only analize the constraint extrema of $f(x, y)$ *at the circumference* $(x - 1)^2 + y^2 = 4$, *which can be calculated using Lagrange multiplyers. The lagrangian function is*

$$L(x, y, \lambda) = x^2 + 3y^2 + \lambda((x - 1)^2 + y^2 - 4)$$

we can conclude that the function has no global maximum or minimum. Observe that the value 0 *of the limit is never reached by the function: the graph approaches the line* $y = 0$ *(horizontal asymptote) but it never touches it.*

Example 6. *Fo the function of example* **??** *considered in the interval* $[1, +\infty[$*, as* $f(1) = 1$*, we conclude that there is a global maximum at* $x = 1$*, with value* 1 *and no global minimum. For the same function considered in the interval* $]0, 1]$ *we conclude that there is no global maximum, but a global minimum at* $x = 1$ *with value* 1*.*

Example 7. *For the function* $f(x) = 1/(x^2 + 1)$ *we find a local maximum at* $x = 0$ *with value* 1*. The limits as* $x \to \pm\infty$ *are* 0*, so we conclude that the function has a global maximum at* $x = 0$ *with value* 1*, but no global minimum.*

2 Functions of two variables

The strategy to be applied for functions of two variables is, in principle, the same, but we must keep into account the following facts.

1. The concept of increasing or decreasing function does not make sense, so we are forced to used the second derivatives test.
2. Computing limits for two variables functions is by no means easy and we have not considered this problem in our course. For this reason we'll be interested only in finding global maximum and minimum points only for functions whose domain is closed and bounded. In the case of two variables, as regards our course, the boundary of such a domain is usually a closed curve like a circumference or an ellipse, or a closed line consisting of a circumference pieces, of segments, of a parabola pieces.

This said, you can procede in a way similar to the case of one variable functions.

1. Compute the two first partial derivatives and check where they are both zero. All the points (stationary points) where these derivatives are both zero are "candidates" to be global maximum or minimum points. Usually you do not need to check whether they are maximum, minimum or saddle points (using the Hessian), unless this is specifically requested by the text of the problem.
2. Check the behaviour of the function at the boundary of the domain (a closed line as mentioned). This is usually the most difficult part of the problem and, in the cases of our interest, can be done using one of the following strategies.
   a) Use in a suitable way the equation of the boundary (or the equations of the boundary in case the are different pieces of lines) to obtain a function of one variable form the function of two variables, and then proceed as mentioned for functions of one variable.
   b) Use the Lagrangian multiplier method.

The Lagrangian multiplier method is, in principle, more general, but the first method is usually preferable, *when possible*. In this note we'll discuss only the first method using some examples.

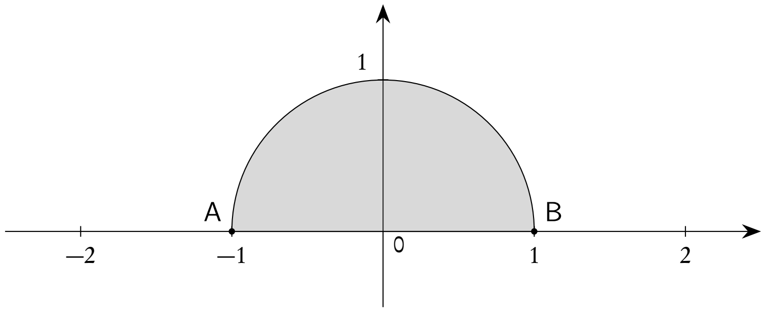
Example 8. *Find the global maximum and minimum of the function*

$$f(x, y) = x^2 - 8x + y^2 + 7$$

in the domain given by the following inequalities

$$\begin{cases} x^2 + y^2 \leq 1 \\ y \geq 0 \end{cases}.$$

Solution. Let's plot the domain.



Now we search the stationary points.

$$\begin{cases} f'_x(x,y) = 2x - 8 = 0 \\ f'_y(x,y) = 2y = 0 \end{cases}, \quad \Rightarrow \quad x = 4 \quad \text{and} \quad y = 0.$$

The only stationary point is $(4,0)$, that is outside our domain (so it is not important for our problem.

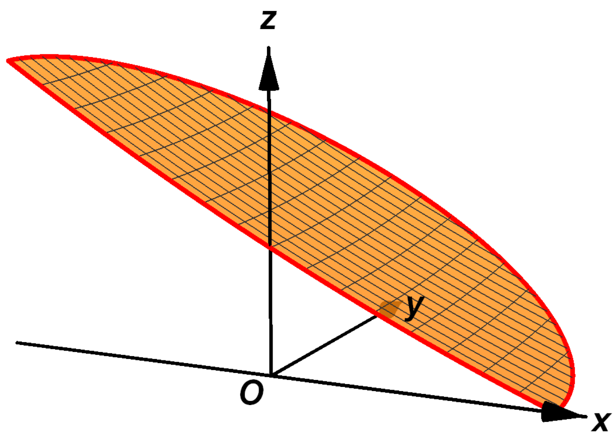
Now we check the boundary.

— Segment $\overline{AB}$. The equation of this segment is $y = 0$, with $-1 \le x \le 1$. By substitution into the function f we obtain the one variable function $g(x) = x^2 - 8x + 7$. This function has a global maximum at $x = -1$ and a global minimum at $x = 1$. These values correspond to the points $(-1,0)$ and $(1,0)$ of the plane. The values of the function g at -1 and 1 are, respectively, 16 and 0, and they are also the values of the function f at $(-1,0)$ and $(1,0)$.

— Arc $\widehat{AB}$. We can write the equation of this arc as $y^2 = 1 - x^2$, remembering, if needed, that $y \ge 0$. By substitution into the function f we obtain the one variable function $h(x) = -8x + 8$. This function has a global maximum at $x = -1$ and a global minimum at $x = 1$, with values 16 and 0 respectively. These points are exactly the same as those found in the segment $\overline{AB}$.

We can conclude that the function has 16 as global maximum and 0 as global minimum, in the given domain. □

Only for the sake of completeness[1], we plot the surface-graph of this function in the given domain (the units of measure are different for the three axes).



Example 9. *Find the global maximum and minimum of the function*

$$f(x,y) = 2x^2 - 8x + y^2 - 8y + 7$$

[1]You will not be requested to plot such a graph!

Solution We know that $\lambda = 0.53$, which tells us that $df/dc = 0.53$. The constraint corresponds to a budget of $\$3.78$ thousand. Therefore increasing the budget by $\$1000$ increases production by about 0.53 units. In order to make the increase in budget profitable, the extra goods produced must sell for more than $\$1000$. Thus, if p is the price of each unit of the good, then $0.53p$ is the revenue from the extra 0.53 units sold. Thus, we need $0.53p \geq 1000$ so $p \geq 1000/0.53 = \$1890$.

The Lagrangian Function

Constrained optimization problems are frequently solved using a *Lagrangian function*, $\mathcal{L}$. For example, to optimize $f(x, y)$ subject to the constraint $g(x, y) = c$, we use the Lagrangian function

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c).$$

To see how the function $\mathcal{L}$ is used, compute the partial derivatives of $\mathcal{L}$:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x}, \\
\frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y}, \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= -(g(x, y) - c).
\end{aligned}$$

Notice that if (x_0, y_0) is an extreme point of $f(x, y)$ subject to the constraint $g(x, y) = c$ and λ_0 is the corresponding Lagrange multiplier, then at the point (x_0, y_0, λ_0) we have

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial y} = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \lambda} = 0.$$

In other words, (x_0, y_0, λ_0) is a critical point for the unconstrained Lagrangian function, $\mathcal{L}(x, y, \lambda)$. Thus, the Lagrangian converts a constrained optimization problem to an unconstrained problem.

Example 4 A company has a production function with three inputs x, y, and z given by

$$f(x, y, z) = 50x^{2/5}y^{1/5}z^{1/5}.$$

The total budget is \$24,000 and the company can buy x, y, and z at \$80, \$12, and \$10 per unit, respectively. What combination of inputs will maximize production?[9]

Solution We need to maximize the objective function

$$f(x, y, z) = 50x^{2/5}y^{1/5}z^{1/5},$$

subject to the constraint

$$g(x, y, z) = 80x + 12y + 10z = 24{,}000.$$

The method for functions of two variables works for functions of three variables, so we construct the Lagrangian function

$$\mathcal{L}(x, y, z, \lambda) = 50x^{2/5}y^{1/5}z^{1/5} - \lambda(80x + 12y + 10z - 24{,}000),$$

[9] Adapted from M. Rosser, *Basic Mathematics for Economists*, p. 363 (New York: Routledge, 1993).

and solve the system of equations we get from grad $\mathcal{L} = \vec{0}$:

$$\frac{\partial \mathcal{L}}{\partial x} = 20x^{-3/5}y^{1/5}z^{1/5} - 80\lambda = 0,$$
$$\frac{\partial \mathcal{L}}{\partial y} = 10x^{2/5}y^{-4/5}z^{1/5} - 12\lambda = 0,$$
$$\frac{\partial \mathcal{L}}{\partial z} = 10x^{2/5}y^{1/5}z^{-4/5} - 10\lambda = 0,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(80x + 12y + 10z - 24{,}000) = 0.$$

Simplifying this system gives

$$\lambda = \frac{1}{4}x^{-3/5}y^{1/5}z^{1/5},$$
$$\lambda = \frac{5}{6}x^{2/5}y^{-4/5}z^{1/5},$$
$$\lambda = x^{2/5}y^{1/5}z^{-4/5},$$
$$80x + 12y + 10z = 24{,}000.$$

Eliminating z from the first two equations gives $x = 0.3y$. Eliminating x from the second and third equations gives $z = 1.2y$. Substituting for x and z into $80x + 12y + 10z = 24{,}000$ gives

$$80(0.3y) + 12y + 10(1.2y) = 24{,}000,$$

so $y = 500$. Then $x = 150$ and $z = 600$, and $f(150, 500, 600) = 4{,}622$ units.

The graph of the constraint, $80x + 12y + 10z = 24{,}000$, is a plane. Since the inputs x, y, z must be nonnegative, the graph is a triangle in the first octant, with edges on the coordinate planes. On the boundary of the triangle, one (or more) of the variables x, y, z is zero, so the function f is zero. Thus production is maximized within the budget using $x = 150$, $y = 500$, and $z = 600$.

Exercises and Problems for Section 15.3

Exercises

In Exercises 1–17, use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint, if such values exist.

1. $f(x, y) = x + y, \quad x^2 + y^2 = 1$

2. $f(x, y) = x + 3y + 2, \quad x^2 + y^2 = 10$

3. $f(x, y) = (x - 1)^2 + (y + 2)^2, \quad x^2 + y^2 = 5$

4. $f(x, y) = x^3 + y, \quad 3x^2 + y^2 = 4$

5. $f(x, y) = 3x - 2y, \quad x^2 + 2y^2 = 44$

6. $f(x, y) = 2xy, \quad 5x + 4y = 100$

7. $f(x_1, x_2) = {x_1}^2 + {x_2}^2, \quad x_1 + x_2 = 1$

8. $f(x, y) = x^2 + y, \quad x^2 - y^2 = 1$

9. $f(x, y, z) = x + 3y + 5z, \quad x^2 + y^2 + z^2 = 1$

10. $f(x, y, z) = x^2 - y^2 - 2z, \quad x^2 + y^2 = z$

11. $f(x, y, z) = xyz, \quad x^2 + y^2 + 4z^2 = 12$

12. $f(x, y) = x^2 + 2y^2, \quad x^2 + y^2 \le 4$

13. $f(x, y) = x + 3y, \quad x^2 + y^2 \le 2$

14. $f(x, y) = xy, \quad x^2 + 2y^2 \le 1$

15. $f(x, y) = x^3 + y, \quad x + y \ge 1$

16. $f(x, y) = (x + 3)^2 + (y - 3)^2, \quad x^2 + y^2 \le 2$

17. $f(x, y) = x^2y + 3y^2 - y, \quad x^2 + y^2 \le 10$

18. Decide whether each point appears to be a maximum, minimum, or neither for the function f constrained by the loop in Figure 15.30.

(a) P **(b)** Q **(c)** R **(d)** S

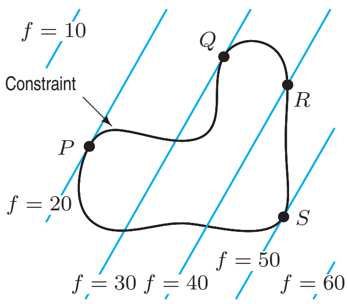


Figure 15.30