21th lesson

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Theory

Definition 1. Let f be a function defined on an open interval I. We say that a function $F\colon I \to \mathbb{R}$ is an *antiderivative of* f *on* I if for each $x \in I$ the derivative $F'(x)$ exists and $F'(x) = f(x)$.

Remarks 2. An antiderivative of f is sometimes called a function primitive to f.
If F is an antiderivative of f on I, then F is continuous on I.

Theorem 3 (Uniqueness of an antiderivative)**.** Let F and G be antiderivatives of f on an open interval I. Then there exists $c \in \mathbb{R}$ such that $F(x) = G(x) + c$ for each $x \in I$.

Theorem 4 (Linearity of antiderivatives)**.** Suppose that f has an antiderivative F on an open interval I, g has an antiderivative G on I, and let $\alpha, \beta \in \mathbb{R}$. Then the function $\alpha F + \beta G$ is an antiderivative of $\alpha f + \beta g$ on I.

Hints

$$x^{-a} = \frac{1}{x^a} \qquad \cos^2 x + \sin^2 x = 1$$

$$x^{a/b} = \sqrt[b]{x^a} \qquad a^b = e^{b \ln a} \qquad a^2 - b^2 = (a+b)(a-b)$$

Exercises

1. Find F - the antiderivative of a function f at the maximal (open) set. (Specify the set.)

   (a) $f(x) = x^{13}$
   (b) $f(x) = \sqrt{x}$
   (c) $f(x) = \dfrac{1}{x^3}$
   (d) $f(x) = \dfrac{1}{x}$
   (e) $f(x) = (1 + \sin x + \cos x)$
   (f) $f(x) = 7\sqrt[3]{x^2} + \dfrac{1}{2}\sin x - \dfrac{2}{1+x^2}$
   (g) $f(x) = \dfrac{2}{\cos^2 x} - e^x$
   (h) $f(x) = \dfrac{1}{\sqrt{1-x^2}} + \dfrac{1}{1+x^2} + 1 + x^2$
   (i) $f(x) = \sqrt{x^3} - \dfrac{1}{\sqrt{x}}$
   (j) $f(x) = \dfrac{3x^2 + 4x + 2}{3x}$
   (k) $f(x) = (1-x)(1-2x)(1-3x)$
   (l) $f(x) = \dfrac{x+1}{\sqrt{x}}$

2. Prove that if $F'(x) = f(x)$, then $\left(\frac{1}{a}F(ax+b) + C\right)' = f(ax+b)$, for $a \neq 0$.

3. Find F - the antiderivative of a function f at the maximal (open) set. (Specify the set.)

(a) $f(x) = \cos(3x)$

(b) $f(x) = \sin(2x - \pi)$

(c) $f(x) = e^{5-3x}$

(d) $f(x) = \dfrac{1}{1+4x^2}$

(e) $f(x) = \dfrac{1}{1-4x}$

(f) $f(x) = (2x+1)^7$

(g) $f(x) = e^{3x} + \dfrac{7}{x}$

(h) $f(x) = (e^{-x} + e^{-2x})$

(i) $f(x) = (3 - x^2)^3$

(j) $f(x) = (\sin 5x - \sin 5\alpha)$

(k) $f(x) = \dfrac{1}{x-2} + (3x+7)^5$

(l) $f(x) = \dfrac{1}{\sin^2\left(2x + \frac{\pi}{4}\right)}$

(m) $f(x) = \dfrac{-2}{\sqrt{1-2x^2}}$

(n) $f(x) = \dfrac{1}{x+A}$

4. Find F - the antiderivative of a function f at the maximal (open) set. (Specify the set.)

(a) $f(x) = \dfrac{e^{2x}-1}{e^x+1} + \dfrac{4}{1-\cos^2 x}$

(b) $f(x) = \dfrac{1}{\sqrt{4-(3x-1)^2}}$

(c) $f(x) = (1-\sqrt{x})^2$

(d) $f(x) = \tan^2 x$

(e) $f(x) = \dfrac{x^2}{1+x^2}$

(f) $f(x) = \dfrac{x^2+3}{x^2-1}$

(g) $f(x) = (2^x + 3^x)^2$

(h) $f(x) = \dfrac{1}{2+3x^2}$

(i) $f(x) = \operatorname{cotg}^2 x$

(j) $f(x) = \dfrac{1}{\sqrt{2-5x}}$

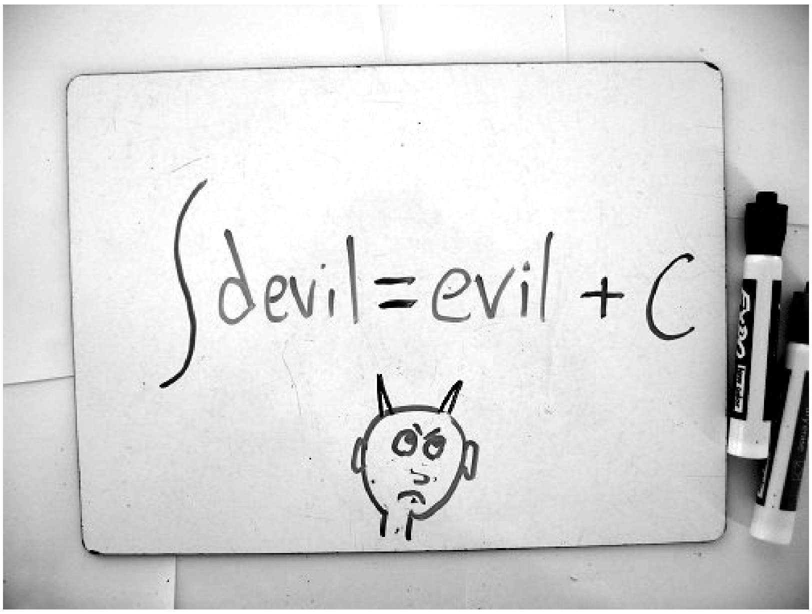
(k) $f(x) = \left(\dfrac{a}{x} + \dfrac{a^2}{x^2} + \dfrac{a^3}{x^3}\right)$, $a \in \mathbb{R}$

5. Find a function f such that $f'(x) = 6x(1-x)$ and $f(0) = 1$.

6. Find mistakes

(a) $\displaystyle\int x^2 e^x \,\mathrm{d}x = \frac{1}{3}x^3 e^x + c$

(b) $\displaystyle\int \frac{x}{\sqrt{1-x^2}} \,\mathrm{d}x = x \int \frac{1}{\sqrt{1-x^2}} \,\mathrm{d}x = x \arcsin x + c$



Source 1: https://mathwithbaddrawings.com/2013/05/27/calculus-joke/

(4a) $e^{2x} - 1 = (e^x - 1)(e^x + 1)$
(4b) $4 - (3x - 1)^2 = 1 - \left(\frac{3x-1}{2}\right)^2$
(4c) $\tan{}^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x}$