23rd lesson

`https://www2.karlin.mff.cuni.cz/~kuncova/en/teaching.php`,
kuncova@karlin.mff.cuni.cz

Theory

Theorem 1 (substitution)**.**

1. Let F be an antiderivative of f on (a,b). Let $\varphi\colon(\alpha,\beta)\to(a,b)$ have a finite derivative at each point of (α,β). Then

$$\int f\big(\varphi(x)\big)\varphi'(x)\,\mathrm{d}x \overset{C}{=} F\big(\varphi(x)\big) \quad \text{on } (\alpha,\beta).$$

2. Let φ be a function with a finite derivative in each point of (α,β) such that the derivative is either everywhere positive or everywhere negative, and such that $\varphi\big((\alpha,\beta)\big)=(a,b)$. Let f be a function defined on (a,b) and suppose that

$$\int f\big(\varphi(t)\big)\varphi'(t)\,\mathrm{d}t \overset{C}{=} G(t) \quad \text{on } (\alpha,\beta).$$

   Then

$$\int f(x)\,\mathrm{d}x \overset{C}{=} G\big(\varphi^{-1}(x)\big) \quad \text{on } (a,b).$$

Theorem 2 (integration by parts)**.** Let I be an open interval and let the functions f and g be continuous on I. Let F be an antiderivative of f on I and G an antiderivative of g on I. Then

$$\int f(x)G(x)\,\mathrm{d}x = F(x)G(x) - \int F(x)g(x)\,\mathrm{d}x \quad \text{on } I.$$

Remarks 3. Per partes can be expressed also as

$$\int u'(x)v(x)\,\mathrm{d}x = u(x)v(x) - \int u(x)v'(x)\,\mathrm{d}x \text{ na } I.$$

Remarks 4. Let $P(x)$ be a polynomial. The following table can help with choosing u' and v.

	$v(x)$	$u'(x)$
$P(x)\cdot e^{kx}$	$P(x)$	e^{kx}
$P(x)\cdot a^{kx}$	$P(x)$	a^{kx}
$P(x)\cdot\sin(kx)$	$P(x)$	$\sin(kx)$
$P(x)\cdot\cos(kx)$	$P(x)$	$\cos(kx)$

	$v(x)$	$u'(x)$
$P(x)\cdot\ln^n x$	$\ln^n x$	$P(x)$
$P(x)\cdot\arcsin\,(kx)$	$\arcsin\,(kx)$	$P(x)$
$P(x)\cdot\arccos\,(kx)$	$\arccos\,(kx)$	$P(x)$
$P(x)\cdot\arctan\,(kx)$	$\arctan\,(kx)$	$P(x)$
$P(x)\cdot\mathrm{arcctg}\,(kx)$	$\mathrm{arcctg}\,(kx)$	$P(x)$

Hints

$$\cot x = \frac{\cos x}{\sin x}$$

$$\frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1 - \sin^2 x}$$

$$\cos^3 x = \cos x \cdot \cos^2 x = \cos x(1 - \sin^2 x)$$

$$\frac{1}{x\sqrt{1+x^2}} = \frac{x}{x^2\sqrt{1+x^2}}$$

Exercises

Find F - the antiderivative of a function f at the maximal (open) set. (Specify the set.)

1. $\displaystyle\int \text{arctan } x \, \mathrm{d}x$
2. $\displaystyle\int \frac{1}{\cos x} \, \mathrm{d}x$
3. $\displaystyle\int \text{cotg } x \, \mathrm{d}x$
4. $\displaystyle\int x \ln \frac{1+x}{1-x} \, \mathrm{d}x$
5. $\displaystyle\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}} \, \mathrm{d}x$
6. $\displaystyle\int \frac{1}{\sqrt{x(1-x)}} \, \mathrm{d}x$
7. $\displaystyle\int x^2 e^{-2x} \, \mathrm{d}x$
8. $\displaystyle\int \frac{\cos^3 x}{\sin x} \, \mathrm{d}x$
9. $\displaystyle\int \frac{1}{x\sqrt{x^2+1}} \, \mathrm{d}x$
10. $\displaystyle\int x\text{arctan } x \, \mathrm{d}x$
11. $\displaystyle\int \ln(x + \sqrt{1+x^2}) \, \mathrm{d}x$
12. $\displaystyle\int \sin(\ln x) \, \mathrm{d}x$
13. $\displaystyle\int x^n \ln x \, \mathrm{d}x, \; n \neq -1$
14. $\displaystyle\int e^{ax} \sin bx \, \mathrm{d}x$

$\underline{x}/\!\wedge = \hslash$ (9)