23rd lesson

Exercises

Find F - the antiderivative of a function f at the maximal (open) set. (Specify the set.)

1. $\displaystyle\int \arctan x \,\mathrm{d}x$

   Solution:

   Per partes: $u' = 1$, $u = x$, $v = \arctan x$, $v' = \frac{1}{1+x^2}$.

   $$\int 1 \cdot \arctan x \, dx = [x\arctan x] - \int \frac{x}{1+x^2} \, dx =$$

   Substitution $y = 1 + x^2$.

   $$= x\arctan x - \frac{1}{2}\int \frac{1}{y} \, dy \stackrel{C}{=} x\arctan x - \frac{1}{2}\ln|y| = x\arctan x - \frac{1}{2}\ln(1+x^2)$$

   $x \in \mathbb{R}$

2. $\displaystyle\int \frac{1}{\cos x} \,\mathrm{d}x$

   Solution: Substitution $y = \sin x$. Then $dy = \cos x \, dx$ and we have

   $$\int \frac{1}{\cos x} \, dx = \int \frac{\cos x}{\cos^2 x} \, dx = \int \frac{\cos x}{1 - \sin^2 x} \, dx = \int \frac{dy}{1-y^2} \stackrel{C}{=} \frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| =$$

   $$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| = \ln\left|\mathrm{tg}\ \frac{x}{2} + \frac{\pi}{4}\right|$$

   $x \in (-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$, $k \in \mathbb{Z}$.

3. $\displaystyle\int \mathrm{cotg}\ x \,\mathrm{d}x$

   Solution: Substitution $y = \sin x$. Then $dy = \cos x \, dx$ and we have

   $$\int \mathrm{cotg}\ x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{dy}{y} \stackrel{C}{=} \ln|y| = \ln|\sin x|$$

   $x \in (0 + k\pi, \pi + k\pi)$, $k \in \mathbb{Z}$.

4. $\displaystyle\int x \ln \frac{1+x}{1-x} \,\mathrm{d}x$

   Solution: Per partes: $u' = x$, $u = \frac{x^2}{2}$, $v = \ln\frac{1+x}{1-x}$, $v' = \frac{2}{1-x^2}$.

   $$\int x \ln\frac{1+x}{1-x} \, dx = \frac{1}{2}x^2 \ln\frac{1+x}{1-x} - \int \frac{x^2}{1-x^2} \, dx = \frac{1}{2}x^2 \ln\frac{1+x}{1-x} - \int \frac{x^2 - 1 + 1}{1-x^2} \, dx =$$

   $$= \frac{1}{2}x^2 \ln\frac{1+x}{1-x} - \int \left(-1 + \frac{1}{1-x^2}\right) dx \stackrel{C}{=} \frac{1}{2}x^2 \ln\frac{1+x}{1-x} + x - \frac{1}{2}\ln\frac{1+x}{1-x}$$

5. $\displaystyle\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}}\, \mathrm{d}x$

Solution: Substitution $y = \sin x - \cos x$. Then $dy = \cos x + \sin x$ and we have

$$\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}}\, dx = \int \frac{dy}{y^{1/3}} \overset{C}{=} \frac{3}{2} y^{2/3} = \frac{3}{2}\sqrt[3]{(\sin x - \cos x)^2} = \frac{3}{2}\sqrt[3]{1 - \sin 2x}$$

$x \in (\frac{\pi}{4} + k\pi, \frac{5\pi}{4} + k\pi)$, $k \in \mathbb{Z}$.

6. $\displaystyle\int \frac{1}{\sqrt{x(1-x)}}\, \mathrm{d}x$

Solution: Substitution $y = \sqrt{x}$. Then $y^2 = x$, $dy = \frac{1}{2\sqrt{x}}\, dx$ and we have

$$\int \frac{1}{\sqrt{x(1-x)}}\, dx = 2\int \frac{1}{\sqrt{1-x}} \frac{dx}{2\sqrt{x}} = 2\int \frac{1}{\sqrt{1-y^2}}\, dy \overset{C}{=} \arcsin y = \arcsin \sqrt{x}$$

$x \in (0, 1)$.

7. $\displaystyle\int x^2 e^{-2x}\, \mathrm{d}x$

Solution:

Per partes: $u' = e^{-2x}$, $u = -\frac{1}{2}e^{-2x}$, $v = x^2$, $v' = 2x$.

$$\int x^2 e^{-2x}\, dx = \left[-\frac{1}{2}x^2 e^{-2x}\right] + \int x e^{-2x}\, dx =$$

Second per partes: $u' = e^{-2x}$, $u = -\frac{1}{2}e^{-2x}$, $v = x$, $v' = 1$.

$$= -\frac{1}{2}x^2 e^{-2x} + \left[-\frac{1}{2}x e^{-2x}\right] + \frac{1}{2}\int e^{-2x}\, dx \overset{C}{=} -\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}x e^{-2x} - \frac{1}{4}\int e^{-2x}$$

8. $\displaystyle\int \frac{\cos^3 x}{\sin x}\, \mathrm{d}x$

Solution: Substitution $y = \sin x$. Then

$$\int \frac{\cos^3 x}{\sin x}\, dx = \int \frac{\cos^2 x\, \cos x}{\sin x}\, dx = \int \frac{(1 - \sin^2 x)}{\sin x} \cos x\, dx =$$

$$= \int \frac{1 - y^2}{y}\, dy = \int \frac{1}{y}\, dy - \int y\, dy \overset{C}{=} \ln|y| - \frac{y^2}{2} = \ln|\sin x| - \frac{\sin^2 x}{2}$$

$x \in (0 + k\pi, \pi + k\pi)$, $k \in \mathbb{Z}$.

9. $\displaystyle\int \frac{1}{x\sqrt{x^2+1}}\,\mathrm{d}x$

Solution: Substitution $y=\sqrt{x^2+1}$. Then $dy = \frac{x}{\sqrt{x^2+1}}\,dx$ a $y^2-1=x^2$ and we have

$$\int \frac{1}{x\sqrt{x^2+1}}\,dx = \int \frac{1}{x^2}\frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{1}{y^2-1}\,dy = -\int \frac{1}{1-y^2}\,dy \stackrel{C}{=} -\frac{1}{2}\ln\frac{1+y}{1-y} =$$

$$-\frac{1}{2}\ln\frac{1+\sqrt{x^2+1}}{1-\sqrt{x^2+1}}$$

$x\in(0,\infty)$ and $x\in(-\infty,0)$.

10. $\displaystyle\int x\mathrm{arctan}\ x\,\mathrm{d}x$

Solution: Per partes: $u'=x$, $u=\frac{x^2}{2}$, $v=\mathrm{arctan}\ x$, $v'=\frac{1}{1+x^2}$.

$$\int x\mathrm{arctan}\ x\,dx = \frac{1}{2}x^2\mathrm{arctan}\ x - \frac{1}{2}\int\frac{x^2}{1+x^2}\,dx = \frac{1}{2}x^2\mathrm{arctan}\ x - \frac{1}{2}\int\frac{x^2+1-1}{1+x^2}\,dx =$$

$$= \frac{1}{2}x^2\mathrm{arctan}\ x - \frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)\,dx \stackrel{C}{=} \frac{1}{2}x^2\mathrm{arctan}\ x - \frac{1}{2}x + \frac{1}{2}\mathrm{arctan}\ x$$

$x\in\mathbb{R}$

11. $\displaystyle\int \ln(x+\sqrt{1+x^2})\,\mathrm{d}x$

Solution: Per partes: $u'=1$, $u=x$, $v=\ln(x+\sqrt{1+x^2})$, $v'=\frac{1}{\sqrt{1+x^2}}$.

$$\int 1\cdot\ln\left(x+\sqrt{1+x^2}\right)\,dx = \left[x\ln\left(x+\sqrt{1+x^2}\right)\right] - \int\frac{x}{\sqrt{1+x^2}}\,dx \stackrel{C}{=}$$

$$x\ln\left(x+\sqrt{1+x^2}\right) - \sqrt{1+x^2}$$

Apply substitution $y=1+x^2$ on the last integral.

$x\in\mathbb{R}$

12. $\displaystyle\int \sin(\ln x)\,\mathrm{d}x$

Solution: Per partes $v'=1$, $u=\sin(\ln x)$. Then $v=x$ and $u'=\cos(\ln x)\cdot\frac{1}{x}$. We have

$$\int 1\cdot\sin(\ln x) = x\sin(\ln x) - \int\cos(\ln x)\,dx =$$

Let us use per partes again $v'=1$ and $u=\cos(\ln x)$ and we have

$$\int 1\cdot\sin(\ln x) = x\sin(\ln x) - \int\cos(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) - \int\sin(\ln x)$$

Put the integral to the left-hand side

$$2\int 1\cdot\sin(\ln x) \overset{C}{=} x\sin(\ln x) - x\cos(\ln x)$$

$$\int \sin(\ln x) \overset{C}{=} \frac{1}{2}x(\sin(\ln x) - \cos(\ln x))$$

13. $\int x^n \ln x \,\mathrm{d}x,\ n \neq -1$

Solution:

Per partes $u' = x^n$, $v = \ln x$. Then $u = x^{n+1}/n+1$ and $v' = \frac{1}{x}$. Then we have

$$\int x^n \ln x\, dx = \frac{x^{n+1}}{n+1}\ln x - \int \frac{x^n}{n+1}\, dx \overset{C}{=} \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2}$$

14. $\int e^{ax}\sin bx \,\mathrm{d}x$

Solution:

Let $b = 0$, then$\int e^{ax}\sin(0x)\ \mathrm{d}x = \int 0\ \mathrm{d}x \overset{C}{=} 1$.

Now let us assume $a \neq 0$, $b \neq 0$. We apply per partes (twice), $u' = e^{ax}$, $v = \sin(bx)$ (for the second per partes $u' = e^{ax}$ and $v = \cos(bx)$.

$$\int e^{ax}\sin bx\ \mathrm{d}x = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\ \mathrm{d}x =$$

$$-\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\ \mathrm{d}x.$$

Putting integral to the left-hand side we obatin

$$\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\sin bx\ \mathrm{d}x \overset{C}{=} -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx$$

$$\int e^{ax}\sin bx\ \mathrm{d}x \overset{C}{=} -\frac{b}{a^2+b^2}e^{ax}\cos bx + \frac{a}{a^2+b^2}e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)$$

It is easy to verify that the result holds also for $a = 0$, $b \neq 0$.