24th lesson

http://www.karlin.mff.cuni.cz/~kuncova/,

1. $f(x) = \dfrac{x}{(x+1)(x+2)(x+3)}$

   Solution:

   We are looking for the decomposion in the form

$$\frac{x}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

   If we multiply both sides, we obtain

$$x = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

   Let us set $x = -1,\ -2,\ -3$. Then we get

$$-1 = 2A \implies A = -\frac{1}{2}$$

$$-2 = -B \implies B = 2$$

$$-3 = 2C \implies C = -\frac{3}{2}$$

   Thus

$$\int \frac{x}{(x+1)(x+2)(x+3)}\,dx = \int \frac{-1}{2}\frac{1}{x+1}\,dx + \int \frac{2}{x+2}\,dx + \int \frac{-3}{2}\frac{1}{x+3}\,dx \overset{C}{=}$$

$$\overset{C}{=} -\frac{1}{2}\ln|x+1| + 2\ln|x+2| - \frac{3}{2}\ln|x+3| = \frac{1}{2}\ln\left|\frac{(x+2)^4}{(x+1)(x+3)^3}\right|$$

2. $f(x) = \dfrac{x}{x^3-1}$

   Solution:

   We have $x^3 - 1 = (x-1)(x^2+x+1)$, whereas the second bracket has no real roots. We are looking for the decomposition:

$$\frac{x}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

   Let us multiply

$$x = A(x^2+x+1) + (Bx+C)(x-1)$$

   For $x = 1$ we get $A = \frac{1}{3}$.

   If we set $A = 1/3$, we obtain

$$x = \frac{1}{3}x^2 + \frac{1}{3}x + \frac{1}{3} + Bx^2 - Bx + Cx - C,$$

which implies $C = \frac{1}{3}$ (constants) and $B = -\frac{1}{3}$ (coefficients with x^2). Thus, the decomposition is of the form

$$\frac{x}{x^3-1} = \frac{1}{3}\frac{1}{x-1} - \frac{1}{3}\frac{x-1}{x^2+x+1}$$

Then

$$\int \frac{1}{3}\frac{1}{x-1}\,\mathrm{d}x \overset{C}{=} \frac{1}{3}\ln|x-1|$$

$$\int \frac{1}{3}\frac{x-1}{x^2+x+1}\,\mathrm{d}x = \int \frac{1}{6}\frac{2x+1}{x^2+x+1}\,\mathrm{d}x - \int \frac{1}{2}\frac{1}{x^2+x+1}\,\mathrm{d}x - \overset{C}{=}$$
$$\frac{1}{6}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\text{arctan}\,\frac{2x+1}{\sqrt{3}}$$

Together

$$\int \frac{x}{x^3-1}\,\mathrm{d}x \overset{C}{=} \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) + \frac{1}{\sqrt{3}}\text{arctan}\,\frac{2x+1}{\sqrt{3}}$$

3. $f(x) = \dfrac{x^3+1}{x^3-5x^2+6x}$

Solution:

The degrees of the polynomials are equal - we need to divide polynomials. However, there is an easier way:

$$\frac{x^3+1}{x^3-5x^2+6x} = \frac{(x^3-5x^2+6x)+5x^2-6x+1}{x^3-5x^2+6x} = 1 + \frac{5x^2-6x+1}{x^3-5x^2+6x}$$

Now we can use the partial fraction decomposition

$$= 1 + \frac{5x^2-6x+1}{x(x-2)(x-3)} = 1 + \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}$$

Let us multiply both sides with the denominator

$$5x^2-6x+1 = A(x-2)(x-3) + Bx(x-3) + Cx(x-2)$$

Let us set $x=0$, $x=2$ and $x=3$. We obtain

$$1 = 6A \implies A = \frac{1}{6}$$

$$9 = -2B \implies B = -\frac{9}{2}$$

$$45 - 18 + 1 = 3C \implies C = \frac{28}{3}$$

Hence

$$\int \frac{x^3+1}{x^3-5x^2+6x}\,dx = \int\left(1 + \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}\right)dx \overset{C}{=} x + \frac{1}{6}\ln|x| - \frac{9}{2}\ln|x-2| + \frac{28}{3}\ln|x-3|$$

4. $f(x) = \dfrac{x^4}{x^4+5x^2+4}$

Solution:

We have

$$\frac{x^4}{x^4+5x^2+4} = \frac{(x^4+5x^2+4)-5x^2-4}{x^4+5x^2+4} = 1 - \frac{5x^2+4}{x^4+5x^2+4} = 1 - \frac{5x^2+4}{(x^2+1)(x^2+4)}$$

The partial decomposition should be in the form

$$\frac{5x^2+4}{(x^2+1)(x^2+4)} = \frac{A+Bx}{x^2+1} + \frac{C+Dx}{x^2+4}$$

However, since there is no x (just x^2), we can consider

$$\frac{5x^2+4}{(x^2+1)(x^2+4)} = \frac{A}{x^2+1} + \frac{C}{x^2+4}$$

It can be helpful to write t instead of x^2. (This is not the integral substitution, only auxiliary calculation.)

$$\frac{5t+4}{(t+1)(t+4)} = \frac{A}{t+1} + \frac{C}{t+4}$$

If we multiply both sides, we obtain

$$5t+4 = A(t+4) + C(t+1)$$

Let us set $t=-4$ and $t=-1$. Then we get

$$-16 = -3C \implies C = \frac{16}{3}$$

$$-1 = 3A \implies A = -\frac{1}{3}$$

Together

$$\frac{5t+4}{(t+1)(t+4)} = -\frac{1}{3}\frac{1}{t+1} + \frac{16}{3}\frac{1}{t+4}$$

and

$$\frac{5x^2+4}{(x^2+1)(x^2+4)} = -\frac{1}{3}\frac{1}{x^2+1} + \frac{16}{3}\frac{1}{x^2+4}$$

Now let us integrate

$$\int \frac{x^4}{x^4+5x^2+4}\,dx = \int \left(1 + \frac{1}{3}\frac{1}{x^2+1} - \frac{16}{3}\cdot\frac{1}{4}\frac{1}{(x/2)^2+1}\right) dx \overset{C}{=}$$

$$\overset{C}{=} x + \frac{1}{3}\text{arctan } x - \frac{4}{3}\cdot\frac{1}{1/2}\text{arctan } \frac{x}{2} = x + \frac{1}{3}\text{arctan } x - \frac{8}{3}\text{arctan } \frac{x}{2}$$

5. $f(x) = \dfrac{x^2+1}{(x+1)^2(x-1)}$

   Solution:

   The partial fraction decomposion

$$\frac{x^2+1}{(x+1)^2(x-1)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x-1}$$

   After multiplying by the denominator

$$x^2+1 = A(x-1) + B(x+1)(x-1) + C(x+1)^2$$

   Let us set $x = 1$ and $x = -1$, then we have

$$2 = 4C \implies C = \frac{1}{2}$$

$$2 = -2A \implies A = -1$$

   Let us set $x = 0$:

$$1 = -A - B + C \implies B = -A + C - 1 = -(-1) + \frac{1}{2} - 1 = \frac{1}{2}$$

   Hence

$$\int \frac{x^2+1}{(x+1)^2(x-1)}\,dx = \int \left(\frac{-1}{(x+1)^2} + \frac{1}{2}\cdot\frac{1}{x+1} + \frac{1}{2}\cdot\frac{1}{x-1} \right) dx \overset{C}{=}$$

$$\overset{C}{=} \frac{1}{x+1} + \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| = \frac{1}{x+1} + \frac{1}{2}\ln|x^2-1|$$

6. $f(x) = \dfrac{1}{x(1+x)(1+x+x^2)}$

   Solution: Quadratic polynomial $x^2+x+1 = (x+\frac{1}{2})^2 + \frac{3}{4}$ has no real roots. Therefore we are looking for the partial fraction decomposion as

$$\frac{1}{x(1+x)(1+x+x^2)} = \frac{A}{x} + \frac{B}{1+x} + \frac{Cx+D}{x^2+x+1}$$

   Let us multiply by the denominator

$$1 = A(1+x)(1+x+x^2) + Bx(1+x+x^2) + (Cx+D)x(x+1)$$

   Setting $x = 0$ we obtain $A = 1$. Further, let $x = -1$, then $B = -1$. Setting A and B gives

$$1 = (1+x)(1+x+x^2) - x(1+x+x^2) + (Cx+D)x(x+1)$$

$$1 = 1 + x + x^2 + Cx^3 + Cx^2 + \mathrm{d}x^2 + \mathrm{d}x$$

which implies $D = -1$ and $C = 0$. Thus

$$\frac{1}{x(1+x)(1+x+x^2)} = \frac{1}{x} - \frac{1}{1+x} - \frac{1}{x^2+x+1}$$

Let us integrate

$$\int \frac{1}{x(1+x)(1+x+x^2)}\,\mathrm{d}x = \ln|x| - \ln|1+x| - \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,\mathrm{d}x \stackrel{C}{=}$$

$$\stackrel{C}{=} \ln\left|\frac{x}{1+x}\right| - \sqrt{\frac{4}{3}}\arctan\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}} = \ln\left|\frac{x}{1+x}\right| - \frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}$$

7. $f(x) = \left(\frac{x}{x^2-3x+2}\right)^2$

Solution:

At first we need to decompose the denominator

$$x^2-3x+2 = (x-1)(x-2)$$

Therefore we are looking for the decomposition of

$$\frac{x^2}{(x-1)^2(x-2)^2}$$

It is in the form of

$$\frac{x^2}{(x-1)^2(x-2)^2} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{(x-2)^2} + \frac{D}{x-2}$$

Let us multiply both sides

$$x^2 = A(x-2)^2 + B(x-1)(x-2)^2 + C(x-1)^2 + D(x-2)(x-1)^2$$

and set $x = 1$ and $x = 2$. We obtain

$$1 = A, \qquad 4 = C$$

Further, let us set $x = 0$ and $x = 3$. We get

$$0 = 4A - 4B + C - 2D = 4 - 4B + 4 - 2D \implies -8 = -4B - 2D$$

$$9 = A + 2B + 4C + 4D = 1 + 2B + 16 + 4D \implies -8 = 2B + 4D$$

It gives $D = -4$ and $B = 4$.

Hence

$$\int \left(\frac{x}{x^2-3x+2}\right)^2 \mathrm{d}x = \int \left(\frac{1}{(x-1)^2} + \frac{4}{x-1} + \frac{4}{(x-2)^2} - \frac{4}{x-2}\right) \mathrm{d}x \stackrel{C}{=}$$

$$\stackrel{C}{=} -\frac{1}{x-1} + 4\ln|x-1| - \frac{4}{x-2} - 4\ln|x-2| = -\frac{x-2+4(x-1)}{(x-1)(x-2)} + 4\ln\left|\frac{x-1}{x-2}\right| =$$

$$-\frac{5x-6}{x^2-3x+2} + 4\ln\left|\frac{x-1}{x-2}\right|$$

8. $f(x) = \dfrac{1}{x^3+1}$

Solution:

It holds $x^3+1 = (x+1)(x^2-x+1) = (x+1)((x-\frac{1}{2})^2+\frac{3}{4})$. The second bracket has no real roots, hence we are looking for

$$\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

After multiplying

$$1 = A(x^2-x+1) + (Bx+C)(x+1)$$

set $x=-1$, then we have $A=\frac{1}{3}$. It gives

$$1 = \frac{1}{3}(x^2-x+1) + Bx^2 + Bx + Cx + C$$

which implies $C=\frac{2}{3}$ (the constants) and $B=-\frac{1}{3}$ (coefficients with x^2). Together

$$\frac{1}{x^3+1} = \frac{1}{3}\frac{1}{x+1} - \frac{1}{3}\frac{x-2}{x^2-x+1}$$

Integration

$$\int \frac{1}{3}\frac{1}{x+1}\,\mathrm{d}x \stackrel{C}{=} \frac{1}{3}\ln|x+1|$$

$$\int \frac{1}{3}\frac{x-2}{x^2-x+1}\,\mathrm{d}x = \int \frac{1}{6}\frac{2x-1}{x^2-x+1}\,\mathrm{d}x - \int \frac{1}{2}\frac{1}{x^2-x+1}\,\mathrm{d}x \stackrel{C}{=}$$

$$\stackrel{C}{=} \frac{1}{6}\ln(x^2-x+1) - \frac{1}{\sqrt{3}}\operatorname{arctan}\frac{2x-1}{\sqrt{3}}$$

Finally

$$\int \frac{1}{x^3+1}\,\mathrm{d}x \stackrel{C}{=} \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\operatorname{arctan}\frac{2x-1}{\sqrt{3}}$$