25th lesson

`https://www2.karlin.mff.cuni.cz/~kuncova/en/teaching.php`,
kuncova@karlin.mff.cuni.cz

Theory

Theorem 1 (Newton-Leibniz formula)**.** Let f be a function continuous on an interval $(a-\varepsilon, b+\varepsilon)$, $a, b \in \mathbb{R}$, $a < b$, $\varepsilon > 0$ and let F be an antiderivative of f on $(a-\varepsilon, b+\varepsilon)$. Then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{1}$$

Theorem 2 (integration by parts)**.** Suppose that the functions f, g, f' a g' are continuous on an interval $[a, b]$. Then

$$\int_a^b f'g = [fg]_a^b - \int_a^b fg'.$$

Theorem 3 (substitution)**.** Let the function f be continuous on an interval $[a, b]$. Suppose that the function φ has a continuous derivative on $[\alpha, \beta]$ and φ maps $[\alpha, \beta]$ into the interval $[a, b]$. Then

$$\int_\alpha^\beta f\big(\varphi(x)\big)\varphi'(x)\,\mathrm{d}x = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,\mathrm{d}t.$$

Exercises

1. Compute the definite integrals:

   (a) $\displaystyle\int_0^\pi \sin x\,\mathrm{d}x$

   (b) $\displaystyle\int_1^2 3x^2 + 2x + 1\,\mathrm{d}x$

   (c) $\displaystyle\int_1^2 2 + \sqrt{x} + \frac{1}{x^2}\,\mathrm{d}x$

   (d) $\displaystyle\int_{-5}^0 \frac{2}{3-4x}\,\mathrm{d}x$

   (e) $\displaystyle\int_{-7}^{-2} \frac{1}{\sqrt{2-x}}\,\mathrm{d}x$

   (f) $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x$

   (g) $\displaystyle\int_2^\infty \frac{1}{x}\,\mathrm{d}x$

   (h) $\displaystyle\int_{-\infty}^0 e^x\,\mathrm{d}x$

   (i) $\displaystyle\int_0^\infty e^x\,\mathrm{d}x$

   (j) $\displaystyle\int_0^\infty \sin x\,\mathrm{d}x$

2. Compute the definite integrals:

(a) $f(x) = \begin{cases} 3, & x < 1 \\ x^3, & x \geq 1 \end{cases}$

i. $\int_{-2}^{0} f(x)$ ii. $\int_{2}^{4} f(x)$ iii. $\int_{-3}^{2} f(x)$ iv. $\int_{-3}^{1} f(x)$

(b) $\int_{-2}^{2} f$,

$$f(x) = \begin{cases} \frac{3}{x}, & x \leq -1 \\ 3 + 4x, & -1 < x < 2 \\ 11e^{2x-4}, & 2 \leq x \end{cases}$$

(c) $\int_{-4}^{2} |x| \, \mathrm{d}x$

(d) $\int_{-2}^{3} \sqrt{x^6} \, \mathrm{d}x$

(e) $\int_{-\pi}^{\frac{3\pi}{2}} |\sin x| \, \mathrm{d}x$

3. Compute the definite integrals:

(a) $\int_{1}^{2} \frac{3x^2}{x^3 + 1} \, \mathrm{d}x$

(b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin x \cos x \, \mathrm{d}x$

(c) $\int_{1}^{2} x \ln x \, \mathrm{d}x$

(d) $\int_{0}^{\pi} x^2 \sin x \, \mathrm{d}x$

(e) $\int_{1}^{e} \frac{\ln^2 x}{x} \, \mathrm{d}x$

(f) $\int_{-1}^{1} \frac{x^2}{1 + x^2} \, \mathrm{d}x$

(g) $\int_{0}^{\infty} \frac{1}{(x + 3)^5}$

(h) $\int_{0}^{1} \frac{e^x}{e^{2x} + 1} + \frac{1}{\cos^2 x} \, \mathrm{d}x$

(i) $\int_{1}^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} \, \mathrm{d}x$

(j) $\int_{a}^{b} \mathrm{sgn} x \, \mathrm{d}x$, $a < 0$, $b > 0$

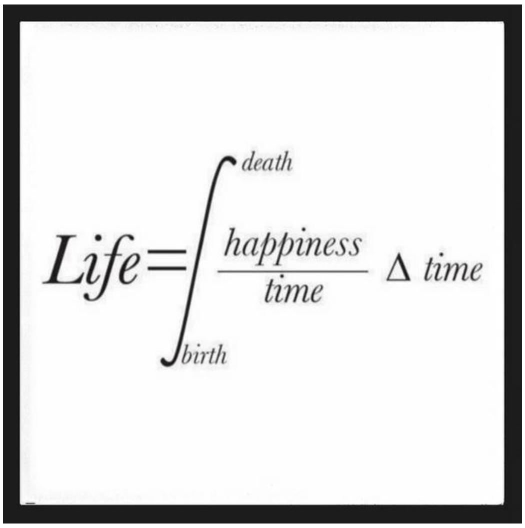
(k) $\int_{1}^{\infty} \frac{\arctan x}{1 + x^2} \, \mathrm{d}x$

(l) $\int_{1}^{2} \frac{1}{x \ln x} \, \mathrm{d}x$

(m) $\int_{0}^{\pi} \frac{\sin x}{\cos^2 x + 1} \, \mathrm{d}x$

(n) $\int_{-1}^{1} x^2 e^{-x} \, \mathrm{d}x$

(o) $\int_{2}^{3} \frac{x^2 - x + 1}{x - 1} \, \mathrm{d}x$


Life = ∫_birth^death happiness/time Δ time