25th lesson

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Theory

Theorem 1 (Newton-Leibniz formula)**.** Let f be a function continuous on an interval $(a-\varepsilon, b+\varepsilon)$, $a, b \in \mathbb{R}$, $a < b$, $\varepsilon > 0$ and let F be an antiderivative of f on $(a-\varepsilon, b+\varepsilon)$. Then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{1}$$

Theorem 2 (integration by parts)**.** Suppose that the functions f, g, f' a g' are continuous on an interval $[a, b]$. Then

$$\int_a^b f'g = [fg]_a^b - \int_a^b fg'.$$

Theorem 3 (substitution)**.** Let the function f be continuous on an interval $[a, b]$. Suppose that the function φ has a continuous derivative on $[\alpha, \beta]$ and φ maps $[\alpha, \beta]$ into the interval $[a, b]$. Then

$$\int_\alpha^\beta f\big(\varphi(x)\big)\varphi'(x)\,\mathrm{d}x = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,\mathrm{d}t.$$

Exercises

1. Compute the definite integrals:

   (a) $\displaystyle\int_0^\pi \sin x\,\mathrm{d}x$

   Solution:

   $$\int_0^\pi \sin x\,\mathrm{d}x = [-\cos x]_0^\pi = -\cos(\pi) - (-\cos(0)) = -(-1) + 1 = 2.$$

   (b) $\displaystyle\int_1^2 3x^2 + 2x + 1\,\mathrm{d}x$

   Solution:

   $$\int_1^2 3x^2 + 2x + 1\,\mathrm{d}x = [x^3 + x^2 + x]_1^2 = (2^3 + 2^2 + 2^1) - (1^3 - 1^2 - 1^1) = 11,$$

   (c) $\displaystyle\int_1^2 2 + \sqrt{x} + \frac{1}{x^2}\,\mathrm{d}x$

   Solution:

   $$\int_1^2 2+\sqrt{x}+\frac{1}{x^2}\,\mathrm{d}x = \left[2x + \frac{2}{3}\sqrt{x^3} - \frac{1}{x}\right]_1^2 = 2\cdot 2+\frac{2}{3}\sqrt{2^3}-\frac{1}{2}-\left(2\cdot 1 + \frac{2}{3}\sqrt{1^3} - \frac{1}{1}\right) = \frac{11}{6}+\frac{4\sqrt{2}}{3}.$$

(d) $\displaystyle\int_{-5}^{0} \frac{2}{3-4x}\,\mathrm{d}x$

Solution:

$$\int_{-5}^{0} \frac{2}{3-4x}\,\mathrm{d}x = \left[2\frac{\ln|3-4x|}{-4}\right]_{-5}^{0} = -\frac{1}{2}(\ln 3 - \ln 23)$$

(e) $\displaystyle\int_{-7}^{-2} \frac{1}{\sqrt{2-x}}\,\mathrm{d}x$

Solution:

$$\int_{-7}^{-2} \frac{1}{\sqrt{2-x}}\,\mathrm{d}x = \left[-2\sqrt{2-x}\right]_{-7}^{-2} = -2(2-3) = 2.$$

(f) $\displaystyle\int_{0}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x$

Solution:

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x = [\arctan x]_0^{\infty} = \lim_{x\to\infty} \arctan x - \lim_{x\to 0+} \arctan x = \frac{\pi}{2} - 0.$$

(g) $\displaystyle\int_{2}^{\infty} \frac{1}{x}\,\mathrm{d}x$

Solution:

$$\int_{2}^{\infty} \frac{1}{x}\,\mathrm{d}x = [\ln x]_2^{\infty} = \lim_{x\to\infty} \ln x - \lim_{x\to 2+} \ln x = \infty - \ln 2 = \infty.$$

(h) $\displaystyle\int_{-\infty}^{0} e^x\,\mathrm{d}x$

Solution:

$$\int_{-\infty}^{0} e^x\,\mathrm{d}x = [e^x]_{-\infty}^{0} = \lim_{x\to 0-} e^x - \lim_{x\to -\infty} e^x = 1 - 0 = 1.$$

(i) $\displaystyle\int_{0}^{\infty} e^x\,\mathrm{d}x$

Solution:

$$\int_{0}^{\infty} e^x\,\mathrm{d}x = [e^x]_0^{\infty} = \lim_{x\to\infty} e^x - \lim_{x\to 0+} e^x = \infty - 1 = \infty.$$

(j) $\displaystyle\int_{0}^{\infty} \sin x\,\mathrm{d}x$

Solution:

$$\int_{0}^{\infty} \sin x\,\mathrm{d}x = [-\cos x]_0^{\infty} = \lim_{x\to\infty}(-\cos x) - \lim_{x\to 0+}(-\cos x)$$

Since the first limit does not exist, the integral $\int_0^\infty \sin x\,\mathrm{d}x$ does not exist either.

2. Compute the definite integrals:

   (a) $f(x) = \begin{cases} 3, & x < 1 \\ x^3, & x \geq 1 \end{cases}$

      i. $\int_{-2}^{0} f(x)$
      Solution:
$$\int_{-2}^{0} f(x) = \int_{-2}^{0} 3 \, \mathrm{d}x = [3x]_{-2}^{0} = 6.$$

      ii. $\int_{2}^{4} f(x)$
      Solution:
$$\int_{2}^{4} f(x) = \int_{2}^{4} x^3 = \left[\frac{x^4}{4}\right]_2^4 = 64 - 4 = 60.$$

      iii. $\int_{-3}^{2} f(x)$
      Solution:
$$\int_{-3}^{2} f(x) = \int_{-3}^{1} 3 \, \mathrm{d}x + \int_{1}^{2} x^3 \, \mathrm{d}x = [3x]_{-3}^{1} + \left[\frac{x^4}{4}\right]_1^2 = 12 + \frac{15}{4} = \frac{63}{4}.$$

      iv. $\int_{-3}^{1} f(x)$
      Solution:
$$\int_{-3}^{1} f(x) = \int_{-3}^{1} 3 \, \mathrm{d}x = [3x]_{-3}^{1} = 12.$$

   (b) $\int_{-2}^{3} f,$

$$f(x) = \begin{cases} \frac{3}{x}, & x \leq -1 \\ 3 + 4x, & -1 < x < 2 \\ 11e^{2x-4}, & 2 \leq x \end{cases}$$

   Solution:
$$\int_{-2}^{3} f = \int_{-2}^{-1} \frac{3}{x} + \int_{-1}^{2} 3 + 4x + \int_{2}^{3} 11e^{2x-4} = [3 \ln |x|]_{-2}^{-1} + \left[3x + 2x^2\right]_{-1}^{2} + \left[\frac{11}{2}e^{2x-4}\right]_2^3$$
$$= 0 - 3\ln 2 + 6 + 8 - (-3) - 2 + \frac{11}{2}e^2 - \frac{11}{2} = -3\ln 2 + \frac{11}{2}e^2 + \frac{19}{2}$$

   (c) $\int_{-4}^{2} |x| \, \mathrm{d}x$
   Solution:
$$\int_{-4}^{2} |x| \, \mathrm{d}x = \int_{-4}^{0} -x \, \mathrm{d}x + \int_{0}^{2} x \, \mathrm{d}x = \left[-\frac{x^2}{2}\right]_{-4}^{0} + \left[\frac{x^2}{2}\right]_0^2 = 8 + 2 = 10.$$

(d) $\int_{-2}^{3} \sqrt{x^6}\,\mathrm{d}x$
Solution:

$$\int_{-2}^{3} \sqrt{x^6}\,\mathrm{d}x = \int_{-2}^{0} -x^3\,\mathrm{d}x + \int_{0}^{3} x^3\,\mathrm{d}x = \left[-\frac{x^4}{4}\right]_{-2}^{0} + \left[\frac{x^4}{4}\right]_{0}^{3} = 4 + \frac{81}{4} = \frac{97}{4}$$

(e) $\int_{-\pi}^{\frac{3\pi}{2}} |\sin x|\,\mathrm{d}x$
Solution:

$$\begin{aligned}\int_{-\pi}^{\frac{3\pi}{2}} |\sin x|\,\mathrm{d}x &= \int_{-\pi}^{0} -\sin x\,\mathrm{d}x + \int_{0}^{\pi} \sin x\,\mathrm{d}x + \int_{\pi}^{\frac{3\pi}{2}} -\sin x\,\mathrm{d}x \\ &= [\cos x]_{-\pi}^{0} + [-\cos x]_{0}^{\pi} + [\cos x]_{\pi}^{\frac{3\pi}{2}} \\ &= 1-(-1)+-(-1)-(-1)+0-(-1) = 5\end{aligned}$$

3. Compute the definite integrals:

(a) $\int_{1}^{2} \frac{3x^2}{x^3+1}\,\mathrm{d}x$
Solution: Substitution $y = x^3+1,\ \mathrm{d}y = 3x^2\,\mathrm{d}x$, limits of integration: 2 and 9.

$$\int_{1}^{2} \frac{3x^2}{x^3+1}\,\mathrm{d}x = \int_{2}^{9} \frac{1}{y}\,\mathrm{d}y = [\ln|y|]_2^9 = \ln 9 - \ln 2.$$

We have $\varphi(x) = x^3+1$. The interval: $(\alpha,\beta) = (1,2)$. The function φ is continuous at $[1,2]$. The derivative $\varphi' = 3x^2$ is continuous at $[1,2]$. Further $\varphi(1) = 2$ and $\varphi(2) = 9$.

(b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin x \cos x\,\mathrm{d}x$
Solution:
Substitution $y = \sin x,\ \mathrm{d}y = \cos x\,\mathrm{d}x$, limits of integration: $\sin(\pi/4) = \sqrt{2}/2$ a $\sin(\pi/3) = \sqrt{3}/2$.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin x \cos x\,\mathrm{d}x = \int_{\sqrt{2}/2}^{\sqrt{3}/2} y\,\mathrm{d}y = \left[\frac{y^2}{2}\right]_{\sqrt{2}/2}^{\sqrt{3}/2} = \frac{1}{2}\left(\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2\right) = \frac{1}{8}.$$

(c) $\int_{1}^{2} x \ln x\,\mathrm{d}x$
Solution: Per partes

$$\int_{1}^{2} x \ln x\,\mathrm{d}x = \left[\frac{x^2}{2}\ln x\right]_1^2 - \int_{1}^{2} \frac{x^2}{2x}\,\mathrm{d}x = 2\ln 2 - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^2 = 2\ln 2 - \frac{3}{4}.$$

(d) $\int_0^{\pi} x^2 \sin x \,\mathrm{d}x$

Solution: Per partes twice:

$$\int_0^{\pi} x^2 \sin x \,\mathrm{d}x = [-x^2 \cos x]_0^{\pi} + \int_0^{\pi} 2x \cos x \,\mathrm{d}x = [-x^2 \cos x + 2x \sin x]_0^{\pi} - 2 \int_0^{\pi} \sin x =$$

$$[-x^2 \cos x + 2x \sin x + 2 \cos x]_0^{\pi} = \pi^2 - 4$$

(e) $\int_1^{e} \frac{\ln^2 x}{x} \,\mathrm{d}x$

Solution: Substitution $y = \ln x,\ \mathrm{d}y = \frac{1}{x}\,\mathrm{d}x$, limits of integration: 0 and 1.

$$\int_1^{e} \frac{\ln^2 x}{x} \,\mathrm{d}x = \int_0^{1} y^2 \,\mathrm{d}y = \left[\frac{y^3}{3}\right]_0^1 = \frac{1}{3}.$$

(f) $\int_{-1}^{1} \frac{x^2}{1+x^2} \,\mathrm{d}x$

Solution:

$$\int_{-1}^{1} \frac{x^2}{1+x^2} \,\mathrm{d}x = \int_{-1}^{1} \frac{x^2+1-1}{1+x^2} \,\mathrm{d}x = \int_{-1}^{1} 1 + \frac{-1}{1+x^2} \,\mathrm{d}x = [x - \arctan\ x]_{-1}^{1} = 2 - \frac{\pi}{2}.$$

(g) $\int_0^{\infty} \frac{1}{(x+3)^5}$

Solution:

$$\int_0^{\infty} \frac{1}{(x+3)^5} = \left[\frac{-1}{4(x+3)^4}\right]_0^{\infty} = \lim_{x\to\infty} \frac{-1}{4(x+3)^4} - \lim_{x\to 0+} \frac{-1}{4(x+3)^4} = \frac{1}{4\cdot 81}$$

(h) $\int_0^{1} \frac{e^x}{e^{2x}+1} + \frac{1}{\cos^2 x} \,\mathrm{d}x$

Solution: Substitution $y = e^x,\ \mathrm{d}y = e^x\,\mathrm{d}x$,

$$\int_0^{1} \frac{e^x}{e^{2x}+1} = \int_1^{e} \frac{1}{y^2+1} = [\arctan\ y]_1^e = \arctan\ e - \arctan\ 1 = \arctan\ e - \frac{\pi}{4}$$

$$\int_0^{1} \frac{1}{\cos^2 x} \,\mathrm{d}x = [\tan\ x]_0^1 = \tan\ 1.$$

Together

$$\int_0^{1} \frac{e^x}{e^{2x}+1} + \frac{1}{\cos^2 x} \,\mathrm{d}x = \arctan\ e - \frac{\pi}{4} + \tan\ 1$$

(i) $\displaystyle\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$

Solution: Substitution $y=\sqrt{x},\ \mathrm{d}y=\frac{1}{2\sqrt{x}}\,\mathrm{d}x$.

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x=\int_1^\infty 2e^{-y}\,\mathrm{d}y=2\left[-e^{-y}\right]_1^\infty=-2(\lim_{y\to\infty}e^{-y}-\lim_{x\to 1+}e^{-y})$$
$$=-2\left(0-\frac{1}{e}\right)=\frac{2}{e}.$$

(j) $\displaystyle\int_a^b \operatorname{sgn}x\,\mathrm{d}x,\ a<0,\ b>0$

Solution: At first we need to divide integral into two parts:

$$\int_a^b \operatorname{sgn}x\,\mathrm{d}x=\int_a^0 -1\,\mathrm{d}x+\int_0^b 1\,\mathrm{d}x=[-x]_a^0+[x]_0^b=a+b.$$

(k) $\displaystyle\int_1^\infty \frac{\arctan\,x}{1+x^2}\,\mathrm{d}x$

Solution: Substitution $y=\arctan\,x,\ \mathrm{d}y=\frac{1}{1+x^2}\,\mathrm{d}x$:

$$\int_1^\infty \frac{\arctan\,x}{1+x^2}\,\mathrm{d}x=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} y\,\mathrm{d}y=\left[\frac{y^2}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}=\frac{\pi^2}{8}-\frac{\pi^2}{32}=\frac{3\pi^2}{32}.$$

(l) $\displaystyle\int_1^2 \frac{\mathrm{d}x}{x\ln x}$

Solution: Substitution $y=\ln x,\ \mathrm{d}y=\frac{1}{x}\,\mathrm{d}x$,

$$\int_1^2 \frac{\mathrm{d}x}{x\ln x}=\int_0^{\ln 2}\frac{1}{y}\,\mathrm{d}y=[\ln y]_0^{\ln 2}=\lim_{y\to\ln 2-}\ln y-\lim_{y\to 0+}\ln y=\ln(\ln 2)-(-\infty)=\infty.$$

(m)

$$\int_0^\pi \frac{\sin x}{\cos^2 x+1}\,\mathrm{d}x$$

Solution: Substitution $y=\cos x,\ \mathrm{d}y=-\sin x\,\mathrm{d}x$

$$\int_0^\pi \frac{\sin x}{\cos^2 x+1}\,\mathrm{d}x=\int_{-1}^1\frac{1}{1+y^2}\,\mathrm{d}y=[\arctan\,y]_{-1}^1=\arctan\,1-\arctan\,(-1)=2\frac{\pi}{4}=\frac{\pi}{2}.$$

(n) $\displaystyle\int_{-1}^1 x^2e^{-x}\,\mathrm{d}x$

Solution: per partes twice:

$$\int_{-1}^1 x^2e^{-x}\,\mathrm{d}x=\left[-x^2e^{-x}\right]_{-1}^1+\int_{-1}^1 2xe^{-x}\,\mathrm{d}x=\left[-x^2e^{-x}\right]_{-1}^1+\left[-2xe^{-x}\right]_{-1}^1+\int_{-1}^1 e^{-x}\,\mathrm{d}x$$
$$=\left[-x^2e^{-x}-2xe^{-x}-2e^{-x}\right]_{-1}^1=e-5e^{-1}$$

(o) $\displaystyle\int_2^3 \frac{x^2 - x + 1}{x - 1}\,\mathrm{d}x$

Solution:

$$\int_2^3 \frac{x^2 - x + 1}{x - 1}\,\mathrm{d}x = \int_2^3 x + \frac{1}{x-1}\,\mathrm{d}x = \left[\frac{x^2}{2} + \ln|x-1|\right]_2^3 = \frac{9}{2} + \ln 2 - \left(\frac{4}{2} + \ln 1\right) = \frac{5}{2} + \ln 2.$$