26th lesson

https://www2.karlin.mff.cuni.cz/~kuncova/en/teaching.php,
kuncova@karlin.mff.cuni.cz

Theorem 1 (Area between curves)**.** Let f and g be continuous functions such that $f(x) \geq g(x)$ at $[a, b]$. Then the area between the curves f and g from a to b is

$$A = \int_a^b f(x) - g(x)\,\mathrm{d}x.$$

Exercise 1. Find the are between the curves $y = x(3-x)$ and $y = x$.

Exercise 2. Find the are between the curves $y = \sqrt{x}$, $y = x - 2$ and the x-axis.

Theorem 2 (Volume of a solid)**.** Let f be continuous and **nonnegative** on $[a, b]$.

Let us define A as a region between the function f and the x-axis between points a and b.

Then the volume of the solid of revolution formed by revolving A around the x-axis is

$$V = \pi \int_a^b f^2(x)\,\mathrm{d}x.$$

Further, let $f'(x)$ be continuous. Then the surface area of the solid of revolution is

$$S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\,\mathrm{d}x.$$

Exercise 3. Find the volume of solid which is generated by the area under the function $y = e^{-x}$ on the interval $[0, 1]$.

Exercise 4. Find the volume of solid which is generated by the area between the functions $y = x^2$ and $y = x$ revolving around the x-axis. (The solid has a hole inside.)

Exercise 5. Find the surface area of a solid which is generated by the function $y = \frac{x^3}{3}$ for $1 \leq x \leq 2$ revolving around the x-axis.

Exercise 6. Find the surface area of a solid which is generated by the function $y = \sqrt{9 - x^2}$ for $1 \leq x \leq 3$ revolving around the x-axis.

Theorem 3 (Length of a smooth curve)**.** Let f is a function such that f' is continuous on $[a, b]$. Then the length of the curve is

$$l = \int_a^b \sqrt{1 + (f'(x))^2}dx.$$

Exercise 7. Find the length of the curve $y = \sqrt{x} - \frac{\sqrt{x^3}}{3}$ between $x = 0$ and $x = 2$.

Exercise 8. Find the length of the curve $y = x^2 - 4|x| - x$ between $x = -4$ and $x = 4$. (Hint: check, if the derivative exists.)
Fact:

$$\int \sqrt{1+t^2}\,\mathrm{d}t = \frac{1}{2}t\sqrt{t^2+1} + \frac{1}{2}\ln(\sqrt{t^2+1}+t)$$

Remarks 4 (Exponential growth and decay)**.** The exponential growth follows the mathematical model

$$y = k_0 e^{kt}, k > 0.$$

The exponential decay follows the mathematical model

$$y = k_0 e^{kt}, k < 0.$$

Exercise 9. Consider the population of bacteria, which grows according to the formula $200e^{0.02t}$, t is time in minutes.

1. How many bacteria we have after 5 hours?
2. When the population reach 100 000 bacteria?

Exercise 10. Let us consider the carbon-14. Its halflife is 5730 years.

1. Now we have 100g of carbon-14. How much of the carbon do we have in 50 years?
2. How old is an artifact, if it originally contained 100g of carbon, but now it contains only 10g?