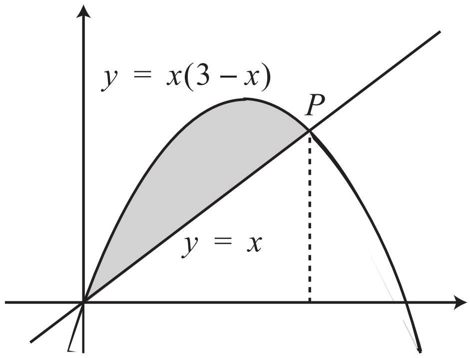
1

Solution

Sketching both curves on the same axes, we can see by setting $y = 0$ that the curve $y = x(3-x)$ cuts the x-axis at $x = 0$ and $x = 3$. Furthermore, the coefficient of x^2 is negative and so we have an inverted U-shape curve. The line $y = x$ goes through the origin and meets the curve $y = x(3-x)$ at the point P. It is this point that we need to find first of all.



At P the y co-ordinates of both curves are equal. Hence:

$$\begin{aligned} x(3-x) &= x \\ 3x - x^2 &= x \\ 2x - x^2 &= 0 \\ x(2-x) &= 0 \end{aligned}$$

so that either $x = 0$, the origin, or else $x = 2$, the x co-ordinate of the point P.

We now need to find the shaded area in the diagram. To do this we need the area under the upper curve, the graph of $y = x(3-x)$, between the x-axis and the ordinates $x = 0$ and $x = 2$. Then we need to subtract from this the area under the lower curve, the line $y = x$, and between the x-axis and the ordinates $x = 0$ and $x = 2$.

The area under the curve is

$$\begin{aligned} \int_0^2 y\,\mathrm{d}x &= \int_0^2 x(3-x)\mathrm{d}x \\ &= \int_0^2 (3x - x^2)\mathrm{d}x \\ &= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^2 \\ &= [6 - \tfrac{8}{3}] - [0] \\ &= 3\tfrac{1}{3}\,, \end{aligned}$$

and the area under the straight line is

$$\begin{aligned} \int_0^2 y\,\mathrm{d}x &= \int_0^2 x\,\mathrm{d}x \\ &= \left[\frac{x^2}{2}\right]_0^2 \\ &= [2] - [0] \\ &= 2\,. \end{aligned}$$

Thus the shaded area is $3\frac{1}{3} - 2 = 1\frac{1}{3}$ units of area.

Since the parabola lies above the line on $[-1, 2]$, the area integrand is $2 - x^2 - (-x)$.

$$A = \int_{-1}^{2} [2 - x^2 - (-x)]\, dx$$

$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2}$$

$$= \frac{9}{2} \text{ units squared}$$

Now try Exercise 5.

EXAMPLE 3 Using a Calculator

Find the area of the region enclosed by the graphs of $y = 2\cos x$ and $y = x^2 - 1$.

SOLUTION

The region is shown in Figure 7.9.

Using a calculator, we solve the equation

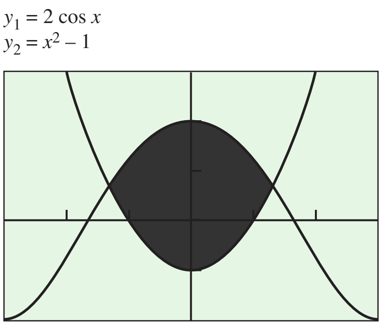
$$2\cos x = x^2 - 1$$

to find the x-coordinates of the points where the curves intersect. These are the limits of integration. The solutions are $x = \pm 1.265423706$. We store the negative value as A and the positive value as B. The area is

$$\text{NINT}\,(2\cos x - (x^2 - 1), x, A, B) \approx 4.994907788.$$

This is the final calculation, so we are now free to round. The area is about 4.99.

Now try Exercise 7.



[–3, 3] by [–2, 3]

Figure 7.9 The region in Example 3.

Boundaries with Changing Functions

If a boundary of a region is defined by more than one function, we can partition the region into subregions that correspond to the function changes and proceed as usual.

Finding Intersections by Calculator

The coordinates of the points of intersection of two curves are sometimes needed for other calculations. To take advantage of the accuracy provided by calculators, use them to solve for the values and *store* the ones you want.

EXAMPLE 4 Finding Area Using Subregions

Find the area of the region R in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

SOLUTION

The region is shown in Figure 7.10.

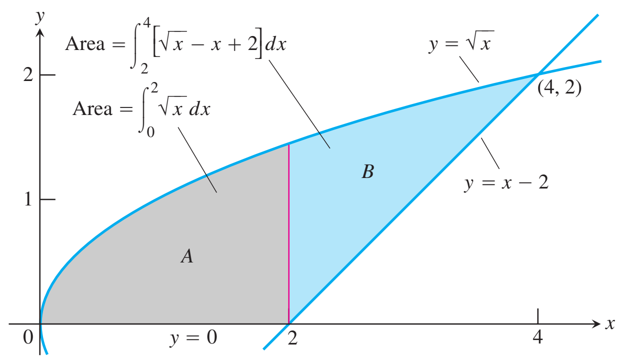


Figure 7.10 Region R split into subregions A and B. (Example 4)

continued

While it appears that no single integral can give the area of R (the bottom boundary is defined by two different curves), we can split the region at $x = 2$ into two regions A and B. The area of R can be found as the sum of the areas of A and B.

$$\text{Area of } R = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 [\sqrt{x} - (x-2)]\,dx}_{\text{area of } B}$$

$$= \frac{2}{3}x^{3/2}\Big]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

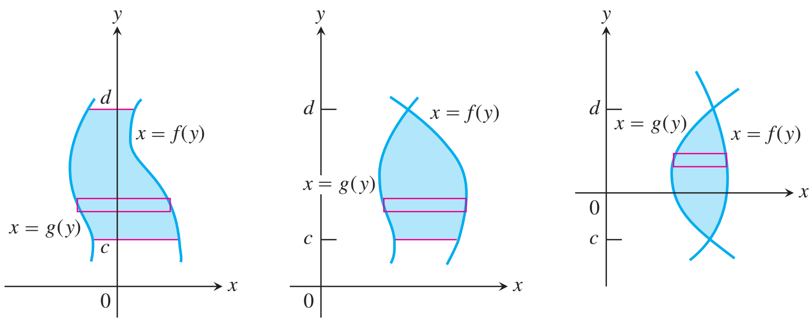
$$= \frac{10}{3} \text{ units squared}$$

Now try Exercise 9.

Integrating with Respect to *y*

Sometimes the boundaries of a region are more easily described by functions of y than by functions of x. We can use approximating rectangles that are horizontal rather than vertical and the resulting basic formula has y in place of x.

For regions like these



use this formula

$$A = \int_c^d [f(y) - g(y)]\,dy.$$

EXAMPLE 5 Integrating with Respect to *y*

Find the area of the region in Example 4 by integrating with respect to y.

SOLUTION

We remarked in solving Example 4 that "it appears that no single integral can give the area of R," but notice how appearances change when we think of our rectangles being summed over y. The interval of integration is $[0, 2]$, and the rectangles run between the same two curves on the entire interval. There is no need to split the region (Figure 7.11).

We need to solve for x in terms of y in both equations:

$$y = x - 2 \quad \text{becomes} \quad x = y + 2,$$

$$y = \sqrt{x} \quad \text{becomes} \quad x = y^2, \quad y \geq 0.$$

continued

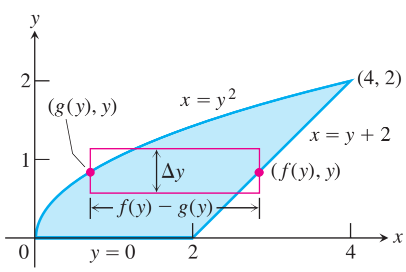
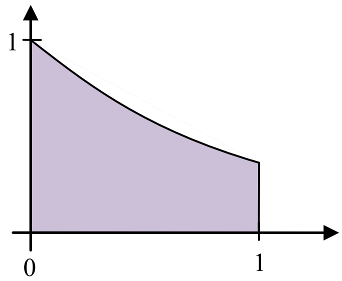


Figure 7.11 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y. (Example 5)

Example 3

3

Find the volume of the solid formed by rotating the area under $f(x) = e^{-x}$ on the interval [0,1] about the x-axis.



This is the region pictured in the earlier example. We substitute in the function and bounds into the formula we derived to set up the definite integral.

Volume = $\int_0^1 \pi\left(e^{-x}\right)^2 dx$

Using exponent rules, the integrand can be simplified. The constant π can be pulled out of the integral.

$\pi\int_0^1 e^{-2x} dx$

Using the substitution u = -2x, we can integrate this function.

$$\pi\int_0^1 e^{-2x} dx = -\frac{1}{2}\pi e^{-2x}\Big]_0^1 = \left(-\frac{1}{2}\pi e^{-2(1)}\right) - \left(-\frac{1}{2}\pi e^{-2(0)}\right) \approx 1.358 \text{ cubic units}$$

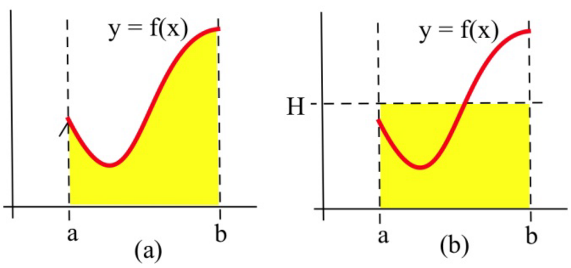
Average Value

We know the average of n numbers, $a_1, a_2, \ldots, a_n$, is their sum divided by n. But what if we need to find the average temperature over a day's time -- there are too many possible temperatures to add them up. This is a job for the definite integral.

The average value of a function f(x) on the interval [a, b] is given by

$$\frac{1}{b-a}\int_a^b f(x)\, dx$$

The average value of a positive f has a nice geometric interpretation. Imagine that the area under f (Fig. a) is a liquid that can "leak" through the graph to form a rectangle with the same area (Fig. b).



If the height of the rectangle is H, then the area of the rectangle is $H\cdot(b-a)$. We know the area of the rectangle is the same as the area under f so $H\cdot(b-a) = \int_a^b f(x)dx$. Then $H = \frac{1}{b-a}\int_a^b f(x)dx$, the average value of f on [a,b].

The average value of a positive function f is the height H of the rectangle whose area is the same as the area under f.

and the volume of a thin "slab" is then

$$(1-x_i^2)\sqrt{3}(1-x_i^2)\Delta x.$$

Thus the total volume is

$$\int_{-1}^{1}\sqrt{3}(1-x^2)^2\,dx=\frac{16}{15}\sqrt{3}.$$

□

One easy way to get "nice" cross-sections is by rotating a plane figure around a line. For example, in figure 9.3.3 we see a plane region under a curve and between two vertical lines; then the result of rotating this around the x-axis, and a typical circular cross-section.

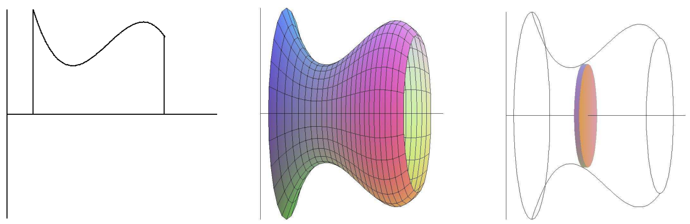


Figure 9.3.3 A solid of rotation. (AP)

Of course a real "slice" of this figure will not have straight sides, but we can approximate the volume of the slice by a cylinder or disk with circular top and bottom and straight sides; the volume of this disk will have the form $\pi r^2\Delta x$. As long as we can write r in terms of x we can compute the volume by an integral.

EXAMPLE 9.3.3 Find the volume of a right circular cone with base radius 10 and height 20. (A right circular cone is one with a circular base and with the tip of the cone directly over the center of the base.) We can view this cone as produced by the rotation of the line $y=x/2$ rotated about the x-axis, as indicated in figure 9.3.4.

At a particular point on the x-axis, say x_i, the radius of the resulting cone is the y-coordinate of the corresponding point on the line, namely $y_i=x_i/2$. Thus the total volume is approximately

$$\sum_{i=0}^{n-1}\pi(x_i/2)^2\,dx$$

and the exact volume is

$$\int_0^{20}\pi\frac{x^2}{4}\,dx=\frac{\pi}{4}\frac{20^3}{3}=\frac{2000\pi}{3}.$$

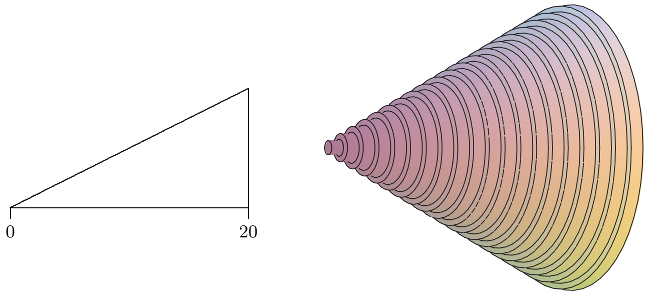


Figure 9.3.4 A region that generates a cone; approximating the volume by circular disks. (AP)

Note that we can instead do the calculation with a generic height and radius:

$$\int_0^h\pi\frac{r^2}{h^2}x^2\,dx=\frac{\pi r^2}{h^2}\frac{h^3}{3}=\frac{\pi r^2 h}{3},$$

giving us the usual formula for the volume of a cone. □

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EXAMPLE 9.3.4 Find the volume of the object generated when the area between $y=x^2$ and $y=x$ is rotated around the x-axis. This solid has a "hole" in the middle; we can compute the volume by subtracting the volume of the hole from the volume enclosed by the outer surface of the solid. In figure 9.3.5 we show the region that is rotated, the resulting solid with the front half cut away, the cone that forms the outer surface, the horn-shaped hole, and a cross-section perpendicular to the x-axis.

We have already computed the volume of a cone; in this case it is $\pi/3$. At a particular value of x, say x_i, the cross-section of the horn is a circle with radius x_i^2, so the volume of the horn is

$$\int_0^1\pi(x^2)^2\,dx=\int_0^1\pi x^4\,dx=\pi\frac{1}{5},$$

so the desired volume is $\pi/3-\pi/5=2\pi/15$.

As with the area between curves, there is an alternate approach that computes the desired volume "all at once" by approximating the volume of the actual solid. We can approximate the volume of a slice of the solid with a washer-shaped volume, as indicated in figure 9.3.5.

The volume of such a washer is the area of the face times the thickness. The thickness, as usual, is Δx, while the area of the face is the area of the outer circle minus the area of

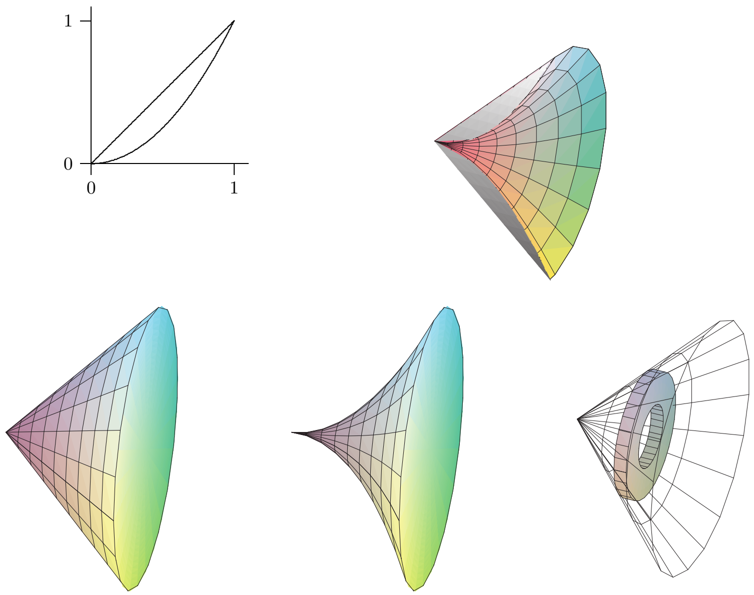

Figure 9.3.5 Solid with a hole, showing the outer cone and the shape to be removed to form the hole. (AP)

4

the inner circle, say $\pi R^2 - \pi r^2$. In the present example, at a particular x_i, the radius R is x_i and r is x_i^2. Hence, the whole volume is

$$\int_0^1 \pi x^2 - \pi x^4\,dx = \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}.$$

Of course, what we have done here is exactly the same calculation as before, except we have in effect recomputed the volume of the outer cone. □

Suppose the region between $f(x) = x + 1$ and $g(x) = (x-1)^2$ is rotated around the y-axis; see figure 9.3.6. It is possible, but inconvenient, to compute the volume of the resulting solid by the method we have used so far. The problem is that there are two "kinds" of typical rectangles: those that go from the line to the parabola and those that touch the parabola on both ends. To compute the volume using this approach, we need to

break the problem into two parts and compute two integrals:

$$\pi\int_0^1 (1+\sqrt{y})^2 - (1-\sqrt{y})^2\,dy + \pi\int_1^4 (1+\sqrt{y})^2 - (y-1)^2\,dy = \frac{8}{3}\pi + \frac{65}{6}\pi = \frac{27}{2}\pi.$$

If instead we consider a typical vertical rectangle, but still rotate around the y-axis, we get a thin "shell" instead of a thin "washer". If we add up the volume of such thin shells we will get an approximation to the true volume. What is the volume of such a shell? Consider the shell at x_i. Imagine that we cut the shell vertically in one place and "unroll" it into a thin, flat sheet. This sheet will be almost a rectangular prism that is Δx thick, $f(x_i) - g(x_i)$ tall, and $2\pi x_i$ wide (namely, the circumference of the shell before it was unrolled). The volume will then be approximately the volume of a rectangular prism with these dimensions: $2\pi x_i(f(x_i) - g(x_i))\Delta x$. If we add these up and take the limit as usual, we get the integral

$$\int_0^3 2\pi x(f(x) - g(x))\,dx = \int_0^3 2\pi x(x + 1 - (x-1)^2)\,dx = \frac{27}{2}\pi.$$

Not only does this accomplish the task with only one integral, the integral is somewhat easier than those in the previous calculation. Things are not always so neat, but it is often the case that one of the two methods will be simpler than the other, so it is worth considering both before starting to do calculations.

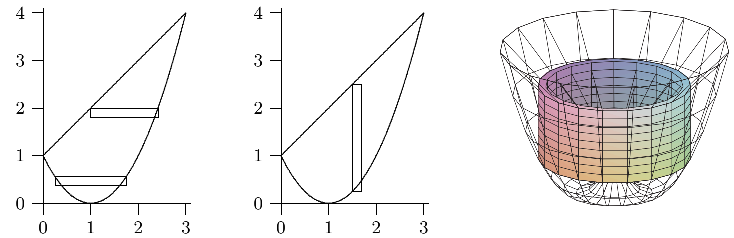

Figure 9.3.6 Computing volumes with "shells". (AP)

EXAMPLE 9.3.5 Suppose the area under $y = -x^2 + 1$ between $x = 0$ and $x = 1$ is rotated around the x-axis. Find the volume by both methods.

Disk method: $\displaystyle\int_0^1 \pi(1-x^2)^2\,dx = \frac{8}{15}\pi.$

Shell method: $\displaystyle\int_0^1 2\pi y\sqrt{1-y}\,dy = \frac{8}{15}\pi.$ □

solid is

$$\begin{aligned}
V(b) &= \int_1^b A(x)\,dx \\
&= \int_1^b \pi \frac{1}{x^6}\,dx \\
&= \pi \left[\frac{-1}{5x^5}\right]_1^b \\
&= \pi \left[\frac{-1}{5b^5 + \frac{1}{5}}\right] \\
&= \pi \left(\frac{1}{5} - \frac{1}{5b^5}\right).
\end{aligned}$$

As b goes to ∞, the term $\frac{1}{5b^5}$ goes to zero rather quickly, so the function $V(b)$ goes to $\frac{\pi}{5}$ as $b \to \infty$.

8. Write down an integral which will compute the length of the part of the curve $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/4$. Don't worry about evaluating this integral.

   Answer: I plan to use the arc length integral, which says that the length of a curve $y = f(x)$ from $x = a$ to $x = b$ is given by

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx,$$

   so I need to figure out $\frac{dy}{dx}$. Using the Chain Rule,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\Big(\ln(\cos x)\Big) \\
&= \frac{1}{\cos x}\frac{d}{dx}(\cos x) \\
&= \frac{1}{\cos x}(-\sin x) \\
&= \frac{-\sin x}{\cos x}.
\end{aligned}$$

   Therefore, the length of the curve from $x = 0$ to $x = \pi/4$ is given by the integral

$$\int_0^{\pi/4} \sqrt{1 + \left(\frac{-\sin x}{\cos x}\right)^2}\,dx = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\,dx = \int_0^{\pi/4} \sec x\,dx.$$

   At this point, we haven't yet learned how to find the antiderivative of $\sec x$, so this is as far as we can go.

5

9. Calculate the surface area of the surface obtained by revolving the curve $y = \frac{x^3}{3}$ around the x-axis for $1 \leq x \leq 2$.

   I plan to use the fact that the surface area of a surface given by revolving the graph of $y = f(x)$ around the x-axis from $x = a$ to $x = b$ is given by

$$\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

5

Therefore, it's important to know $f'(x)$ (or, saying the same thing, $\frac{dy}{dx}$). But of course $f'(x) = x^2$, so the surface area between $x = 1$ and $x = 2$ will be

$$\int_1^2 2\pi \frac{x^3}{3}\sqrt{1+(x^2)^2}\,dx = \int_1^2 2\pi\frac{x^3}{3}\sqrt{1+x^4}\,dx.$$

Let $u = 1 + x^4$. Then $du = 4x^3\,dx$ and we can write the above integral as

$$\begin{aligned}\frac{2\pi}{3}\cdot\frac{1}{4}\int_1^2 4x^3\sqrt{1+x^4}\,dx &= \frac{\pi}{6}\int_2^{17}\sqrt{u}\,du\\ &= \frac{\pi}{6}\left[\frac{2}{3}u^{3/2}\right]_2^{17}\\ &= \frac{\pi}{3}\left[\frac{17\sqrt{17}}{3} - \frac{2\sqrt{2}}{3}\right]\\ &= \underline{\frac{\pi}{9}\left(17\sqrt{17} - 2\sqrt{2}\right)}\end{aligned}$$

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10. Calculate the surface area of the surface obtained by revolving the curve $y = \sqrt{9-x^2}$ around the x-axis for $1 \le x \le 3$.

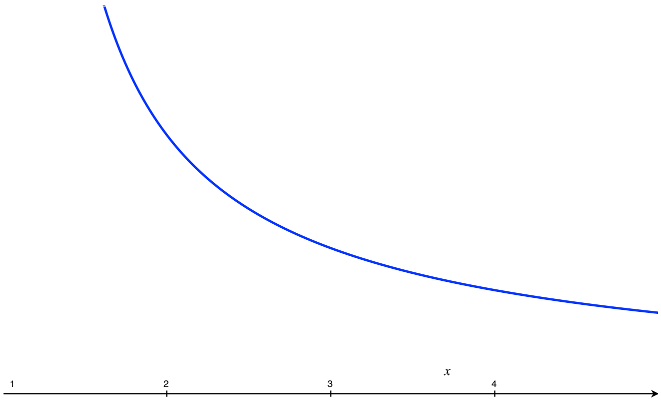
Answer: Again, I intend to use the surface area integral, so I need to know $\frac{dy}{dx}$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\sqrt{9-x^2}\right)\\ &= \frac{d}{dx}\left[(9-x^2)^{1/2}\right]\\ &= \frac{1}{2}(9-x^2)^{-1/2}\cdot\frac{d}{dx}\left(9-x^2\right)\\ &= \frac{1}{2\sqrt{9-x^2}}\cdot(-2x)\\ &= \frac{-x}{\sqrt{9-x^2}}.\end{aligned}$$

Therefore, the surface area of the surface is given by

$$\begin{aligned}\int_1^3 2\pi\sqrt{9-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{9-x^2}}\right)^2}\,dx &= 2\pi\int_1^3\sqrt{9-x^2}\sqrt{1+\frac{x^2}{9-x^2}}\,dx\\ &= 2\pi\int_1^3\sqrt{9-x^2}\sqrt{\frac{9-x^2}{9-x^2}+\frac{x^2}{9-x^2}}\,dx\\ &= 2\pi\int_1^3\sqrt{9-x^2}\sqrt{\frac{9}{9-x^2}}\,dx\\ &= 2\pi\int_1^3\sqrt{9-x^2}\frac{3}{\sqrt{9-x^2}}\,dx\\ &= 2\pi\int_1^3 3\,dx\\ &= 2\pi\Big[3x\Big]_1^3\\ &= 2\pi(9-3)\\ &= \underline{12\pi}\end{aligned}$$

11. Find the area between $y = \frac{x}{x^2-1}$ and the x-axis for $2 \le x \le 4$.



Answer: Clearly, we just need to integrate $\frac{x}{x^2-1} - 0 = \frac{x}{x^2-1}$ from $x = 2$ to $x = 4$ to find this area; in symbols:

$$\int_2^4 \frac{x}{x^2-1}\, dx.$$

To compute this integral, I will use u-substitution. Let $u = x^2 - 1$. Then $du = 2x\, dx$, and so the above integral is equal to

$$\begin{aligned}
\frac{1}{2}\int_2^4 \frac{2x}{x^2-1} &= \frac{1}{2}\int_3^{15} \frac{du}{u} \\
&= \frac{1}{2}\Big[\ln u\Big]_3^{15} \\
&= \frac{1}{2}[\ln 15 - \ln 3] \\
&= \frac{1}{2}\ln 5 \\
&= \ln\sqrt{5}.
\end{aligned}$$

7

12. What is the length of the curve $y = \sqrt{x} - \frac{x^{3/2}}{3}$ between $x = 0$ and $x = 2$?

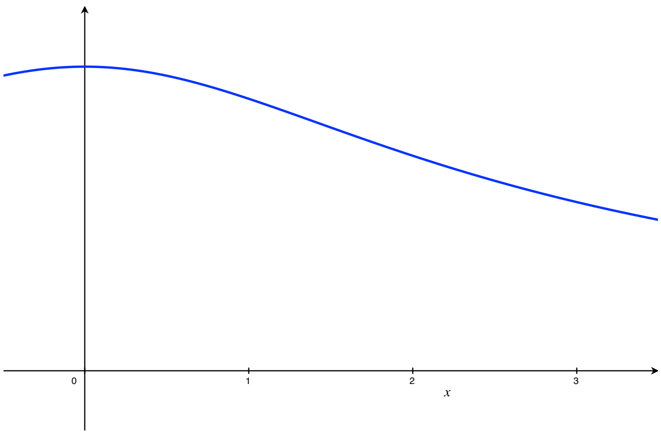
Answer: Once again, I plan to use the arc length integral, so I need to know $\frac{dy}{dx}$:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\sqrt{x} - \frac{x^{3/2}}{3}\right) \\
&= \frac{1}{2\sqrt{x}} - \frac{1}{2}\sqrt{x}.
\end{aligned}$$

Therefore, the length of the curve from $x = 0$ to $x = 2$ is given by

$$\begin{aligned}
\int_0^2 \sqrt{1 + \left(\frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2}\right)^2}\, dx &= \int_0^2 \sqrt{1 + \left(\frac{1}{4x} - \frac{1}{2} + \frac{x}{4}\right)}\, dx \\
&= \int_0^2 \sqrt{\frac{1}{4x} + \frac{1}{2} + \frac{x}{4}}\, dx \\
&= \int_0^2 \sqrt{\left(\frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{2}\right)^2}\, dx \\
&= \int_0^2 \left(\frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{2}\right) dx \\
&= \left[\sqrt{x} + \frac{x^{3/2}}{3}\right]_0^2 \\
&= \sqrt{2} + \frac{2\sqrt{2}}{3} \\
&= \frac{5\sqrt{2}}{3}.
\end{aligned}$$

13. What is the volume obtained by rotating the region between $y = \frac{1}{\sqrt{x^2+4}}$ and the x-axis for $0 \le x \le 3$ around the y-axis?



Answer: Since the axis of rotation is vertical, washers will be horizontal and cylindrical shells will have vertical sides. Clearly, finding the area of washers will be a problem, since some will have outer edge given by the line $x = 3$ and some will have outer edge given by the curve. Instead, I will use cylindrical shells. Since the shells change as x changes, I will be expressing everything in terms of x.

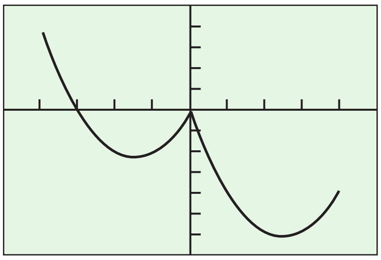
For a given x, the corresponding shell has radius x and height given by the distance from the curve to the x-axis, which is just $\frac{1}{\sqrt{x^2+4}} - 0 = \frac{1}{\sqrt{x^2+4}}$. Therefore, the volume of the solid will be

$$\int_0^3 2\pi x \left(\frac{1}{\sqrt{x^2+4}}\right) dx = \pi \int_0^3 \frac{2x\, dx}{\sqrt{x^2+4}}.$$

What happens if you fail to notice that dy/dx is undefined at $x = 0$ and ask your calculator to compute

$$\text{NINT}\left(\sqrt{1 + \left((1/3)\,x^{-2/3}\right)^2},\ x,\ -8,\ 8\right)?$$

This actually depends on your calculator. If, in the process of its calculations, it tries to evaluate the function at $x = 0$, then some sort of domain error will result. If it tries to find convergent Riemann sums near $x = 0$, it might get into a long, futile loop of computations that you will have to interrupt. Or it might actually produce an answer—in which case you hope it would be sufficiently bizarre for you to realize that it should not be trusted.



[–5, 5] by [–7, 5]

Figure 7.38 The graph of

$y = x^2 - 4|x| - x,\ -4 \le x \le 4,$

has a corner at $x = 0$ where neither dy/dx nor dx/dy exists. We find the lengths of the two smooth pieces and add them together. (Example 4)

EXAMPLE 4 Getting Around a Corner

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Find the length of the curve $y = x^2 - 4|x| - x$ from $x = -4$ to $x = 4$.

SOLUTION

We should always be alert for abrupt slope changes when absolute value is involved. We graph the function to check (Figure 7.38).

There is clearly a corner at $x = 0$ where neither dy/dx nor dx/dy can exist. To find the length, we split the curve at $x = 0$ to write the function *without* absolute values:

$$x^2 - 4|x| - x = \begin{cases} x^2 + 3x & \text{if} \quad x < 0, \\ x^2 - 5x & \text{if} \quad x \ge 0. \end{cases}$$

Then,

$$L = \int_{-4}^{0} \sqrt{1 + (2x + 3)^2}\, dx + \int_{0}^{4} \sqrt{1 + (2x - 5)^2}\, dx$$

$$\approx 19.56. \qquad \text{By NINT}$$

Now try Exercise 27.

Finally, cusps are handled the same way corners are: split the curve into smooth pieces and add the lengths of those pieces.

Quick Review 7.4 *(For help, go to Sections 1.3 and 3.2.)*

In Exercises 1–5, simplify the function.

1. $\sqrt{1 + 2x + x^2}$ on $[1, 5]$ $\quad x + 1$

2. $\sqrt{1 - x + \dfrac{x^2}{4}}$ on $[-3, -1]$ $\quad \dfrac{2 - x}{2}$

3. $\sqrt{1 + (\tan x)^2}$ on $[0, \pi/3]$ $\quad \sec x$

4. $\sqrt{1 + (x/4 - 1/x)^2}$ on $[4, 12]$ $\quad \dfrac{x^2 + 4}{4x}$

5. $\sqrt{1 + \cos 2x}$ on $[0, \pi/2]$ $\quad \sqrt{2}\cos x$

In Exercises 6–10, identify all values of x for which the function fails to be differentiable.

6. $f(x) = |x - 4|$ $\quad 4$

7. $f(x) = 5x^{2/3}$ $\quad 0$

8. $f(x) = \sqrt[5]{x + 3}$ $\quad -3$

9. $f(x) = \sqrt{x^2 - 4x + 4}$ $\quad 2$

10. $f(x) = 1 + \sqrt[3]{\sin x}$ $\quad k\pi$, k any integer

Time(min)	Population Size (no. of bacteria)
110	1805
120	2205

Note that we are using a continuous function to model what is inherently discrete behavior. At any given time, the real-world population contains a whole number of bacteria, although the model takes on noninteger values. When using exponential growth models, we must always be careful to interpret the function values in the context of the phenomenon we are modeling.

Example 4.9.1: Population Growth

Consider the population of bacteria described earlier. This population grows according to the function $f(t)=200e^{0.02t}$, where t is measured in minutes. How many bacteria are present in the population after 5 hours (300 minutes)? When does the population reach 100,000 bacteria?

Solution

We have $f(t)=200e^{0.02t}$. Then

$$f(300)=200e^{0.02(300)}\approx 80,686.$$

There are 80,686 bacteria in the population after 5 hours.

To find when the population reaches 100,000 bacteria, we solve the equation

$$\begin{aligned} 100,000 &= 200e^{0.02t} \\ 500 &= e^{0.02t} \\ \ln 500 &= 0.02t \\ t &= \frac{\ln 500}{0.02}\approx 310.73. \end{aligned}$$

The population reaches 100,000 bacteria after 310.73 minutes.

Exercise 4.9.1

Consider a population of bacteria that grows according to the function $f(t)=500e^{0.05t}$, where t is measured in minutes. How many bacteria are present in the population after 4 hours? When does the population reach 100 million bacteria?

Answer

Use the process from the previous example.

Answer

There are 81,377,396 bacteria in the population after 4 hours. The population reaches 100 million bacteria after 244.12 minutes.

Let's now turn our attention to a financial application: **compound interest**. Interest that is not compounded is called simple interest. ***Simple interest*** is paid once, at the end of the specified time period (usually 1 year). So, if we put \$1000 in a savings account earning 2 simple interest per year, then at the end of the year we have

$$1000(1+0.02)=\$1020. \tag{4.9.10}$$

Compound interest is paid multiple times per year, depending on the compounding period. Therefore, if the bank compounds the interest every 6 months, it credits half of the year's interest to the account after 6 months. During the second half of the year, the account earns interest not only on the initial \$1000, but also on the interest earned during the first half of the year. Mathematically speaking, at the end of the year, we have

$$\begin{aligned} 155 &= 130e^{(\ln 11 - \ln 13/2)t} + 70 \\ 85 &= 130e^{(\ln 11 - \ln 13)t} \\ \frac{17}{26} &= e^{(\ln 11 - \ln 13)t} \\ \ln 17 - \ln 26 &= \left(\frac{\ln 11 - \ln 13}{2}\right) t \\ t &= \frac{2(\ln 17 - \ln 26)}{\ln 11 - \ln 13} \\ &\approx 5.09. \end{aligned}$$

The coffee is too cold to be served about 5 minutes after it is poured.

Exercise 4.9.4

Suppose the room is warmer $(75\,^\circ F)$ and, after 2 minutes, the coffee has cooled only to $185\,^\circ F$. When is the coffee first cool enough to serve? When is the coffee be too cold to serve? Round answers to the nearest half minute.

Hint

Use the process from the previous example.

Answer

The coffee is first cool enough to serve about 3.5 minutes after it is poured. The coffee is too cold to serve about 7 minutes after it is poured.

Just as systems exhibiting exponential growth have a constant doubling time, systems exhibiting exponential decay have a constant half-life. To calculate the half-life, we want to know when the quantity reaches half its original size. Therefore, we have

$$\begin{aligned} \frac{y_0}{2} &= y_0 e^{-kt} \\ \frac{1}{2} &= e^{-kt} \\ -\ln 2 &= -kt \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

Note: This is the same expression we came up with for doubling time.

Definition: Half-Life

If a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by

$$\text{Half-life} = \frac{\ln 2}{k}. \tag{4.9.31}$$

Example 4.9.5: Radiocarbon Dating

One of the most common applications of an exponential decay model is carbon dating. Carbon-14 decays (emits a radioactive particle) at a regular and consistent exponential rate. Therefore, if we know how much carbon was originally present in an object and how much carbon remains, we can determine the age of the object. The half-life of carbon-14 is approximately 5730 years—meaning, after that many years, half the material has converted from the original carbon-14 to

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the new nonradioactive nitrogen-14. If we have 100 g carbon-14 today, how much is left in 50 years? If an artifact that originally contained 100 g of carbon now contains 10 g of carbon, how old is it? Round the answer to the nearest hundred years.

Solution

We have

$$5730 = \frac{\ln 2}{k} \tag{4.9.32}$$

$$k = \frac{\ln 2}{5730}. \tag{4.9.33}$$

So, the model says

$$y = 100e^{-(\ln 2/5730)t}. \tag{4.9.34}$$

In 50 years, we have

$$y = 100e^{-(\ln 2/5730)(50)} \approx 99.40.$$

Therefore, in 50 years, 99.40 g of carbon-14 remains.

To determine the age of the artifact, we must solve

$$10 = 100e^{-(\ln 2/5730)t} \tag{4.9.35}$$

$$\frac{1}{10} = e^{-(\ln 2/5730)t} \tag{4.9.36}$$

$$t \approx 19035. \tag{4.9.37}$$

The artifact is about $19,000$years old.

Exercise 4.9.5: Carbon-14 Decay

If we have 100 g of carbon-14 , how much is left after. years? If an artifact that originally contained 100 g of carbon now contains 20 g of carbon, how old is it? Round the answer to the nearest hundred years.

Example 4.9.2: Growth of Bacteria in a Culture

Suppose the rate of growth of bacteria in a Petri dish is given by $q(t) = 3^t$, where t is given in hours and $q(t)$ is given in thousands of bacteria per hour. If a culture starts with 10,000 bacteria, find a function $Q(t)$ that gives the number of bacteria in the Petri dish at any time t. How many bacteria are in the dish after 2 hours?

Solution

We have

$$Q(t) = \int 3^t dt = \frac{3^t}{\ln 3} + C. \tag{4.9.38}$$

Then, at $t = 0$ we have $Q(0) = 10 = \frac{1}{\ln 3} + C$, so $C \approx 9.090$ and we get

$$Q(t) = \frac{3^t}{\ln 3} + 9.090. \tag{4.9.39}$$

At time $t = 2$, we have

$$Q(2) = \frac{3^2}{\ln 3} + 9.090 \tag{4.9.40}$$

$$= 17.282. \tag{4.9.41}$$

