2nd lesson

https://www2.karlin.mff.cuni.cz/∼kuncova/en/teachIM.php
kunck6am@natur.cuni.cz

Algorithm

1. Find conditions (dividing by zero).
2. Move every expressions and numbers to the left side. On the right side should stay only zero.
3. Decompose the numerator and denominator. Find critical points, where expressions are equal to 0 or points, where expressions are not defined.
4. Separate intervals and make a table/sketch of signs. Check conditions again.
5. Write down the solution. (Check conditions;))

Warning

If it is possible, do not multiply (or divide) by expressions with x. This operation can change the sign in the equation or You can multiply by zero expression. Be careful.

Hints

For any quadratic equation $ax^2 + bx + c = 0$, $a, b, c \in \mathbb{R}$, there is a quadratic formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quadratic equation then can be factored into:

$$(x - x_1)(x - x_2) = 0.$$

(This operation sometimes involved complex roots.)

Exercises

1. Under what condition is the graph of the quadratic function described by $f(x) = ax^2 + bx + c$ concave down?
   (a) $a < 0$ - TRUE
   (b) $b < 0$
   (c) $c < 0$
   (d) More than one of the above.
   (e) None of the above.

2. If $f(x) = ax^2 + bx + c$ is a quadratic function, then the lowest point on the graph of $f(x)$ occurs at $x = -b/2a$.

   (a) True - TRUE if $a > 0$ $x = -b/2a$ is the highest point for $a < 0$

   (b) False

   From:

   http://mathquest.carroll.edu/libraries/PRE.student.01.06.pdf

3. Solve inequations

   (a) $x^2 + 4x \geq 21$

   (b) $4x^2 \leq 15 - 17x$

   (c) $x^2 + 34 > 12x$

   (d) $x^2 - 2x + 1 \leq 0$

4. Solve inequations

   (a) $\dfrac{x^2 + 5x - 6}{x - 3} \geq 0$

   (b) $\dfrac{2x}{2x^2 + 5x + 2} > \dfrac{1}{x + 1}$

   (c) $\dfrac{x^2 + 2x - 3}{x + 1} \geq 0$

   (d) $\dfrac{x^2 - 9}{x + 2} \geq 0$

   (e) $\dfrac{4x^2 + 5x - 9}{x^2 - x - 6} \geq 0$

   (f) $\dfrac{(x + 4)^2(x + 6)}{(x^2 + 7)(x - 2)^3} < 0$

   (g) $\dfrac{x^3 + 1}{x^2 - 9} < 0$

   (h) $\dfrac{x + 1}{(x - 2)(x + 3)} \leq 1 - \dfrac{2}{x - 2}$

   (i) $\dfrac{5}{x^2 - 2x - 15} > 0$

   (j) $\dfrac{1}{3} - \dfrac{2}{x^2} < \dfrac{5}{3x}$

Bonus

5. Find a quadratic inequation (like in the 3rd exercise) such that its solution is

   (a) $x \in (-\infty, -3] \cup [2, \infty)$

   Solution: $(x + 3)(x - 2) \geq 0$, $x^2 + x - 6 \geq 0$

   (b) $x \in (-1, 5)$

   Solution: $(x + 1)(x - 5) < 0$, $x^2 - 4x - 5 < 0$

   (c) $x = -6$

   Solution: $(x + 6)^2 \leq 0$, $x^2 + 12x + 36 \leq 0$

   (d) $\emptyset$

   Solution: For example $x^2 + 5 < 0$.

6. Find the values of a parameter $c \in \mathbb{R}$ such that the range of the function

$$f(x) = \frac{x^2 + 2x + c}{x^2 + 4x + 3c}$$

   is equal to $\mathbb{R}$.

Section 2-12 : Polynomial Inequalities

1. Solve the following inequality.

$$u^2 + 4u \geq 21$$

Step 1

The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the polynomial.

$$\begin{gathered} u^2 + 4u - 21 \geq 0 \\ (u+7)(u-3) \geq 0 \end{gathered}$$

Hint : Where are the only places where the polynomial might change signs?

Step 2

Despite the fact that this is an inequality we first need to know where the polynomial is zero. From the factored from we can quickly see that the polynomial will be zero at,

$$u = -7 \qquad u = 3$$

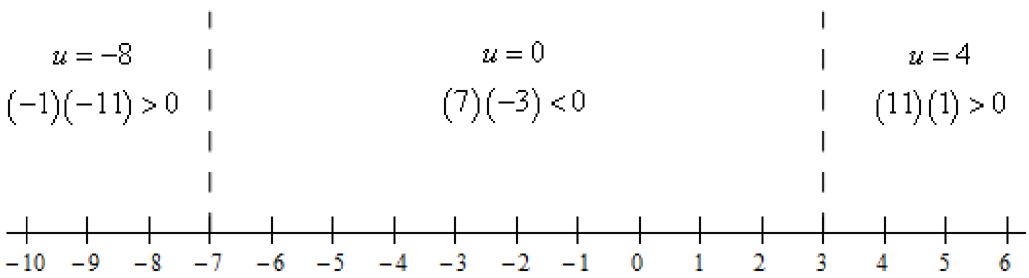
Remember that these points are important because they are the only places where the polynomial on the left side of the inequality might change sign. Given that we want to know where the polynomial is zero (which we now know) or positive knowing where the polynomial might change sign will help considerably with determining the answer we're looking for.

Hint : Knowing that the polynomial can only change sign at the points above how can we quickly determine if the polynomial is positive or negative in the ranges between those points?

Step 3

Recall from the discussion in the notes for this section that because the points from the previous step are the only places where the polynomial might change sign we can quickly determine if the polynomial is positive/negative in the ranges between each of these points simply by plugging in "test points" from each region into the polynomial to check the sign.

So, let's sketch a quick number line with the points where the polynomial is zero graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4
All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\boxed{\begin{array}{lcl} u \le -7 & \text{and} & u \ge 3 \\ \left(-\infty,-7\right] & \text{and} & \left[3,\infty\right) \end{array}}$$

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2. Solve the following inequality.

$$x^2 + 8x + 12 < 0$$

Step 1
The first thing we need to do is get a zero on one side of the inequality (which is already done for this problem) and then, if possible, factor the polynomial.

$$\left(x+6\right)\left(x+2\right) < 0$$

Hint : Where are the only places where the polynomial might change signs?

Step 2
Despite the fact that this is an inequality we first need to know where the polynomial is zero. From the factored from we can quickly see that the polynomial will be zero at,

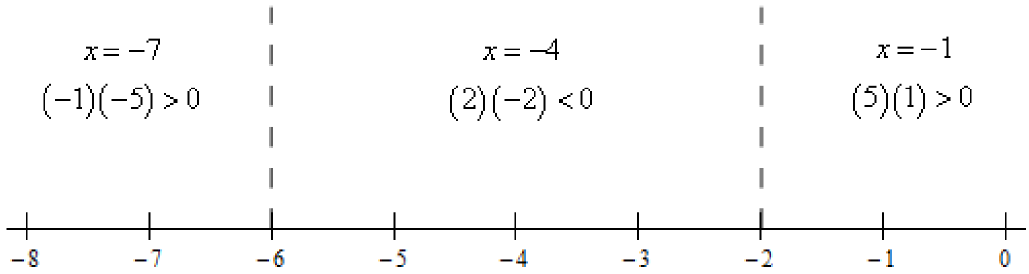
$$x = -6 \qquad\qquad x = -2$$

Remember that these points are important because they are the only places where the polynomial on the left side of the inequality might change sign. Given that we want to know where the polynomial is negative knowing where the polynomial might change sign will help considerably with determining the answer we're looking for.

Hint : Knowing that the polynomial can only change sign at the points above how can we quickly determine if the polynomial is positive or negative in the ranges between those points?

Step 3
Recall from the discussion in the notes for this section that because the points from the previous step are the only places where the polynomial might change sign we can quickly determine if the polynomial is positive/negative in the ranges between each of these points simply by plugging in "test points" from each region into the polynomial to check the sign.

So, let's sketch a quick number line with the points where the polynomial is zero graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4
All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\boxed{\begin{array}{c}-6<x<-2\\(-6,-2)\end{array}}$$

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3. Solve the following inequality.

$$4t^2 \le 15-17t$$

Step 1
The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the polynomial.

$$\begin{array}{c}4t^2+17t-15\le 0\\(t+5)(4t-3)\le 0\end{array}$$

Hint : Where are the only places where the polynomial might change signs?

Step 2
Despite the fact that this is an inequality we first need to know where the polynomial is zero. From the factored from we can quickly see that the polynomial will be zero at,

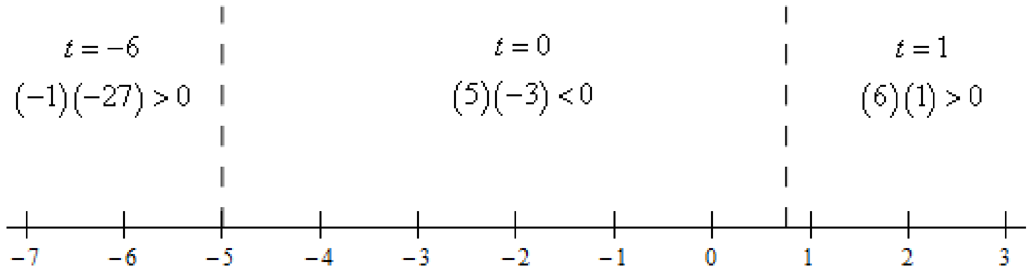
$$t=-5 \qquad\qquad t=\frac{3}{4}$$

Remember that these points are important because they are the only places where the polynomial on the left side of the inequality might change sign. Given that we want to know where the polynomial is zero (which we now know) or negative knowing where the polynomial might change sign will help considerably with determining the answer we're looking for.

Hint : Knowing that the polynomial can only change sign at the points above how can we quickly determine if the polynomial is positive or negative in the ranges between those points?

Step 3
Recall from the discussion in the notes for this section that because the points from the previous step are the only places where the polynomial might change sign we can quickly determine if the polynomial is positive/negative in the ranges between each of these points simply by plugging in "test points" from each region into the polynomial to check the sign.

So, let's sketch a quick number line with the points where the polynomial is zero graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4
All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\boxed{\begin{array}{c} -5 \le t \le \dfrac{3}{4} \\ \left[-5, \dfrac{3}{4}\right] \end{array}}$$

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4. Solve the following inequality.

$$z^2 + 34 > 12z$$

Step 1
The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the polynomial.

$$z^2 - 12z + 34 > 0$$

In this case the polynomial doesn't factor.

Hint : Where are the only places where the polynomial might change signs?

Step 2
Despite the fact that this is an inequality we first need to know where the polynomial is zero. Because the polynomial didn't factor we'll need to use the quadratic formula to determine where it's zero.

$$z = \frac{12 \pm \sqrt{144 - 4(1)(34)}}{2(1)} = \frac{12 \pm \sqrt{8}}{2} \quad \Rightarrow \quad z = 4.5858,\ 7.4142$$

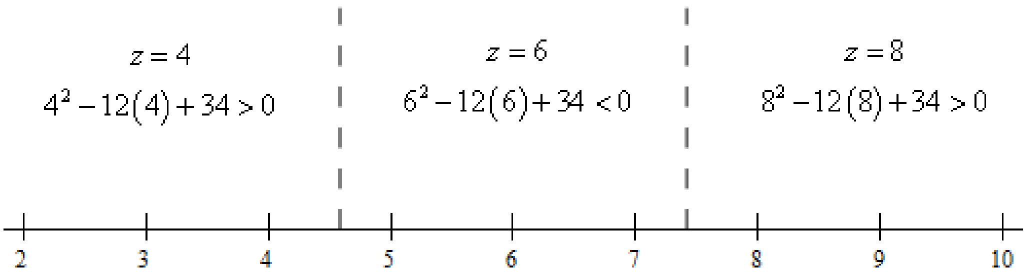
We'll need these points in decimal form to make the rest of the problem easier.

Remember that these points are important because they are the only places where the polynomial on the left side of the inequality might change sign. Given that we want to know where the polynomial is positive knowing where the polynomial might change sign will help considerably with determining the answer we're looking for.

Hint : Knowing that the polynomial can only change sign at the points above how can we quickly determine if the polynomial is positive or negative in the ranges between those points?

Step 3
Recall from the discussion in the notes for this section that because the points from the previous step are the only places where the polynomial might change sign we can quickly determine if the polynomial is positive/negative in the ranges between each of these points simply by plugging in "test points" from each region into the polynomial to check the sign.

So, let's sketch a quick number line with the points where the polynomial is zero graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4
All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\boxed{\begin{array}{lcl} z < 4.5858 & \text{and} & z > 7.4142 \\ (-\infty, 4.5858) & \text{and} & (7.4142, \infty) \end{array}}$$

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5. Solve the following inequality.

$$y^2 - 2y + 1 \le 0$$

Step 1

The first thing we need to do is get a zero on one side of the inequality (which is already done for this problem) and then, if possible, factor the polynomial.

$$(y-1)^2 \le 0$$

Hint : Where are the only places where the polynomial might change signs?

Step 2
Despite the fact that this is an inequality we first need to know where the polynomial is zero. From the factored from we can quickly see that the polynomial will be zero at,

$$y = 1$$

Hint : Is it possible for the polynomial to ever be negative?

Step 3
This problem works a little differently than the others in this section. Because the polynomial is a perfect square we know that it can never be negative! It is only possible for it to be zero or positive.

We are being asked to determine where the polynomial is negative or zero. As noted however it isn't possible for it to be negative. Therefore the only solution we can get for this inequality is where it is zero and we found that in the previous step.

The answer is then,

$$\boxed{y = 1}$$

In this case the answer is a single number and not an inequality. This happens on occasion and we shouldn't worry about these kinds of "unusual" answers.

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6. Solve the following inequality.

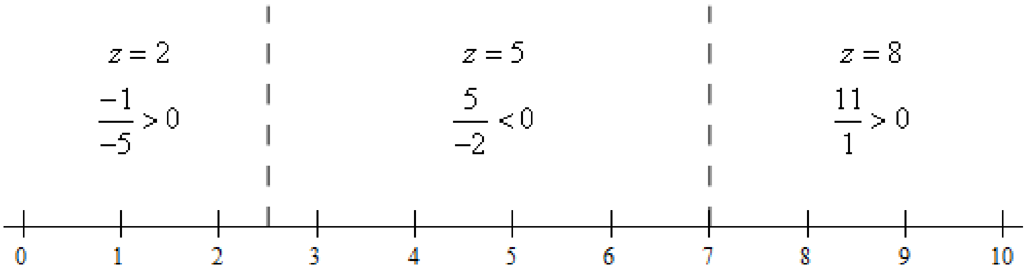
$$t^4 + t^3 - 12t^2 < 0$$

Step 1
The first thing we need to do is get a zero on one side of the inequality (which is already done for this problem) and then, if possible, factor the polynomial.

$$t^2\left(t^2 + t - 12\right) < 0$$
$$t^2\left(t+4\right)\left(t-3\right) < 0$$

Hint : Where are the only places where the polynomial might change signs?

Step 2



Step 4
All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\boxed{\begin{array}{c}\dfrac{5}{2} \le z < 7 \\ \left[\dfrac{5}{2}, 7\right)\end{array}}$$

Be careful with the endpoints for this problem. Because we have an equal sign in the original inequality we need to include $z = \frac{5}{2}$ because the numerator and hence the rational expression will be zero there. However, we can't include $z = 7$ because the denominator is zero there and so the rational expression has division by zero at that point!

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3. Solve the following inequality.

$$\frac{w^2 + 5w - 6}{w - 3} \ge 0$$

Step 1
The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the numerator and denominator as much as possible.

For this problem we already have zero on one side of the inequality but we do need to factor the numerator.

$$\frac{(w+6)(w-1)}{w-3} \ge 0$$

Hint : Where are the only places where the rational expression might change signs?

Step 2

Recall from the discussion in the notes for this section that the rational expression can only change sign where the numerator is zero and/or where the denominator is zero.

We can see that the numerator will be zero at,

$$w = -6 \qquad\qquad w = 1$$

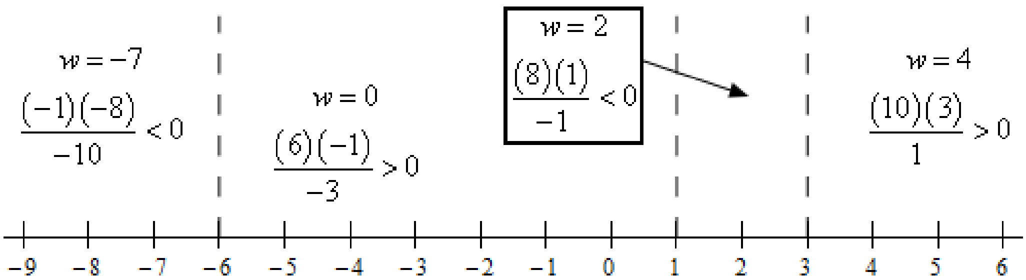
and the denominator will be zero at,

$$w = 3$$

Hint : Knowing that the rational expression can only change sign at the points above how can we quickly determine if the rational expression is positive or negative in the ranges between those points?

Step 3
Just as we did with polynomial inequalities all we need to do is check the rational expression at test points in each region between the points from the previous step. The rational expression will have the same sign as the sign at the test point since it can only change sign at those points.

Here is a sketch of a number line with the points from the previous step graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4
All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\boxed{\begin{array}{ccc} -6 \le w \le 1 & \text{and} & w > 3 \\ \left[-6,1\right] & \text{and} & \left(3,\infty\right) \end{array}}$$

Be careful with the endpoints for this problem. Because we have an equal sign in the original inequality we need to include $w = -6$ and $w = 1$ because the numerator and hence the rational expression will be zero there. However, we can't include $w = 3$ because the denominator is zero there and so the rational expression has division by zero at that point!

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4. Solve the following inequality.

46

Example- 26

Solve for $x : \frac{2x}{2x^2+5x+2} > \frac{1}{x+1}$

Solution: It would be a mistake tocross-multiply and solve since $(x+1)$ and $(2x^2+5x+2)$ are not necessarily positive. We proceed as follows:

$$\frac{2x}{2x^2+5x+2} - \frac{1}{x+1} > 0$$

$$\Rightarrow \quad \frac{2x(x+1)-(2x^2+5x+2)}{(x+1)(2x^2+5x+2)} > 0$$

$$\Rightarrow \quad \frac{-(3x+2)}{(x+1)(2x+1)(x+2)} > 0$$

$$\Rightarrow \quad \frac{(3x+2)}{(x+1)(2x+1)(x+2)} < 0$$

We mark out the roots (of both the numerator and denominator) on a number line and pick the intervals where the expression is negative.

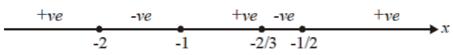


Fig - 29

The required values of *x* are:

$$x \in (-2,-1) \cup \left(-\frac{2}{3}, -\frac{1}{2}\right)$$

Example- 27

Find the values of *x* which satisfy $\frac{8x^2+16x-51}{(2x-3)(x+4)} < 3$

Solution: Here again, do not be tempted to cross multiply by the denominator on both sides since the denominator is not necessarily positive

$$\frac{8x^2+16x-51}{(2x-3)(x+4)} - 3 < 0$$

$$\Rightarrow \quad \frac{(8x^2+16x-51)-3(2x^2+5x-12)}{(2x-3)(x+4)} < 0$$

$$\Rightarrow \quad \frac{2x^2+x-15}{(2x-3)(x+4)} < 0$$

$$\Rightarrow \quad \frac{(2x-5)(x+3)}{(2x-3)(x+4)} < 0$$

We now follow the same procedure as in the previous example (the alternate-interval method)

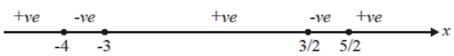


Fig - 30

The required values of *x* are

$$x \in (-4,-3) \cup \left(\frac{3}{2}, \frac{5}{2}\right)$$

Example– 28

Solve the following inequalities for *x*:

(a) $\frac{10x}{x^2+9} \le 1$ **(b)** $\left|\frac{x^2-5x+4}{x^2-4}\right| \le 1$ **(c)** $\left|\frac{x^2-1}{x^2+x+1}\right| < 1$ **(d)** $\log_{10}(x^2-3x+3) \ge 0$

Solution: (a) For this case, notice that the denominator x^2+9 is always positive so that we can directly cross- multiply the denominator to get

$$x^2+9 \ge 10x$$

$$\Rightarrow \quad x^2-10x+9 \ge 0$$

$$\Rightarrow \quad (x-1)(x-9) \ge 0$$

$$\Rightarrow \quad x \le 1 \text{ or } x \ge 9$$

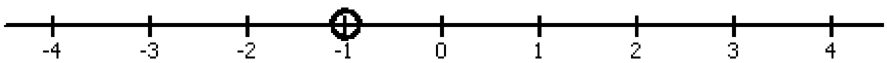
41c

Example 4 **Solving Rational Inequalities**

Rational inequalities can also be solved using a sign analysis procedure. With rational inequalities, however, there is an additional area of consideration – values of x that make the rational expression undefined. Consider the rational inequality below:

$$\frac{x^2+2x-3}{x+1} \geq 0$$

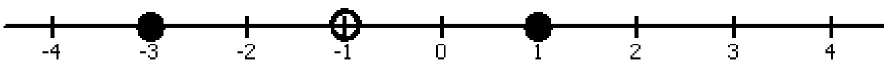
The rational expression is undefined whenever $x+1=0$. Therefore, the value $x=-1$ **cannot** be included in our solution to this inequality. We will display this in our sign analysis procedure by placing an open circle at $x=-1$.



Now we to find the values of x that will make the rational expression equal to 0. That is, we set the numerator equal to zero and solve. These values **will be included** in the solution set since the inequality specifies all values of x that produce a value greater than or equal to 0. On the number line, we show these values as closed circles.

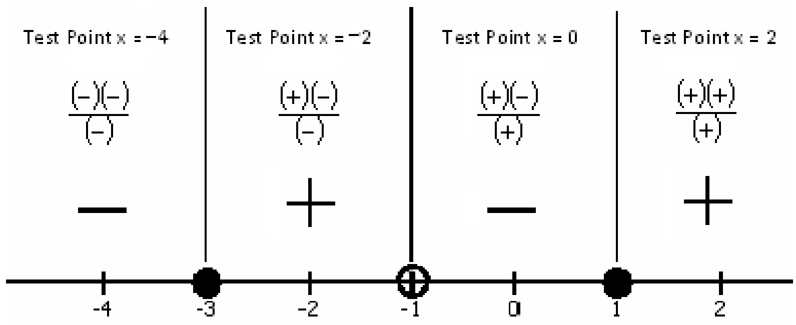
$$\frac{x^2+2x-3}{x+1} = 0 \quad \rightarrow \quad x^2+2x-3=0 \quad \rightarrow \quad (x+3)(x-1)=0$$

$$x=-3 \text{ or } x=1$$



Complete the sign analysis procedure by choosing test points and determining the sign of the rational expression. **Using the factored form of the rational expression will be easiest to determine the sign of the expression**.

$$\frac{x^2+2x-3}{x+1} \geq 0 \quad \rightarrow \quad \frac{(x+3)(x-1)}{x+1} \geq 0$$



The solution to the inequality $\frac{x^2+2x-3}{x+1} \geq 0$ is $-3 \leq x < -1$ or $x \geq 1$.

Example 4

4d

Solve the inequality $\frac{x^2-9}{x+2} \geq 0$. Write the solution set in the interval notation and represent it on the number line.

Solution

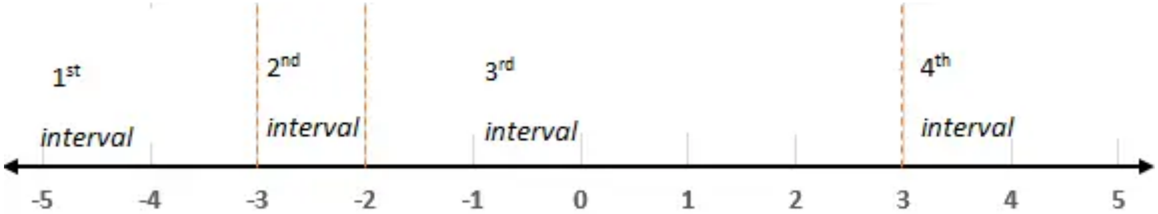
Simplify the quadratic inequality $\frac{x^2-9}{x+2} \geq 0$ by factoring. The resulting equation will be:

$$\frac{(x-3)(x+3)}{x+2} \geq 0.$$

$x - 3 = 0$, $x + 3 = 0$ and $x + 2 = 0$

$x = 3$, $x = -3$ and $x = -2$

The denominator cannot be equal to 0, so -2 is not part of the solution. Using the above roots we will divide the number line into these intervals as shown below: [Remember that, unlike linear functions, quadratics are represented by a parabola].



To write the solution set in the interval notation, we will take points from each of the above interval and see their results in the form of a table.

Show entries Search:

	x=-4	x=-3	x=-2	x=0	x=3	x=4
$\frac{x^2-9}{x+2}$	<0	0	Undefined	<0	0	>0

	x=-4	x=-3	x=-2	x=0	x=3	x=4
Satisfies or not satisfies	Not satisfies	Satisfies	Undefined	Not satisfies	Satisfies	Satisfies

Showing 1 to 2 of 2 entries
Previous Next

We can conclude the following points from the above table:

- Value -4 does not satisfy the inequality, which shows that all the points in the first interval do not satisfy the inequality.
- -3 satisfies but -2 makes the inequality undefined. Hence, these points are expressed as $[-3, 2)$.
- 0 does not satisfy, so all numbers in the third interval are excluded from the solution set except 3.
- 3 and 4 in the fourth interval satisfy the equation which shows all numbers above 3 are included in the solution set. This can be expressed as $[3, \infty)$.

We will express the final result through a union of two sets described above.

The interval notation will be:

$$[-3, 2) U [3, \infty)$$

We will express the inequality on the number line like this:

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4e

Questions c)

$$\frac{4x^2+5x-9}{x^2-x-6} \geq 0$$

Solution

We need to find the zeros of the numerator and denominator by factoring (or any other method). Factor the numerator and the denominator.

4 x^2 + 5x - 9 = (4x + 9)(x - 1) and x^2 - x - 6 = (x - 3)(x + 2)

Rewrite the given inequality as

$$\frac{(4x+9)(x-1)}{(x-3)(x+2)} \geq 0$$

Arrange the zeros of the numerator and the denominator on the number and in order from smallest to the largest as follows.

-∞ - 9 / 4 - 2 1 3 +∞

Select a value of x in the interval (-∞ , - 9/4) and use it to find the sign of the rational expression. Example for x = - 3, the rational expression ((4x + 9)(x - 1)) / ((x - 3)(x + 2)) = 2. Hence the rational expression ((4x + 9)(x - 1)) / ((x - 3)(x + 2)) is positive on the interval (-∞ , - 9/4) .

All the zeros are of odd multiplicity and therefore the sign of ((4x + 9)(x - 1)) / ((x - 3)(x + 2)) will change at all zeros. Hence the signs of the expression ((4x + 9)(x - 1)) / ((x - 3)(x + 2)) as we go from left to right are

-∞ **+** - 9 / 4 **-** - 2 **+** 1 **-** 3 **+** +∞

The solution set of the inequality is given by the union of all intervals where ((4x + 9)(x - 1)) / ((x - 3)(x + 2)) is positive or equal to 0. Hence the solution set for the above inequality, in interval notation, is given by:

(-∞ , - 9/4] ∪ (- 2 , 1] ∪ (3 , +∞)

Questions d)

4f

$$\frac{(x+4)^2(x+6)}{(x^2+7)(x-2)^3} < 0$$

Solution

-∞ - 6 - 4 2 +∞

Select a value of x in the interval (-∞ , - 6) and use it to find the sign of the rational expression. Example for x = - 7, the rational expression ($(x+4)^2(x+6)$) / ($(x^2+7)(x-2)^3$) = 1/4536. Hence the rational expression ($(x+4)^2(x+6)$) / ($(x^2+7)(x-2)$

^{3}) is positive on the interval (-∞ , - 6) .

The sign of ((x+4) ^{2}(x+6)) / ((x ^{2}+7)(x-2) ^{3}) will change at the zeros -6 and 2 because they are of odd multiplicity. The sign of ((x+4) ^{2}(x+6)) / ((x ^{2}+7)(x-2) ^{3}) will not change at - 4 because it is a zero of even multiplicity. Hence the signs of the expression ((x+4) ^{2}(x+6)) / ((x ^{2}+7)(x-2) ^{3}) as we go from left to right are

-∞ **+** - 6 **-** - 4 **-** 2 **+** +∞

The solution set of the inequality is given by the union of all intervals where ((x+4) ^{2}(x+6)) / ((x ^{2}+7)(x-2) ^{3}) is negative. Hence the solution set for the above inequality, in interval notation, is given by:

(- 6 , - 4) ∪ (- 4 , 2)

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4 - 9 - 8

Questions e)

$$\frac{x^3+1}{x^2-9} < 0$$

Solution:

Factor the numerator and the denominator of the given inequality.

x ^{3} + 1 = (x + 1)(x ^{2} - x + 1) and x ^{2} - 9 = (x - 3)(x + 3)

Rewrite the given inequality as follows

$$\frac{(x+1)(x^2-x+1)}{(x-3)(x+3)} < 0$$

The rational expression on the left side of the inequality has the zeros: - 3, - 1 and 3 (x ^{2} - x + 1 has no real zeros). Arrange all the three zeros on the number and in order from smallest to the largest as follows.

-∞ - 3 - 1 3 +∞

Select a value of x in the interval (-∞ , - 3) and use it to find the sign of the rational expression. Example for x = - 4, the rational expression (x ^{3} + 1) / ((x ^{2} - 9)) = - 9. Hence the rational expression on the left side of the given inequality is negative on the interval (-∞ , - 3) .

The sign of the rational expression on the left side of the given inequality will change at the zeros - 1 and 3 because the are of odd multiplicity. Hence the signs of the expression rational expression as we go from left to right are

-∞ **-** - 3 **+** - 1 **-** 3 **+** +∞

The solution set of the inequality is given by the union of all intervals where rational expression on the left side of the given inequality is negative. Hence the solution set for the above inequality, in interval notation, is given by:

(-∞ , - 3) ∪ (- 1 , 3)

48

Questions f)

$$\frac{x+1}{(x-2)(x+3)} \leq 1 - \frac{2}{x-2}$$

Solution

We first rewrite the given inequality with the right side equal to zero.

$$\frac{x+1}{(x-2)(x+3)} - 1 + \frac{2}{x-2} \leq 0$$

Find the LCD (least common denominator) of (x - 2)(x + 3) and x - 2 which is (x - 2)(x + 3).

Rewrite with common LCD.

$$\frac{x+1}{(x-2)(x+3)} - \frac{(x-2)(x+3)}{(x-2)(x+3)} + \frac{2(x+3)}{(x-2)(x+3)} \leq 0$$

Add the rational expressions on the left and simplify.

$$\frac{-x^2+2x+13}{(x-2)(x+3)} \leq 0$$

The rational expression on the left side of the inequality has the zeros at : - 3, 1 - √14 , 2 and 1 + √14. Arrange all the zeros on the number and in order from smallest to the largest as follows.

-∞ - 3 1 - √14 2 1 + √14 +∞

Select a value of x in the interval (-∞ , - 3) and use it to find the sign of the rational expression. Example for x = - 4, the rational expression $(-x^2 + 2x + 13) / ((x-2)(x+3)) = -11/6$. Hence the rational expression on the left side of the given inequality is negative on the interval (-∞ , - 3) .

The sign of the rational expression on the left side of the given inequality will change at all the zeros because they all have odd multiplicity. Hence the signs of the expression rational expression as we go from left to right are

-∞ **-** - 3 **+** 1 - √14 **-** 2 **+** 1 + √14 **-** +∞

The solution set of the inequality is given by:

(-∞ , - 3) ∪ [1 - √14 , 2) ∪ [1 + √14 , +∞)

Example 9.7.3

4i

Solve and write the solution in interval notation: $\frac{5}{x^2-2x-15} > 0$.

Solution

The inequality is in the correct form.

$$\frac{5}{x^2-2x-15} > 0$$

Factor the denominator.

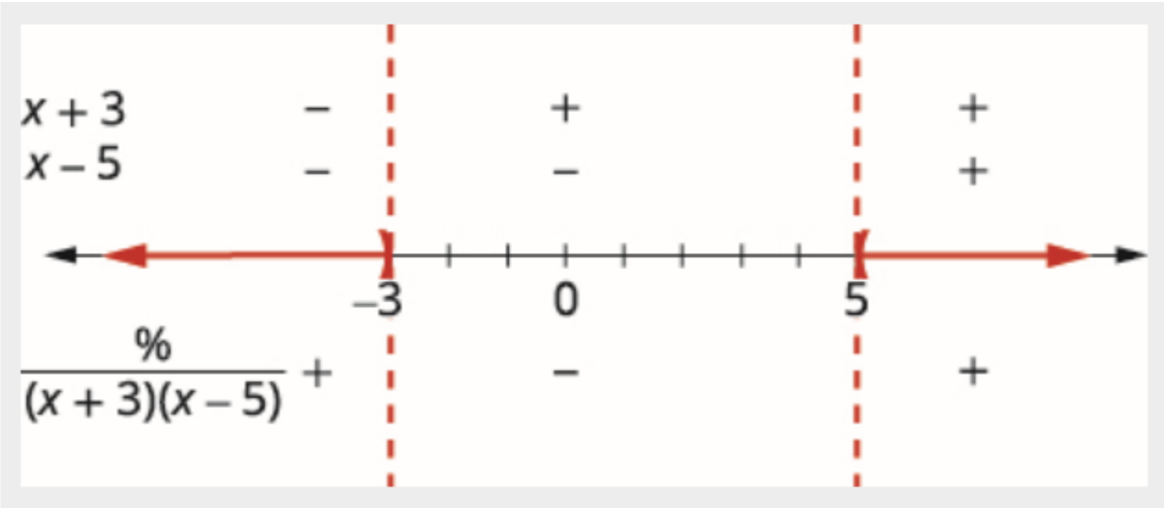
$$\frac{5}{(x+3)(x-5)} > 0$$

Find the critical points. The quotient is 0 when the numerator is 0. Since the numerator is always 5, the quotient cannot be 0.

The quotient will be undefined when the denominator is zero.

$$(x+3)(x-5) = 0$$
$$x = -3, x = 5$$

Use the critical points to divide the number line into intervals.



Test values in each interval. Above the number line show the sign of each factor of the denominator in each interval. Below the number line, show the sign of the quotient.

Write the solution in interval notation.

$$(-\infty, -3) \cup (5, \infty)$$

Example 9.7.4

4j

Solve and write the solution in interval notation: $\frac{1}{3}-\frac{2}{x^2}<\frac{5}{3x}$.

Solution

$$\frac{1}{3}-\frac{2}{x^2}<\frac{5}{3x}$$

Subtract $\frac{5}{3x}$ to get zero on the right.

$$\frac{1}{3}-\frac{2}{x^2}-\frac{5}{3x}<0$$

Rewrite to get each fraction with the LCD $3x^2$.

$$\frac{1\cdot x^2}{3\cdot x^2}-\frac{2\cdot 3}{x^2\cdot 3}-\frac{5\cdot x}{3x-x}<0$$

Simplify.

$$\frac{x^2}{3x^2}-\frac{6}{3x^2}-\frac{5x}{3x^2}<0$$

Subtract the numerators and place the difference over the common denominator.

$$\frac{x^2-5x-6}{3x^2}<0$$

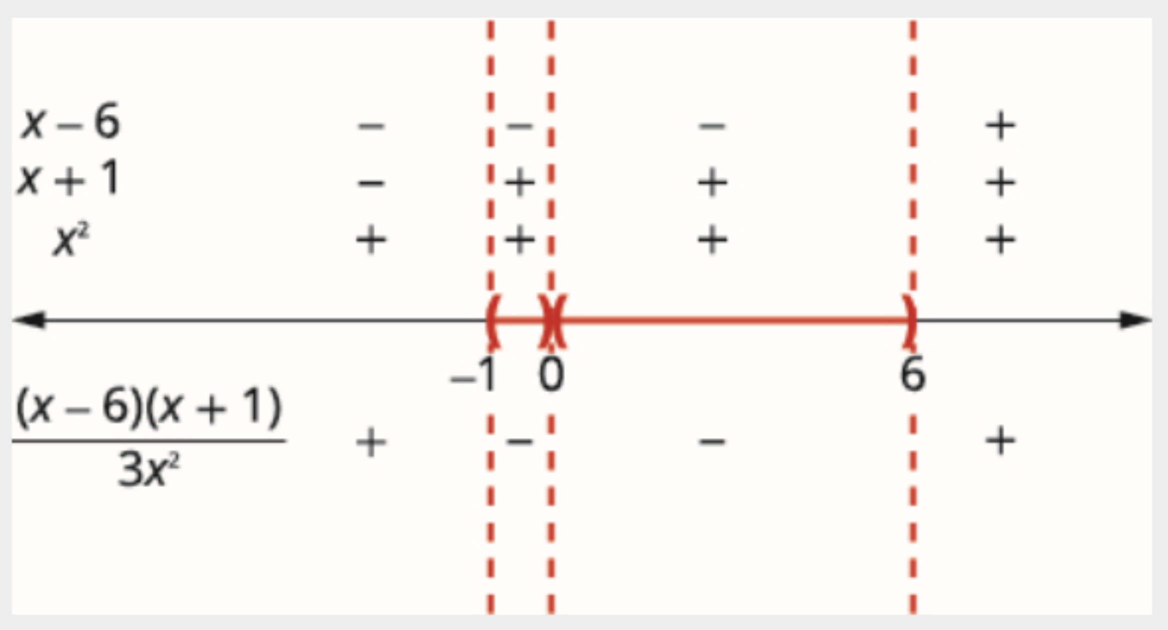
Factor the numerator.

$$\frac{(x-6)(x+1)}{3x^2}<0$$

Find the critical points.

$$\begin{array}{rrr} 3x^2=0 & x-6=0 & x+1=0 \\ x=0 & x=6 & x=-1 \end{array}$$

Use the critical points to divide the number line into intervals.



Above the number line show the sign of each factor in each interval. Below the number line, show the sign of the quotient.

Since, 0 is excluded, the solution is the two $(-1, 0) \cup (0, 6)$ intervals, $(-1, 0)$ and $(0, 6)$.

2. If $f(x) = ax^2 + bx + c$ is a quadratic function, then the lowest point on the graph of $f(x)$ occurs at $x = -b/2a$.

   (a) True - TRUE

   (b) False

   From:

   http://mathquest.carroll.edu/libraries/PRE.student.01.06.pdf

3. Solve inequations

   (a) $x^2 + 4x \geq 21$

   (b) $4x^2 \leq 15 - 17x$

   (c) $x^2 + 34 > 12x$

   (d) $x^2 - 2x + 1 \leq 0$

4. Solve inequations

   (a) $\dfrac{x^2 + 5x - 6}{x - 3} \geq 0$

   (b) $\dfrac{2x}{2x^2 + 5x + 2} > \dfrac{1}{x + 1}$

   (c) $\dfrac{x^2 + 2x - 3}{x + 1} \geq 0$

   (d) $\dfrac{x^2 - 9}{x + 2} \geq 0$

   (e) $\dfrac{4x^2 + 5x - 9}{x^2 - x - 6} \geq 0$

   (f) $\dfrac{(x + 4)^2(x + 6)}{(x^2 + 7)(x - 2)^3} < 0$

   (g) $\dfrac{x^3 + 1}{x^2 - 9} < 0$

   (h) $\dfrac{x + 1}{(x - 2)(x + 3)} \leq 1 - \dfrac{2}{x - 2}$

   (i) $\dfrac{5}{x^2 - 2x - 15} > 0$

   (j) $\dfrac{1}{3} - \dfrac{2}{x^2} < \dfrac{5}{3x}$

Bonus

5. Find a quadratic inequation (like in the 3rd exercise) such that its solution is

   (a) $x \in (-\infty, -3] \cup [2, \infty)$
   Solution: $(x + 3)(x - 2) \geq 0$, $x^2 + x - 6 \geq 0$

   (b) $x \in (-1, 5)$
   Solution: $(x + 1)(x - 5) < 0$, $x^2 - 4x - 5 < 0$

   (c) $x = -6$
   Solution: $(x + 6)^2 \leq 0$, $x^2 + 12x + 36 \leq 0$

   (d) $\emptyset$
   Solution: For example $x^2 + 5 < 0$.

6. Find the values of a parameter $c \in \mathbb{R}$ such that the range of the function

$$f(x) = \frac{x^2 + 2x + c}{x^2 + 4x + 3c}$$

   is equal to $\mathbb{R}$.

Example- 5

Find the set of real values of the parameter *c* so that $\frac{x^2+2x+c}{x^2+4x+3c}$ can take all real values for $x \in \mathbb{R}$.

Solution: We want the given expression to assume all real values (for appropriate *x*), i.e, we want the range of the given expression to be $\mathbb{R}$.

$$\frac{x^2+2x+c}{x^2+4x+3c} = y$$

$$\Rightarrow \quad (1-y)\,x^2 + (2-4y)\,x + c\,(1-3y) = 0.$$

For *x* to be real, the *D* for this equation should be non-negative

$$\Rightarrow 4(1-2y)^2 \geq 4c\,(1-y)\,(1-3y)$$

$$\Rightarrow 4y^2 - 4y + 1 \geq c\left(3y^2 - 4y + 1\right)$$

$$\Rightarrow (4-3c)\,y^2 + (4c-4)\,y + (1-c) \geq 0 \ldots (i)$$

Now comes the crucial step. Since we want the range of *y* to be $\mathbb{R}$. , the constraint (i) should be satisfied by each real value of *y*. This means that the parabola for the left-hand side of (i) should not go below the axis for any value of *y*.

$$\Rightarrow \quad \text{The discriminant for the left hand side of (i) cannot be positive}$$

$$\Rightarrow \quad D\,of\,(i) \leq 0$$
$$\Rightarrow \quad 16(1-c)^2 \leq 4\,(1-c)\,(4-3c)$$

$$\Rightarrow \quad 4c^2 - 8c + 4 \leq 3c^2 - 7c + 4$$

$$\Rightarrow \quad c^2 - c \leq 0$$

$$\Rightarrow \quad c\,(c-1) \leq 0$$

$$\Rightarrow \quad 0 \leq c \leq 1$$