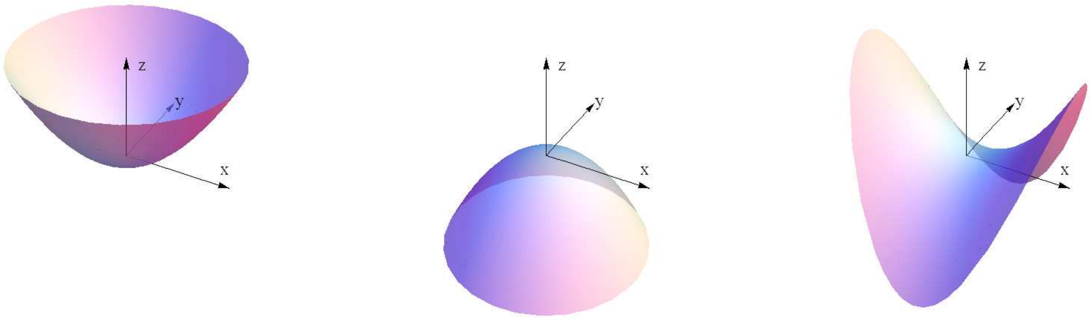
- Here are plots of the three examples:



- We would first like to determine where a function f can have a minimum or maximum value.
  - If $f_x(P) > 0$, then by moving slightly in the positive x-direction the value of f will increase, and by moving slightly in the negative x-direction the value of f will decrease.
  - Inversely, if $f_x(P) < 0$, then by moving slightly in the negative x-direction the value of f will increase, and by moving slightly in the positive x-direction the value of f will decrease.
  - Thus, f can only have a local minimum or maximum at P if $f_x(P) = 0$. By the same reasoning with y in place of x, we must also have $f_y(P) = 0$ at a local minimum or maximum.

- Definition: A critical point of the function $f(x, y)$ is a point (x_0, y_0) such that $f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$, or either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ is undefined.
  - By the observations above, a local minimum or maximum of a function can only occur at a critical point.

- Example: Find the critical points of the function $g(x, y) = x^2 + y^2 + 3$.
  - We have $g_x = 2x$ and $g_y = 2y$. Since both partial derivatives are defined everywhere, the only critical points will occur when $g_x = g_y = 0$.
  - We see $g_x = 0$ precisely when $x = 0$ and $g_y = 0$ precisely when $y = 0$.
  - Thus, there is a unique critical point $(x, y) = \boxed{(0, 0)}$.

- Example: Find the critical points of the function $h(x, y) = x^2 - 4x + y^2 + 2y$.

1a

  - We have $h_x = 2x - 4$ and $h_y = 2y + 2$. Since both partial derivatives are defined everywhere, the only critical points will occur when $h_x = h_y = 0$.
  - We see $h_x = 0$ precisely when $x = 2$ and $h_y = 0$ precisely when $y = -1$.
  - Thus, there is a unique critical point $(x, y) = \boxed{(2, -1)}$.

- Example: Find the critical points of the function $q(x, y) = x^2 + 3xy + 2y^2 - 5x - 8y + 4$.

1a

  - We have $q_x = 2x + 3y - 5$ and $q_y = 3x + 4y - 8$. Since both partial derivatives are defined everywhere, the only critical points will occur when $q_x = q_y = 0$.
  - This yields the equations $2x + 3y - 5 = 0$ and $3x + 4y - 8 = 0$.
  - Setting $2x + 3y - 5 = 0$ and solving for y yields $y = (5 - 2x)/3$. Plugging into the second equation and simplifying eventually gives $x/3 - 4/3 = 0$, so that $x = 4$ and then $y = -1$.
  - Therefore, there is a single critical point $(x, y) = \boxed{(4, -1)}$.

- Example: Find the critical points of the function $p(x, y) = x^3 + y^3 - 3xy$.
  - We have $p_x = 3x^2 - 3y$ and $p_y = 3y^2 - 3x$. Since both partial derivatives are defined everywhere, the only critical points will occur when $p_x = p_y = 0$.

MA2: Solved problems—Functions of more variables: Extrema

1. Find and identify local extrema of $f(x,y) = 2x^3 + 9xy^2 + 15x^2 + 27y^2$.

2. Find and identify local extrema of $f(x,y,z) = x^3 - 2x^2 + y^2 + z^2 - 2xy + xz - yz + 3z$.

3. Find the global extrema of $f(x,y) = x^2 + 2y^2$ given the condition
$$x^2 - 2x + 2y^2 + 4y = 0.$$

4. Find the point on the plane given by $x + y - z = 1$ that is closest to the point $P = (0,-3,2)$ and calculate their distance. Use Lagrange multipliers.

5. A certain line in 3D is given by the equations
$$x + y + z = 1, \qquad 2x - y + z = 3.$$
Find the distance between this line and the point $P = (1,2,-1)$.

6. Find the global extrema of $f(x,y) = x^2 + 4y^2$ on the finite region M bounded by the curves $x^2 + (y+1)^2 = 4$, $y = -1$ and $y = x + 1$.

7. Find the global extrema of $f(x,y) = x^2 + y^2 - 6x + 6y$ on the disk of radius 2, centred at the origin.

8. The equation $y^2 + 2xy = 2x - 4x^2$ defines an implicit function $y(x)$. Find and classify its local extrema.

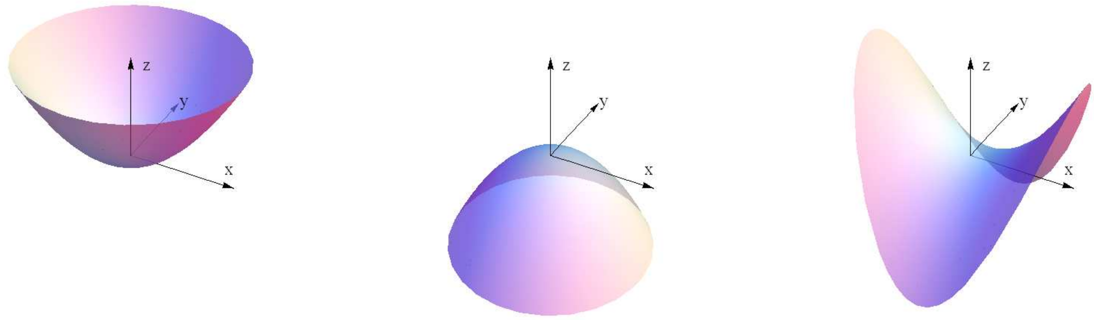
Solutions:

1. First we find stationary points. Partial derivatives:
$$\frac{\partial f}{\partial x} = 6x^2 + 9y^2 + 30x, \qquad \frac{\partial f}{\partial y} = 18xy + 54y.$$
We have to make them equal to zero. We get the system

he

$$2x^2 + 3y^2 + 10x = 0 \qquad xy + 3y = 0.$$
The second equation looks promising, since we can write it as $y(x+3) = 0$. Thus there are two possibilities:
1) $y = 0$. Then the first equation reads $x^2 + 5x = 0$, which yields $x = 0$ and $x = -5$. This possibility therefore leads to points $(0,0)$ and $(-5,0)$
2) $x = -3$. Then the first equation reads $y^2 = 4$, which yields $y = \pm 2$ and points $(-3,\pm 2)$.
Thus we obtain four stationary points: $(0,0)$, $(-5,0)$, $(-3,2)$, and $(-3,-2)$.
To classify them we need to find second order partial derivatives and form the Hess matrix:
$$\frac{\partial^2 f}{\partial x^2} = 12x + 30, \qquad \frac{\partial^2 f}{\partial x \partial y} = 18y, \qquad \frac{\partial^2 f}{\partial y^2} = 18x + 54$$
$$H = \begin{pmatrix} 12x + 30 & 18y \\ 18y & 18x + 54 \end{pmatrix}$$
Now the classification.
For $(0,0)$ we get $H = \begin{pmatrix} 30 & 0 \\ 0 & 54 \end{pmatrix}$. Determinants of principal minors (subdeterminants) are $\Delta_1 = a_{11} = 30$ and $\Delta_2 = \det(H) = 30 \cdot 54 = 1620$. Their signs are $\Delta_1 > 0$, $\Delta_2 > 0$, which shows that the point $f(0,0) = 0$ is a local minimum.
For $(-5,0)$ we get $H = \begin{pmatrix} -30 & 0 \\ 0 & -26 \end{pmatrix}$. Subdeterminants are $\Delta_1 = -30$ and $\Delta_2 = 780$. Their signs are $\Delta_1 < 0$, $\Delta_2 > 0$, which shows that the point $f(-5,0) = 125$ is a local maximum.
For $(-3,-2)$ we get $H = \begin{pmatrix} -6 & -36 \\ -36 & 0 \end{pmatrix}$. Subdeterminants are $\Delta_1 = -6$ and $\Delta_2 = -(-36)^2$. Their signs are $\Delta_1 < 0$, $\Delta_2 < 0$, this does not follow pattern for any local extreme. But from $\Delta_2 < 0$ we conclude that the point $f(-3,2) = 81$ is a saddle point.

- Here are plots of the three examples:



- We would first like to determine where a function f can have a minimum or maximum value.
  - If $f_x(P) > 0$, then by moving slightly in the positive x-direction the value of f will increase, and by moving slightly in the negative x-direction the value of f will decrease.
  - Inversely, if $f_x(P) < 0$, then by moving slightly in the negative x-direction the value of f will increase, and by moving slightly in the positive x-direction the value of f will decrease.
  - Thus, f can only have a local minimum or maximum at P if $f_x(P) = 0$. By the same reasoning with y in place of x, we must also have $f_y(P) = 0$ at a local minimum or maximum.

- Definition: A critical point of the function $f(x, y)$ is a point (x_0, y_0) such that $f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$, or either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ is undefined.
  - By the observations above, a local minimum or maximum of a function can only occur at a critical point.

- Example: Find the critical points of the function $g(x, y) = x^2 + y^2 + 3$.
  - We have $g_x = 2x$ and $g_y = 2y$. Since both partial derivatives are defined everywhere, the only critical points will occur when $g_x = g_y = 0$.
  - We see $g_x = 0$ precisely when $x = 0$ and $g_y = 0$ precisely when $y = 0$.
  - Thus, there is a unique critical point $(x, y) = \boxed{(0, 0)}$.

- Example: Find the critical points of the function $h(x, y) = x^2 - 4x + y^2 + 2y$.
  - We have $h_x = 2x - 4$ and $h_y = 2y + 2$. Since both partial derivatives are defined everywhere, the only critical points will occur when $h_x = h_y = 0$.
  - We see $h_x = 0$ precisely when $x = 2$ and $h_y = 0$ precisely when $y = -1$.
  - Thus, there is a unique critical point $(x, y) = \boxed{(2, -1)}$.

- Example: Find the critical points of the function $q(x, y) = x^2 + 3xy + 2y^2 - 5x - 8y + 4$.
  - We have $q_x = 2x + 3y - 5$ and $q_y = 3x + 4y - 8$. Since both partial derivatives are defined everywhere, the only critical points will occur when $q_x = q_y = 0$.
  - This yields the equations $2x + 3y - 5 = 0$ and $3x + 4y - 8 = 0$.
  - Setting $2x + 3y - 5 = 0$ and solving for y yields $y = (5 - 2x)/3$. Plugging into the second equation and simplifying eventually gives $x/3 - 4/3 = 0$, so that $x = 4$ and then $y = -1$.
  - Therefore, there is a single critical point $(x, y) = \boxed{(4, -1)}$.

- Example: Find the critical points of the function $p(x, y) = x^3 + y^3 - 3xy$.

2a
  - We have $p_x = 3x^2 - 3y$ and $p_y = 3y^2 - 3x$. Since both partial derivatives are defined everywhere, the only critical points will occur when $p_x = p_y = 0$.

- This gives the two equations $3x^2 - 3y = 0$ and $3y^2 - 3x = 0$, or, equivalently, $x^2 = y$ and $y^2 = x$.
- Plugging the first equation into the second yields $x^4 = x$: thus, $x^4 - x = 0$, and factoring yields $x(x-1)(x^2+x+1) = 0$.
- The only real solutions are $x = 0$ (which then gives $y = x^2 = 0$) and $x = 1$ (which then gives $y = x^2 = 1$).
- Therefore, there are two critical points: $\boxed{(0,0) \text{ and } (1,1)}$.

- Example: Find the critical points of the function $f(x,y) = x\, e^{y^2-2x^2}$.
  - We have $f_x = e^{y^2-2x^2} + xe^{y^2-2x^2} \cdot (-4x) = (1-4x^2)e^{y^2-2x^2}$ and $f_y = x\, e^{y^2-2x^2} \cdot (2y) = 2xy\, e^{y^2-2x^2}$. Since both partial derivatives are defined everywhere, the only critical points will occur when $f_x = f_y = 0$.
  - Since exponentials are never zero, we see that $f_x = 0$ when $1 - 4x^2 = 0$ so that $x = \pm\frac{1}{2}$, while $f_y = 0$ when $2xy = 0$ so that $x = 0$ or $y = 0$. Since x cannot be zero by the first equation (since $x = \pm\frac{1}{2}$) we must have $y = 0$.
  - Therefore, there are two critical points: $\boxed{(\frac{1}{2}, 0) \text{ and } (-\frac{1}{2}, 0)}$.

1.3.2 Classifying Critical Points

- Now that we have a list of critical points (namely, the places that a function could potentially have a minimum or maximum value) we would like to know whether those points actually are minima or maxima of f.
- Definition: The discriminant (also called the Hessian) at a critical point is the value $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$, where each of the second-order partials is evaluated at the critical point.
  - One way to remember the definition of the discriminant is as the determinant of the matrix of the four second-order partials: $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$. (We are implicitly using the fact that $f_{xy} = f_{yx}$.)
  - Example: For $g(x,y) = x^2 + y^2$ we have $g_{xx} = g_{yy} = 2$ and $g_{xy} = 0$ so $D = 4$ at the origin.
  - Example: For $h(x,y) = x^2 - y^2$ we have $h_{xx} = 2$, $h_{yy} = -2$, and $h_{xy} = 0$ so $D = -4$ at the origin.
  - Remark: The reason this value is named "discriminant" can be seen by computing D for the function $p(x,y) = ax^2 + bxy + cy^2$: the result is $D = 4ac - b^2$, which is -1 times the quantity $b^2 - 4ac$, the famous discriminant for the quadratic polynomial $ax^2 + bx + c$. (Recall that the discriminant of $ax^2 + bx + c$ determines how many real roots the polynomial has.)
- Theorem (Second Derivatives Test): Suppose P is a critical point of $f(x,y)$, and let D be the value of the discriminant $f_{xx}f_{yy} - f_{xy}^2$ at P. If $D > 0$ and $f_{xx} > 0$, then the critical point is a minimum. If $D > 0$ and $f_{xx} < 0$, then the critical point is a maximum. If $D < 0$, then the critical point is a saddle point. (If $D = 0$, then the test is inconclusive.)
  - Proof (outline): Assume for simplicity that P is at the origin. Then one may show that the function $f(x,y) - f(P)$ is closely approximated by the polynomial $ax^2 + bxy + cy^2$, where $a = \frac{1}{2}f_{xx}$, $b = f_{xy}$, and $c = \frac{1}{2}f_{yy}$. If $D \neq 0$, then the behavior of $f(x,y)$ near the critical point P will be the same as that quadratic polynomial. Completing the square and examining whether the resulting quadratic polynomial has any real roots and whether it opens or downwards yields the test.
- We can combine the above results to yield a procedure for finding and classifying the critical points of a function $f(x,y)$:
  - Step 1: Compute both partial derivatives f_x and f_y.
  - Step 2: Find all points (x,y) where *both* partial derivatives are zero, or where (at least) *one* of the partial derivatives is undefined.

Example 4 ($f(x,y) = x^2 - 2xy + 2y^2 + 2x - 6y + 12$)

2c

Find all critical points of $f(x,y) = x^2 - 2xy + 2y^2 + 2x - 6y + 12$.

Solution. As a preliminary calculation, we find the two first order partial derivatives of $f(x,y)$.

$$f_x(x,y) = 2x - 2y + 2$$
$$f_y(x,y) = -2x + 4y - 6$$

So the critical points are the solutions of the pair of equations $2x - 2y + 2 = 0$, $-2x + 4y - 6$, or equivalently (dividing by two and moving the constants to the right hand side)

$$x - y = -1 \tag{1a}$$
$$-x + 2y = 3 \tag{1b}$$

One strategy for solving a system of two equations in two unknowns (x and y) like this is to

- First use one of the equations to solve for one of the unkowns in terms of the other unknown. For example (1a) tells us that $y = x + 1$. This expresses y in terms of x. We say that we have solved for y in terms of x.
- Then substitute the result, $y = x + 1$ in our case, into the other equation, (1b). In our case, this gives
$$-x + 2(x+1) = 3 \iff x + 2 = 3 \iff x = 1$$
- We have now found that $x = 1$, $y = x + 1 = 2$ is the only solution. So the only critical point is $(1,2)$.

An alternative strategy for solving a system of two equations in two unknowns like (1) is to

- add equations (1a) and (1b) together. This gives
$$\text{(1a)} + \text{(1b)} : \quad (1-1)x + (-1+2)y = -1 + 3 \iff y = 2$$
The point here is that adding equations (1a) and (1b) together eliminates the unknown x, leaving us with one equation in the unknown y, which is easily solved. For other systems of equations you might have multiply the equations by some numbers before adding them together.
- We now know that $y = 2$. Substituting it into (1a) gives us
$$x - 2 = -1 \implies x = 1$$
- Once again we have found that the only critical point is $(1,2)$.

Example 4

Example 5 ($f(x,y) = 2x^3 - 6xy + y^2 + 4y$)

Find all critical points of $f(x,y) = 2x^3 - 6xy + y^2 + 4y$.

2f

Solution. The first order partial derivatives are

$$f_x = 6x^2 - 6y \qquad f_y = -6x + 2y + 4$$

So the critical points are the solutions of

$$6x^2 - 6y = 0 \qquad -6x + 2y + 4 = 0$$

We can rewrite the first equation as $y = x^2$, which expresses y as a function of x. We can then substitute $y = x^2$ into the second equation, giving

$$\begin{aligned} -6x + 2y + 4 = 0 &\iff -6x + 2x^2 + 4 = 0 \iff x^2 - 3x + 2 = 0 \iff (x-1)(x-2) = 0 \\ &\iff x = 1 \text{ or } 2 \end{aligned}$$

When $x = 1$, $y = 1^2 = 1$ and when $x = 2$, $y = 2^2 = 4$. So, there are two critical points: $(1,1)$, $(2,4)$.

Example 5

Example 6 ($f(x,y) = xy(5x + y - 15)$)

Find all critical points of $f(x,y) = xy(5x + y - 15)$.

2g

Solution. The first order partial derivatives of $f(x,y) = xy(5x + y - 15)$ are

$$\begin{aligned} f_x(x,y) &= y(5x + y - 15) + xy(5) &= y(5x + y - 15) + y(5x) &= y(10x + y - 15) \\ f_y(x,y) &= x(5x + y - 15) + xy(1) &= x(5x + y - 15) + x(y) &= x(5x + 2y - 15) \end{aligned}$$

The critical points are the solutions of $f_x(x,y) = f_y(x,y) = 0$ or

$$y(10x + y - 15) = 0 \qquad \text{and} \qquad x(5x + 2y - 15) = 0 \tag{2}$$

The first equation, $y(10x + y - 15) = 0$, is satisfied if either of the two factors y, $(10x + y - 15)$ is zero. So the first equation is satisfied if either of the two equations

$$y = 0 \tag{3a}$$
$$10x + y = 15 \tag{3b}$$

is satisfied. The second equation, $x(5x + 2y - 15) = 0$, is satisfied if either of the two factors x, $(5x + 2y - 15)$ is zero. So the first equation is satisfied if either of the two equations

$$x = 0 \tag{4a}$$
$$5x + 2y = 15 \tag{4b}$$

is satisfied. So both critical point equations (2) are satisfied if one of (3a), (3b) is satisfied and in addition one of (4a), (4b) is satisfied. There are four possibilities:

- (3a) and (4a) are satisfied if and only if $x = y = 0$
- (3a) and (4b) are satisfied if and only if $y = 0,\ 5x + 2y = 15 \iff y = 0,\ 3x = 15$
- (3b) and (4a) are satisfied if and only if $10x + y = 15,\ x = 0 \iff y = 15,\ x = 0$
- (3b) and (4b) are satisfied if and only if $10x + y = 15,\ 5x + 2y = 15$. We can use, for example, the second of these equations to solve for x in terms of y: $x = \frac{1}{5}(15 - 2y)$. When we substitute this into the first equation we get $2(15 - 2y) + y = 15$, which we can solve for y. This gives $-3y = 15 - 30$ or $y = 5$ and then $x = \frac{1}{5}(15 - 2 \times 5) = 1$.

In conclusion, the critical points are $(0,0)$, $(3,0)$, $(0,15)$ and $(1,5)$.

A more compact way to write what we have just done is

$$\begin{array}{rccc} & f_x(x,y) = 0 & \text{and} & f_y(x,y) = 0 \\ \iff & y(10x + y - 15) = 0 & \text{and} & x(5x + 2y - 15) = 0 \\ \iff & \big\{y = 0 \text{ or } 10x + y = 15\big\} & \text{and} & \big\{x = 0 \text{ or } 5x + 2y = 15\big\} \\ \iff & \multicolumn{3}{l}{\big\{x = y = 0\big\} \text{ or } \big\{y = 0,\ x = 3\big\} \text{ or } \big\{x = 0,\ y = 15\big\} \text{ or } \big\{x = 1,\ y = 5\big\}} \end{array}$$

Example 6

Example 7

In a certain community, there are two breweries in competition, so that sales of each negatively affect the profits of the other. If brewery A produces x litres of beer per month and brewery B produces y litres per month, then the profits of the two breweries are given by

$$P = 2x - \frac{2x^2 + y^2}{10^6} \qquad Q = 2y - \frac{4y^2 + x^2}{2 \times 10^6}$$

respectively. Find the sum of the two profits if each brewery independently sets its own production level to maximize its own profit and assumes that its competitor does likewise. Find the sum of the two profits if the two breweries cooperate so as to maximize that sum.

Solution. If A adjusts x to maximize P (for y held fixed) and B adjusts y to maximize Q (for x held fixed) then x and y are determined by

$$\begin{array}{rcr} P_x = 2 - \frac{4x}{10^6} = 0 & \implies & x = \frac{1}{2}10^6 \\ Q_y = 2 - \frac{8y}{2 \times 10^6} = 0 & \implies & y = \frac{1}{2}10^6 \\ & \implies & P + Q = 2(x + y) - \frac{1}{10^6}\big(\frac{5}{2}x^2 + 3y^2\big) \\ & \implies & = 10^6\big(1 + 1 - \frac{5}{8} - \frac{3}{4}\big) = \frac{5}{8}10^6 \end{array}$$

On the other hand if (A, B) adjust (x, y) to maximize $P + Q = 2(x + y) - \frac{1}{10^6}\big(\frac{5}{2}x^2 + 3y^2\big)$, then x and y are determined by

- This gives the two equations $3x^2 - 3y = 0$ and $3y^2 - 3x = 0$, or, equivalently, $x^2 = y$ and $y^2 = x$.
- Plugging the first equation into the second yields $x^4 = x$: thus, $x^4 - x = 0$, and factoring yields $x(x-1)(x^2+x+1) = 0$.
- The only real solutions are $x = 0$ (which then gives $y = x^2 = 0$) and $x = 1$ (which then gives $y = x^2 = 1$).
- Therefore, there are two critical points: $\boxed{(0,0) \text{ and } (1,1)}$.

- Example: Find the critical points of the function $f(x,y) = x\,e^{y^2-2x^2}$.
  - We have $f_x = e^{y^2-2x^2} + xe^{y^2-2x^2} \cdot (-4x) = (1-4x^2)e^{y^2-2x^2}$ and $f_y = x\,e^{y^2-2x^2} \cdot (2y) = 2xy\,e^{y^2-2x^2}$. Since both partial derivatives are defined everywhere, the only critical points will occur when $f_x = f_y = 0$.
  - Since exponentials are never zero, we see that $f_x = 0$ when $1 - 4x^2 = 0$ so that $x = \pm\frac{1}{2}$, while $f_y = 0$ when $2xy = 0$ so that $x = 0$ or $y = 0$. Since x cannot be zero by the first equation (since $x = \pm\frac{1}{2}$) we must have $y = 0$.
  - Therefore, there are two critical points: $\boxed{(\frac{1}{2}, 0) \text{ and } (-\frac{1}{2}, 0)}$.

1.3.2 Classifying Critical Points

- Now that we have a list of critical points (namely, the places that a function could potentially have a minimum or maximum value) we would like to know whether those points actually are minima or maxima of f.
- Definition: The discriminant (also called the Hessian) at a critical point is the value $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$, where each of the second-order partials is evaluated at the critical point.
  - One way to remember the definition of the discriminant is as the determinant of the matrix of the four second-order partials: $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$. (We are implicitly using the fact that $f_{xy} = f_{yx}$.)
  - Example: For $g(x,y) = x^2 + y^2$ we have $g_{xx} = g_{yy} = 2$ and $g_{xy} = 0$ so $D = 4$ at the origin.
  - Example: For $h(x,y) = x^2 - y^2$ we have $h_{xx} = 2$, $h_{yy} = -2$, and $h_{xy} = 0$ so $D = -4$ at the origin.
  - Remark: The reason this value is named "discriminant" can be seen by computing D for the function $p(x,y) = ax^2 + bxy + cy^2$: the result is $D = 4ac - b^2$, which is -1 times the quantity $b^2 - 4ac$, the famous discriminant for the quadratic polynomial $ax^2 + bx + c$. (Recall that the discriminant of $ax^2 + bx + c$ determines how many real roots the polynomial has.)
- Theorem (Second Derivatives Test): Suppose P is a critical point of $f(x,y)$, and let D be the value of the discriminant $f_{xx}f_{yy} - f_{xy}^2$ at P. If $D > 0$ and $f_{xx} > 0$, then the critical point is a minimum. If $D > 0$ and $f_{xx} < 0$, then the critical point is a maximum. If $D < 0$, then the critical point is a saddle point. (If $D = 0$, then the test is inconclusive.)
  - Proof (outline): Assume for simplicity that P is at the origin. Then one may show that the function $f(x,y) - f(P)$ is closely approximated by the polynomial $ax^2 + bxy + cy^2$, where $a = \frac{1}{2}f_{xx}$, $b = f_{xy}$, and $c = \frac{1}{2}f_{yy}$. If $D \neq 0$, then the behavior of $f(x,y)$ near the critical point P will be the same as that quadratic polynomial. Completing the square and examining whether the resulting quadratic polynomial has any real roots and whether it opens or downwards yields the test.
- We can combine the above results to yield a procedure for finding and classifying the critical points of a function $f(x,y)$:
  - Step 1: Compute both partial derivatives f_x and f_y.
  - Step 2: Find all points (x,y) where *both* partial derivatives are zero, or where (at least) *one* of the partial derivatives is undefined.

For $(-3,2)$ we get $H=\begin{pmatrix}-6 & -36\\ -36 & 0\end{pmatrix}$. Subdeterminants are $\Delta_1=-6$ and $\Delta_2=-36^2$. As above, from $\Delta_2<0$ we conclude that the point $f(-3,-2)=81$ is a saddle point.

Some people prefer a different approach that might be simpler if the derivatives are not too bad, it is also somewhat more organized.
First we evaluate those subdeterminants in general, we obtain $\Delta_1=12x+30$ and
$\Delta_2=(12x+30)(18x+54)-(18y)^2=36(6x^2+23x+45-9y^2)$. Then we substitute the stationary points and reach conclusions:

point:	$(0,0)$	$(-5,0)$	$(-3,2)$	$(-3,-2)$
Δ_1:	$+$	$-$	$-$	$-$
Δ_2:	$+$	$+$	$-$	$-$
conclusion:	loc. min.	loc. max.	saddle	saddle

Then one has to write the answer: $f(0,0)=0$ is a local minimum, $f(-5,0)=125$ is a local maximum, $f(-3,2)=f(-3,-2)=81$ are saddle points.

2.

2. First we find stationary points. Partial derivatives:

$$\frac{\partial f}{\partial x}=3x^2-4x-2y+z,\qquad \frac{\partial f}{\partial y}=2y-2x-z,\qquad \frac{\partial f}{\partial z}=2z+x-y+3.$$

We have to solve the system

$$\begin{aligned}3x^2-4x-2y+z&=0\\ 2y-2x-z&=0\\ 2z+x-y+3&=0\end{aligned}$$

Now noe of the equations has the convenient form of a product, so the method used in the previous problem does not help. Another popular method is elimination.
Since there is x^2 in the first equation, we will try to use the others to get rid of y and z in this first equation and then apply the quadratic rule. We can express $z=2y-2x$ from the second equation and put into the first and the third, obtaining $3x^2-6x=0$ and $3y-3x=-3$. What a piece of luck, the first one already features only x, the third one will also come handy when we express $y=x-1$.
The equation $3x^2-6x=0$ has two solutions: $x=0$ and $x=2$.
If $x=0$, then $y=-1$ and $z=-2$. If $x=2$, then $y=1$ and $z=-2$. Thus we have two stationary points, $(0,-1,-2)$ and $(2,1,-2)$.
Now we use the second derivative test. First we need second partial derivatives arranged into the Hess matrix.

$$H=\begin{pmatrix}6x-4 & -2 & 1\\ -2 & 2 & -1\\ 1 & -1 & 2\end{pmatrix}.$$

Calculating subdeterminant in general does not sound very appealing (but you can try this approach), we handle each point separately.
For $(0,-1,-2)$ we get

$$H=\begin{pmatrix}-4 & -2 & 1\\ -2 & 2 & -1\\ 1 & -1 & 2\end{pmatrix}\Longrightarrow \Delta_1=-4,\ \Delta_2=\begin{vmatrix}-4 & -2\\ -2 & 2\end{vmatrix}=-12,\ \Delta_3=|H|=-26.$$

Since $\Delta_2<0$, at the stationary point $(0,-1,-2)$ there is no local extreme but a saddle point.
Better answer: $f(0,-1,2)=13$ is a saddle point (we give more information this way).
(Some authors do not use the notion of saddle in cases of more than two variables, they would just say that this point is not a local extreme.)

2a

The equation $g(x,y)=k$ becomes $x+2y=7$. Therefore, the system of equations that needs to be solved is

$$2x-2=\lambda \tag{13.10.9}$$

$$8y+8=2\lambda \tag{13.10.10}$$

$$x+2y=7. \tag{13.10.11}$$

3. This is a linear system of three equations in three variables. We start by solving the second equation for λ and substituting it into the first equation. This gives $\lambda=4y+4$, so substituting this into the first equation gives

$$2x-2=4y+4.$$

Solving this equation for x gives $x=2y+3$. We then substitute this into the third equation:

$$(2y+3)+2y=7$$

$$4y=4$$

$$y=1.$$

Since $x=2y+3$, this gives $x=5.$

4. Next, we evaluate $f(x,y)=x^2+4y^2-2x+8y$ at the point $(5,1)$,

for the maximum value of f, subject to this constraint.

2. So, we calculate the gradients of both f and g:

$$\vec{\nabla} f(x,y) = (48 - 2x - 2y)\hat{\mathbf{i}} + (96 - 2x - 18y)\hat{\mathbf{j}}$$

$$\vec{\nabla} g(x,y) = 5\hat{\mathbf{i}} + \hat{\mathbf{j}}.$$

The equation $\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$ becomes

$$(48 - 2x - 2y)\hat{i} + (96 - 2x - 18y)\hat{\mathbf{j}} = \lambda(5\hat{\mathbf{i}} + \hat{\mathbf{j}}),$$

which can be rewritten as

$$(48 - 2x - 2y)\hat{\mathbf{i}} + (96 - 2x - 18y)\hat{\mathbf{j}} = \lambda 5\hat{\mathbf{i}} + \lambda\hat{\mathbf{j}}.$$

We then set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$48 - 2x - 2y = 5\lambda$$

$$96 - 2x - 18y = \lambda.$$

The equation $g(x,y) = k$ becomes $5x + y = 54$. Therefore, the system of equations that needs to be solved is

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$$48 - 2x - 2y = 5\lambda$$

$$96 - 2x - 18y = \lambda$$

$$5x + y = 54.$$

3. We use the left-hand side of the second equation to replace λ in the first equation:

$$48 - 2x - 2y = 5(96 - 2x - 18y)$$

$$48 - 2x - 2y = 480 - 10x - 90y$$

$$8x = 432 - 88y$$

$$x = 54 - 11y.$$

Then we substitute this into the third equation:

$$5(54 - 11y) + y = 54$$

$$270 - 55y + y = 54$$

$$216 = 54y$$

$$y = 4.$$

Since $x = 54 - 11y$, this gives $x = 10.$

Let's choose $x = y = 1$. No reason for these values other than they are "easy" to work with. Plugging these into the constraint gives,

$$1 + z + z = 32 \qquad \rightarrow \qquad 2z = 31 \qquad \rightarrow \qquad z = \frac{31}{2}$$

So, this is a set of dimensions that satisfy the constraint and the volume for this set of dimensions is,

$$V = f\left(1,1,\frac{31}{2}\right) = \frac{31}{2} = 15.5 < 34.8376$$

So, the new dimensions give a smaller volume and so our solution above is, in fact, the dimensions that will give a maximum volume of the box are $x = y = z = 3.266$

Notice that we never actually found values for λ in the above example. This is fairly standard for these kinds of problems. The value of λ isn't really important to determining if the point is a maximum or a minimum so often we will not bother with finding a value for it. On occasion we will need its value to help solve the system, but even in those cases we won't use it past finding the point.

Example 2 Find the maximum and minimum of $f(x,y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$.

Solution
This one is going to be a little easier than the previous one since it only has two variables. Also, note that it's clear from the constraint that region of possible solutions lies on a disk of radius $\sqrt{136}$ which is a closed and bounded region, $-\sqrt{136} \le x, y \le \sqrt{136}$, and hence by the Extreme Value Theorem we know that a minimum and maximum value must exist.

2c

Here is the system that we need to solve.

$$\begin{aligned} 5 &= 2\lambda x \\ -3 &= 2\lambda y \\ x^2 + y^2 &= 136 \end{aligned}$$

Notice that, as with the last example, we can't have $\lambda = 0$ since that would not satisfy the first two equations. So, since we know that $\lambda \neq 0$ we can solve the first two equations for *x* and *y* respectively. This gives,

$$x = \frac{5}{2\lambda} \qquad\qquad y = -\frac{3}{2\lambda}$$

Plugging these into the constraint gives,

$$\frac{25}{4\lambda^2} + \frac{9}{4\lambda^2} = \frac{17}{2\lambda^2} = 136$$

We can solve this for λ.

$$\lambda^2 = \frac{1}{16} \qquad \Rightarrow \qquad \lambda = \pm\frac{1}{4}$$

Now, that we know λ we can find the points that will be potential maximums and/or minimums.

If $\lambda = -\frac{1}{4}$ we get,

$$x = -10 \qquad\qquad y = 6$$

and if $\lambda = \frac{1}{4}$ we get,

$$x = 10 \qquad\qquad y = -6$$

To determine if we have maximums or minimums we just need to plug these into the function. Also recall from the discussion at the start of this solution that we know these will be the minimum and maximums because the Extreme Value Theorem tells us that minimums and maximums will exist for this problem.

Here are the minimum and maximum values of the function.

$$f(-10,6) = -68 \qquad \text{Minimum at } (-10,6)$$
$$f(10,-6) = 68 \qquad \text{Maximum at } (10,-6)$$

In the first two examples we've excluded $\lambda = 0$ either for physical reasons or because it wouldn't solve one or more of the equations. Do not always expect this to happen. Sometimes we will be able to automatically exclude a value of λ and sometimes we won't.

Let's take a look at another example.

Example 3 Find the maximum and minimum values of $f(x,y,z) = xyz$ subject to the constraint $x + y + z = 1$. Assume that $x, y, z \geq 0$.

Solution

First note that our constraint is a sum of three positive or zero number and it must be 1. Therefore, it is clear that our solution will fall in the range $0 \leq x, y, z \leq 1$ and so the solution must lie in a closed and bounded region and so by the Extreme Value Theorem we know that a minimum and maximum value must exist.

Here is the system of equation that we need to solve.

$$yz = \lambda \tag{10}$$
$$xz = \lambda \tag{11}$$
$$xy = \lambda \tag{12}$$

By the way, comparing values at the three points we can guess that $f\big(\frac{2}{3}\sqrt{5},\frac{1}{3}\big)=\frac{8}{3}$ is a local minimum of f with respect to the arc we investigate here.
2b) The next part to explore is the oblique segment given by $y=x+1$, $-2\le x\le 0$. It has two endpoints, $(0,1)$ we already listed and the other is $f(-2,-1)=8$, another candidate for extrema on M.
To see what happens in the middle of this curve we can again use Lagrange multipliers, this time applied to $g(x,y)=y-x=1$:

$$\left.\begin{aligned}2x&=-\lambda\\ 8y&=\lambda\\ y-x&=1\end{aligned}\right\}\implies\left.\begin{aligned}4y&=-x\\ y-x&=1\end{aligned}\right\}\implies 5y=1\implies y=\tfrac{1}{5},$$

then $x=-\frac{4}{5}$. We have another candidate, $f\big(-\frac{4}{5},\frac{1}{5}\big)=\frac{4}{5}$ (which seems to be a local minimum of f with respect to that oblique line).
Since the condition is so simple, one might be tempted to simply substitute the expression $y=x+1$ into f, obtaining

$$\varphi(x)=f(x,x+1)=x^2+4(x+1)^2=5x^2+8x+4.$$

The one can use the usual tools of one-variable calculus to find candidates for extrema of this function over the interval $[-2,0]$, arriving at the same results as above.
2c) Finally we check on the horizontal segment, which is given by $y=-1$, $-2\le x\le 2$. Again, candidates come as endpoints, but we already included them above, and local extrema from the middle. Here the simplest way is to substitute, we are interested in extrema of $\varphi(x)=f(x,-1)=x^2+4$ over $-2\le x\le 2$. Endpoints are already done, local extrema are given by the condition $\varphi'(x)=0$, which gives $x=0$. We have another candidate to consider, $f(0,-1)=4$.

Now we put it all together. Candidates are $f(0,0)=0$, $f(2,-1)=8$, $f(0,1)=4$, $f(-2,-1)=8$, $f\big(\frac{2}{3}\sqrt{5},\frac{1}{3}\big)=\frac{8}{3}$, $f\big(-\frac{4}{5},\frac{1}{5}\big)=\frac{4}{5}$, and $f(0,-1)=4$. Comparing values we arrive at the answer: Global maximum of f over M is $f(-2,-1)=f(2,-1)=8$, global minimum is $f(0,0)=0$.
Just out of curiosity, global minimum with respect to the whole border is $f\big(-\frac{4}{5},\frac{1}{5}\big)=\frac{4}{5}$.

Remark: If we wanted to express M using set notation, we could start with the disc given by the first condition and intersect it with two half-planes given by the lines:

$$M=\{(x,y)\in\mathbb{R}^2;\ x^2+(y+1)^2\le 4\ \text{ and }\ y\ge -1\ \text{ and }\ y\le x+1\}.$$

7. The set M can be described as $M=\{(x,y)\in\mathbb{R}^2;\ x^2+y^2\le 4\}$. To find the global extrema we first look at what is happening inside, that is, we check on stationary points of f:

$$\left.\begin{aligned}\tfrac{\partial f}{\partial x}&=0\\ \tfrac{\partial f}{\partial y}&=0\end{aligned}\right\}\implies\left.\begin{aligned}2x-6&=0\\ 2y+6&=0\end{aligned}\right\}\implies x=3,\ y=-3.$$

However, the point $(3,-3)$ is not in M and therefore we disregard it.
Next we have to check on the boundary, that is, we have to find extrema of f given the condition $x^2+y^2=4$. This calls for Lagrange multipliers:

2)

$$\left.\begin{aligned}\tfrac{\partial f}{\partial x}&=\lambda\tfrac{\partial g}{\partial x}\\ \tfrac{\partial f}{\partial y}&=\lambda\tfrac{\partial g}{\partial y}\\ g&=4\end{aligned}\right\}\implies\left.\begin{aligned}2x-6&=\lambda\cdot 2x\\ 2y+6&=\lambda\cdot 2y\\ x^2+y^2&=4\end{aligned}\right\}\implies\begin{aligned}x-3&=\lambda x\\ y+3&=\lambda y\\ x^2+y^2&=4\end{aligned}$$

Now we would like to eliminate λ from the first two equations. Could it happen that $x=0$? Then the first equation would read $-3=0$, not possible; similarly we have $y\ne 0$. Thus we can express λ from the first equation, substitute into the second and eventually obtain $y=-x$. Putting this into the constraint equation we get $x=\pm\sqrt{2}$. Thus there are two candidates, $(\sqrt{2},-\sqrt{2})$ and $(-\sqrt{2},\sqrt{2})$. We put them into f: $f(\sqrt{2},-\sqrt{2})=4-12\sqrt{2}$, $f(-\sqrt{2},\sqrt{2})=4+12\sqrt{2}$, so it seems that the first is the minimum, the second is the maximum.

$$\frac{25}{4\lambda^2}+\frac{9}{4\lambda^2}=\frac{17}{2\lambda^2}=136$$

We can solve this for λ.

$$\lambda^2=\frac{1}{16} \qquad \Rightarrow \qquad \lambda=\pm\frac{1}{4}$$

Now, that we know λ we can find the points that will be potential maximums and/or minimums.

If $\lambda=-\frac{1}{4}$ we get,

$$x=-10 \qquad\qquad y=6$$

and if $\lambda=\frac{1}{4}$ we get,

$$x=10 \qquad\qquad y=-6$$

To determine if we have maximums or minimums we just need to plug these into the function. Also recall from the discussion at the start of this solution that we know these will be the minimum and maximums because the Extreme Value Theorem tells us that minimums and maximums will exist for this problem.

Here are the minimum and maximum values of the function.

$$f(-10,6)=-68 \qquad \text{Minimum at } (-10,6)$$
$$f(10,-6)=68 \qquad \text{Maximum at } (10,-6)$$

In the first two examples we've excluded $\lambda=0$ either for physical reasons or because it wouldn't solve one or more of the equations. Do not always expect this to happen. Sometimes we will be able to automatically exclude a value of λ and sometimes we won't.

Let's take a look at another example.

Example 3 Find the maximum and minimum values of $f(x,y,z)=xyz$ subject to the constraint $x+y+z=1$. Assume that $x,y,z\ge 0$.

Solution

First note that our constraint is a sum of three positive or zero number and it must be 1. Therefore, it is clear that our solution will fall in the range $0\le x,y,z\le 1$ and so the solution must lie in a closed and bounded region and so by the Extreme Value Theorem we know that a minimum and maximum value must exist.

2e

Here is the system of equation that we need to solve.

$$yz=\lambda \tag{10}$$
$$xz=\lambda \tag{11}$$
$$xy=\lambda \tag{12}$$

$$x + y + z = 1 \qquad (13)$$

Let's start this solution process off by noticing that since the first three equations all have λ they are all equal. So, let's start off by setting equations (10) and (11) equal.

$$yz = xz \qquad \Rightarrow \qquad z(y - x) = 0 \qquad \Rightarrow \qquad z = 0 \text{ or } y = x$$

So, we've got two possibilities here. Let's start off with by assuming that $z = 0$. In this case we can see from either equation (10) or (11) that we must then have $\lambda = 0$. From equation (12) we see that this means that $xy = 0$. This in turn means that either $x = 0$ or $y = 0$.

So, we've got two possible cases to deal with there. In each case two of the variables must be zero. Once we know this we can plug into the constraint, equation (13), to find the remaining value.

$$\begin{aligned} z = 0,\ x = 0: &\qquad \Rightarrow \qquad y = 1 \\ z = 0,\ y = 0: &\qquad \Rightarrow \qquad x = 1 \end{aligned}$$

So, we've got two possible solutions $(0,1,0)$ and $(1,0,0)$.

Now let's go back and take a look at the other possibility, $y = x$. We also have two possible cases to look at here as well.

This first case is $x = y = 0$. In this case we can see from the constraint that we must have $z = 1$ and so we now have a third solution $(0,0,1)$.

The second case is $x = y \neq 0$. Let's set equations (11) and (12) equal.

$$xz = xy \qquad \Rightarrow \qquad x(z - y) = 0 \qquad \Rightarrow \qquad x = 0 \text{ or } z = y$$

Now, we've already assumed that $x \neq 0$ and so the only possibility is that $z = y$. However, this also means that,

$$x = y = z$$

Using this in the constraint gives,

$$3x = 1 \qquad \Rightarrow \qquad x = \frac{1}{3}$$

So, the next solution is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

We got four solutions by setting the first two equations equal.

To completely finish this problem out we should probably set equations (10) and (12) equal as well as setting equations (11) and (12) equal to see what we get. Doing this gives, For $(2,1,-2)$ we get

$$H = \begin{pmatrix} 8 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \implies \Delta_1 = 8,\ \Delta_2 = \begin{vmatrix} 8 & -2 \\ -2 & 2 \end{vmatrix} = 12,\ \Delta_3 = |H| = 28.$$

Since always $\Delta_i > 0$, we conclude that $f(2,1,-2) = -7$ is a local minimum.
Recall that for a local maximum we need $\Delta_1 < 0$, $\Delta_2 > 0$, and $\Delta_3 < 0$.

3. Since expressing y from the constraint would be messy, this calls for Lagrange multipliers with $g(x,y) = x^2 - 2x + 2y^2 + 4y$. Equations to solve are $\nabla f = \lambda \nabla g$ and $g = 0$, that is,

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$$\left.\begin{aligned} \tfrac{\partial f}{\partial x} &= \lambda \tfrac{\partial g}{\partial x} \\ \tfrac{\partial f}{\partial y} &= \lambda \tfrac{\partial g}{\partial y} \\ g &= 0 \end{aligned}\right\} \implies \left.\begin{aligned} 2x &= \lambda(2x-2) \\ 4y &= \lambda(4y+4) \\ x^2 - 2x + 2y^2 + 4y &= 0 \end{aligned}\right\} \implies \begin{aligned} x &= \lambda(x-1) \\ y &= \lambda(y+1) \\ x^2 - 2x + 2y^2 + 4y &= 0 \end{aligned}$$

A typical strategy is to eliminate λ from the first two equations in order to obtain some relationship between the variables x, y, this is then used with condition $g = 0$ to find the desired points.
We would like to isolate λ from the first equation. Can we have $x = 1$? The first equation then reads $1 = 0$, which is not true. Thus for sure $x \neq 1$ and we can write $\lambda = \frac{x}{x-1}$. Putting it into the second equation and multiplying out we get $y = -x$. Now this can be put into the constraint, we obtain $3x^2 - 6x = 0$ and two solutions, $x = 0$ and $x = 2$. Thus there are two suspicious points: $(0,0)$ and $(2,-2)$. We substitute them into f: $f(0,0) = 0$, $f(2,-2) = 12$. Comparing values we guess that the former is a local minimum and the latter is a local maximum.
Determining global extrema usually involves some analysis of the situation. We have two local extrema, but we do not know whether they give global extrema. In general, we find global extrema by comparing values at local extrema and also values at "borders" of the set. Thus we need to know more about M, the set determined by the given condition where we look at f.
A frequent trouble arises when the given set is not bounded, since then we have to ask what happens to f when points of M run away to some infinity. Could it happen that x tends to infinity within this set? Since points from M satisfy $2y^2 + 4y = 2x - x^2$, this would force the expression $2y^2 + 4y$ to tend to minus infinity, but that is not possible. Similarly we argue that also y cannot go to infinity and we thus have a bounded set M.
Another source of trouble is if the set M is a curve that has some endpoints, then we would have to check on those. How does M actually look like? In fact, rewriting the condition as

$$(x-1)^2 + 2(y+1)^2 = 3$$

we see that M is an ellipse. This is a close curve without any end, so whatever important happens to values of f on it, it must happen at one of the points we found earlier. Thus we can conclude that $f(0,0) = 0$ is a minimum and $f(2,-2) = 12$ is a maximum of f on the given set.

4. The unknown point $Q = (x,y,z)$ satisfies $x + y - z = 1$, that would be the constraint with $g(x,y,z) = x + y - z$. The function to minimize should be the distance between P and Q, but that would mean a square root. It will be easier to minimize the distance squared, which is equivalent (think about it). Thus we have $f(x,y,z) = \mathrm{dist}(P,Q)^2 = x^2 + (y+3)^2 + (z-2)^2$.
We use Lagrange multipliers, the equations $\nabla f = \lambda \nabla g$ and $g = 1$ now give

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$$\left.\begin{aligned} \tfrac{\partial f}{\partial x} &= \lambda \tfrac{\partial g}{\partial x} \\ \tfrac{\partial f}{\partial y} &= \lambda \tfrac{\partial g}{\partial y} \\ \tfrac{\partial f}{\partial z} &= \lambda \tfrac{\partial g}{\partial z} \\ g &= 1 \end{aligned}\right\} \implies \left.\begin{aligned} 2x &= \lambda \cdot 1 \\ 2(y+3) &= \lambda \cdot 1 \\ 2(z-2) &= \lambda \cdot (-1) \\ x + y - z &= 1 \end{aligned}\right\} \implies \begin{aligned} x &= \tfrac{1}{2}\lambda \\ y + 3 &= \tfrac{1}{2}\lambda \\ z - 2 &= -\tfrac{1}{2}\lambda \\ x + y - z &= 1 \end{aligned}$$

Again, we start by eliminating λ from the first three equations, for instance by substituting for $\frac{1}{2}\lambda$ from the first equation into the next two. Then

$$\left.\begin{aligned} y+3&=x\\ 2-z&=x\\ x+y-z&=1 \end{aligned}\right\} \implies \left.\begin{aligned} y&=x-3\\ z&=2-x\\ x+y-z&=1 \end{aligned}\right\} \implies x+(x-3)+(2-x)=1 \implies x=2.$$

We easily find the other unknowns and obtain a suspicious point $Q=(2,-1,0)$. Is the function f and hence the distance really minimal, not for instance maximal at Q? We try another point from the plane. Say, the point $R=(0,1,0)$ has $\text{dist}(R,P)=\sqrt{4^2+2^2}=\sqrt{20}$. On the other hand, the distance from Q to P is $\text{dist}(Q,P)=\sqrt{2^2+2^2+2^2}=\sqrt{12}$, so it looks like the desired minimum.

We can also argue that it is possible to go to infinity within the given plane, we can easily let $x\to\infty$ and the other coordinates adjust, then also the distance goes to infinity (this is obvious when we imagine the situation) and thus the value we found cannot be the maximum.

Alternative: The first three equations offer the possibility of easily expressing all variables using λ (say, $z=2-\frac{1}{2}\lambda$). When we do it and substitute into the constrain, we get an equation with one unknown λ, namely $\frac{3}{2}\lambda=6$. For this we have $\lambda=4$ and we now have exactly the same $x=2$ etc. as before.

Note: Instead of Lagrange multipliers one could use the constraint to get $x=1-y+z$ and substitute into f, obtaining $F(y,z)=(1-y+z)^2+(y+3)^2+(z-2)^2$. We find its local extrema:

$$\left.\begin{aligned} \tfrac{\partial F}{\partial y}&=0\\ \tfrac{\partial F}{\partial z}&=0 \end{aligned}\right\} \implies \left.\begin{aligned} -2(1-y+z)+2(y+3)&=0\\ 2(1-y+z)+2(z-2)&=0 \end{aligned}\right\} \implies y=-1,\ z=0.$$

5. The distance between a point and a line is given as the distance between the given point and the closest point of the line, so we have to find that point.

We have two constraints, one given by $g(x,y,z)=x+y+z=1$, the other by $h(x,y,z)=2x-y+z=3$. We want a point $Q=(x,y,z)$ satisfying these constraints such that its distance from P is minimal, we will minimize the distance squared $f(x,y,z)=(x-1)^2+(y-2)^2+(z+1)^2$. Now there will be two Lagrange multipliers, we call them λ and μ (it is easier to write g,h and λ,μ rather then g_1,g_2 and λ_1,λ_2 as in the theorem). The equations are $\nabla f=\lambda\nabla g+\mu\nabla h$, $g=1$ and $h=3$, that is,

$$\left.\begin{aligned} \tfrac{\partial f}{\partial x}&=\lambda\tfrac{\partial g}{\partial x}+\mu\tfrac{\partial h}{\partial x}\\ \tfrac{\partial f}{\partial y}&=\lambda\tfrac{\partial g}{\partial y}+\mu\tfrac{\partial h}{\partial y}\\ \tfrac{\partial f}{\partial z}&=\lambda\tfrac{\partial g}{\partial z}+\mu\tfrac{\partial h}{\partial z}\\ g&=1\\ h&=3 \end{aligned}\right\} \implies \left.\begin{aligned} 2(x-1)&=\lambda\cdot1+\mu\cdot2\\ 2(y-2)&=\lambda\cdot1+\mu\cdot(-1)\\ 2(z+1)&=\lambda\cdot1+\mu\cdot1\\ x+y+z&=1\\ 2x-y+z&=3 \end{aligned}\right\} \implies \begin{aligned} 2(x-1)&=\lambda+2\mu\\ 2(y-2)&=\lambda-\mu\\ 2(z+1)&=\lambda+\mu\\ x+y+z&=1\\ 2x-y+z&=3 \end{aligned}$$

We will again try to eliminate the multipliers from the first three equations. For instance, we add the second and the third equation, get $\lambda=y+z-1$, putting it back into the third equation we get $\mu=z-y+3$. Substituting λ,μ into the first equation we get $2x+y-3z=6$. This is typical, we had 3 equations with 5 unknowns, so after using up two equations we end up with one and only three unknowns.

Now we also take into account the two constraints, so we get

$$\left.\begin{aligned} 2x+y-3z&=6\\ x+y+z&=1\\ 2x-y+z&=3 \end{aligned}\right\} \implies x=2,\ y=0,\ z=-1.$$

We then calculate the distance from $Q=(2,0,-1)$ to P: $\text{dist}(P,Q)=\sqrt{5}$. Just to make sure

(2a) $|x-1| + |x-3| + |x-5| = 4$

(a) $x \in (-\infty, -1)$

$-x+1 \quad -x+3 \quad -x+5 = 4$

$-3x = -5$

$x = \frac{5}{3}$

$5/3 \notin (-\infty, -1)$

(b) $x \in [1, 3)$

$x-1 \quad -x+3 \quad -x+5 = 4$

$-x = -3$

$x = 3$

$3 \notin [1, 3)$

(c) $x \in [3, 5)$

$x-1 + x-3 - x+5 = 4$

$\underline{\underline{x = 3}}$

(d) $x \in [5, \infty)$

$x-1 + x-3 + x-5 = 4$

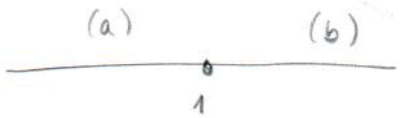
$3x = 13$

$x = 13/3$

$13/3 \notin [5, \infty)$

Conclusion: $x = 3$

(2b) $||x-1| - 2| < 1$

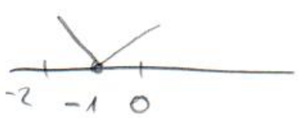


(a) $x \in (-\infty, 1)$

$|-x+1-2| < 1$

$|\underbrace{-x-1}_{(-1)(x+1)}| < 1$

$|x+1| < 1$

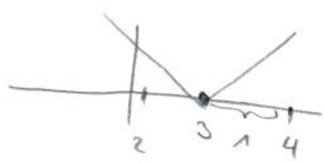


$x \in \underline{\underline{(-2, 0)}} \quad (\cap (-\infty, 1))$

(b) $x \in [1, \infty)$

$|x-1-2| < 1$

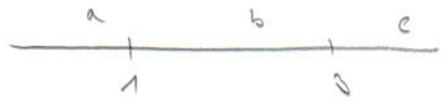
$|x-3| < 1$



$x \in \underline{\underline{(2, 4)}} \quad (\cap [1, \infty))$

Conclusion $x \in (-2, 0) \cup (2, 4)$

(2c) $|x-1| - |x-3| > x$



(a) $x \in (-\infty, 1)$

$$-x-1-(-x+3) > x$$
$$-x-1+x-3 > x$$
$$-4 > x$$

$x \in (-\infty, -4)$

(b) $x \in [1, 3)$

$$x-1-(-x+3) > x$$
$$x-1+x-3 > x$$
$$x > 4$$

impossible

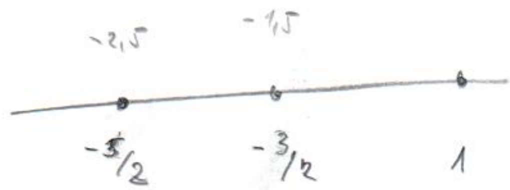
(c) $x \in [3, \infty)$

$$x-1-(x-3) > x$$
$$2 > x$$

Impossible

conclusion $x \in (-\infty, -4)$

(2d) $|2x+3| + |2x+5| > |x-1|$



(a) $x \in (-\infty, -5/2)$

$$-2x-3 \quad -2x-5 > -x+1$$
$$-3x > 9$$
$$-9 > 3x$$
$$-3 > x$$

$x \in (-\infty, -3)$

(b) $x \in [-5/2, -3/2)$

$$-2x-3 \quad +2x+5 > -x+1$$
$$x > -1$$

impossible

(c) $x \in [-3/2, 1)$

$$2x+3+2x+5 > -x+1$$
$$5x > -7$$
$$x > -\frac{7}{5}$$

$x \in (-7/5, 1)$

(d) $x \in [1, \infty)$

$$2x+3+2x+5 > x-1$$
$$3x > -9$$
$$x > -3$$

$x \in [1, \infty)$

conclusion: $x \in (-7/5, \infty) \cup (-\infty, -3)$

(2e)

(1) $|x+2| > |x| - x$

		-2		0	
$x+2$	$-$	$\times$	$+$		$+$
x	$-$		$-$	$\times$	$+$

(a) $x \in (-\infty, -2)$

$-x-2 > -x-x$

$x > 2$

impossible

(b) $x \in (-2, 0)$

$x+2 > -x-x$

$3x > -2$

$x > -\frac{2}{3}$

$\underline{x \in (-\frac{2}{3}, 0)}$

(c) $x \in (0, \infty)$

$x+2 > x - x$

$x > -2$

$\underline{x \in (0, \infty)}$

(d) borders:

$x = -2$

$0 > 2-(-2)$

$0 > 4$ no

$x = 0$

$2 > 0$ ✓

conclusion $\underline{x \in (-\frac{2}{3}, \infty)}$

2f

(1) $|x+2| > |x+1| + x$

		-2		-1	
$x+2$	$-$	$\times$	$+$		$+$
$x+1$	$-$		$-$	$\times$	$+$

(a) $x \in (-\infty, -2)$

$-x-2 > -x-1+x$

$\boxed{-1 > x}$

$x \in (-\infty, -2)$

(b) $x \in (-2, -1)$

$x+2 > -x-1+x$

$\boxed{x > -3}$

$x \in (-2, -1)$

(c) $x \in (-1, \infty)$

$x+2 > x+1+x$

$\boxed{1 > x}$

$x \in (-1, 1)$

(d) borders

$x = -2$

$0 > |-2+1| - 2$

$0 > -1$ ✓

$x = -1$

$|-1+2| > -1$

$1 > -1$ ✓

Conclusion $\boxed{x \in (-\infty, 1)}$

2g

(1) $|x - |x+2|| < x$

(a) $x+2 \geq 0$

$\boxed{x \geq -2}$

$|x - x - 2| < x$

$\boxed{2 < x}$

(b) $x+2 < 0$

$\boxed{x < -2}$

$|x + x + 2| < x$

$|2(x+1)| < x$

(b.1) $x+1 \geq 0$

$x \geq -1$

impossible

(b.2) $x+1 < 0$

$\boxed{x < -1}$

$-2x - 2 < x$

$-2 < 3x$

$\boxed{-\frac{2}{3} < x}$

impossible

Conclusion $x \in (2, \infty)$

2a

(1) $|x + |x+2|| < 4x$

(a) $x+2 \geq 0$

$\boxed{x \geq -2}$

$|x + x+2| < 4x$

$|2(x+1)| < 4x$

(a1) $x+1 \geq 0$

$\boxed{x \geq -1}$

$2x+2 < 4x$

$2 < 2x$

$\boxed{1 < x}$

(a2) $x+1 < 0$

$\boxed{x < -1}$

$-2x-2 < 4x$

$-2 < 6x$

$-\frac{1}{3} < x$

impossible

(b) $x+2 < 0$

$\boxed{x < -2}$

$|x - x-2| < 4x$

$|-2| < 4x$

$2 < 4x$

$\boxed{\frac{1}{2} < x}$

impossible

conclusion $\boxed{x > 1}$