12th lesson - Composition of functions

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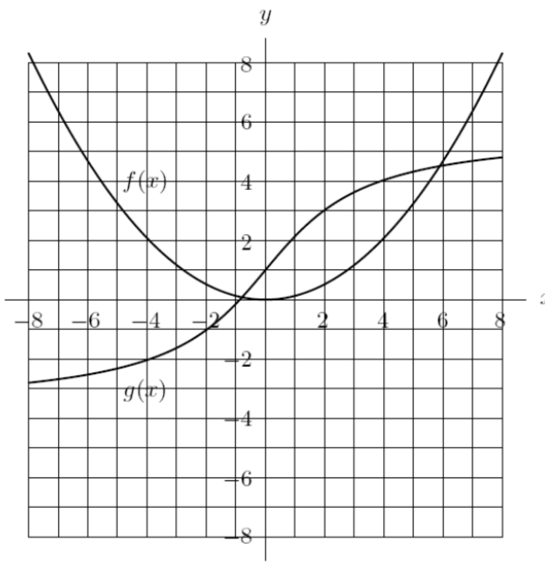
Exercises

Compositions

1. Let $f(x) = x^2$ and $g(x) = x - 2$. Find
   (a) $f(g(3)) = 1$
   (b) $g(f(3)) = 7$
   (c) $f(g(x))(x-2)^2$
   (d) $g(f(x)) = x^2 - 2$
2. Let $f(x) = 4 - x^2$ and $g(x) = \sqrt{x}$. Find
   (a) $f(g(x)) = 4 - (\sqrt{x})^2 = 4 - |x|$
   (b) $g(f(x)) = \sqrt{4 - x^2}$
3. Let $f(x) = 3x - 8$ and $g(x) = \frac{x+8}{3}$. Find
   (a) $f(g(x)) = 3\left(\frac{x+8}{3}\right) - 8 = x$
   (b) $g(f(x)) = \frac{(3x-8)+8}{3} = x$
4. Express the following functions as composition:
   (a) $(1 + x^3)^{27}$ **Solution:** $f(g(x))$ where $f(y) = y^{27}$, $g(x) = 1 + x^3$
   (b) e^{-x^2} **Solution:** $f(g(x))$ where $f(y) = e^y$, $g(x) = -x^2$
   (c) $-(e^x)^2$ **Solution:** $f(g(x))$ where $f(y) = -y^2$, $g(x) = e^x$

5. Find $g(f(3))$, if the f and g are at the picture:



1: http://nebula2.deanza.edu/~karl/Classes/Files/Math12/ch01.pdf

Solution: Since $f(3) = 1$ and $g(1) = 2$, we obtain $g(f(3)) = 2$.

6. The values of functions f and g can be found in the table. Find $f(g(0))$.

x	-2	-1	0	1	2
$f(x)$	1	0	-2	2	-1
$g(x)$	-1	1	2	0	-2

Solution: Since $g(0) = 2$ and $f(2) = -1$, we obtain $f(g(0)) = -1$.

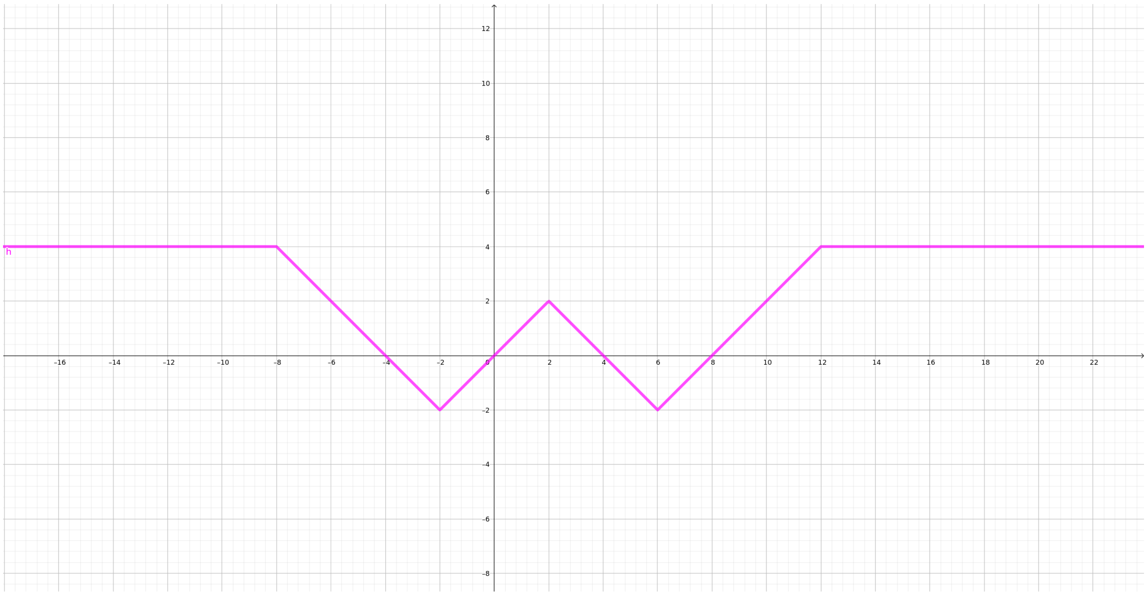
7. The values of functions f and g can be found in the table. Find x, if $f(g(x)) = 1$.

x	-2	-1	0	1	2
$f(x)$	1	0	-2	2	-1
$g(x)$	-1	1	2	0	-2

8. Look at function $h(x)$ (the pink one)
https://www.geogebra.org/calculator/zu6td6rv

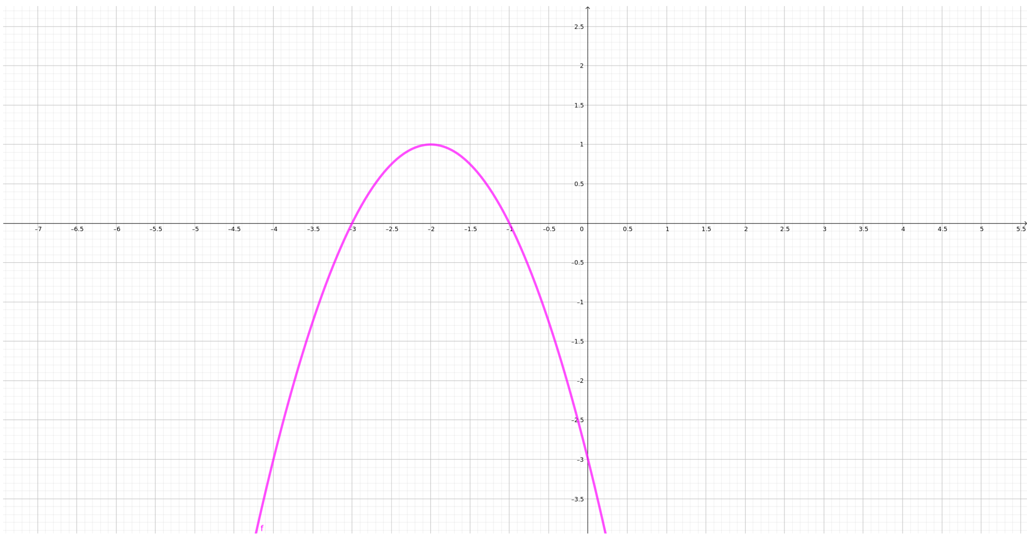
Sketch

(a) $\frac{h(x)}{2}$ (b) $3h(x)$ (c) $-2h(x)$ (d) $h(-x)$ (e) $h(3x)$ (f) $h(x/2)$



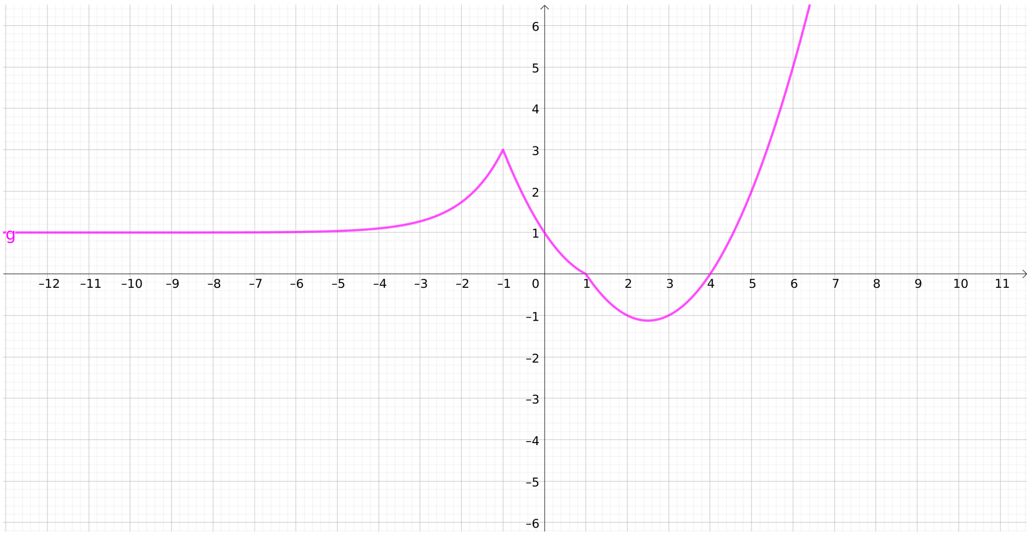
9. Look at function $f(x)$ (the pink one)
   https://www.geogebra.org/calculator/rcd6wsup
   Sketch

   (a) $f(x+1)$ (b) $f(x-1)$ (c) $f(x)+1$ (d) $f(x)-1$



10. Look at function $g(x)$ (the pink one)
    https://www.geogebra.org/calculator/sa3h5jad
    Sketch

    (a) $|g(x)|$ (b) $g(|x|)$ (c) $-g(|x|)$ (d) $g(-|x|)$



11. Look at function $f(x)$ (the pink one)
    https://www.geogebra.org/calculator/ksyvvk9z
    Sketch $\frac{1}{f(x)}$

Properties

12. Let $f(x)$ and $g(x)$ be functions (with suitable domains and images).
    TRUE or FALSE?
    (a) Let f and g be odd. Then
        i. $f+g$ is odd.
           Solution: True. $(f+g)(x) = f(-x)+g(-x) = -f(x)+-g(x) = -(f+g)(x)$
        ii. fg is odd.
           Solution: False, fg is even. $(fg)(-x) = -f(x)\cdot(-(g(x))) = f(x)g(x)$

iii. $f(g)$ is odd.
Solution: True. $f(g(-x)) = f(-g(x)) = -f(g(x))$

(b) Let f be even, g be odd. Then

i. fg is even.
Solution: False, is odd. $(fg)(-x) = f(x) \cdot (-g(x)) = -(fg)(x)$

ii. $f(g)$ is even.
Solution: True. $f(g(-x)) = f(-g(x)) = f(g(x))$

iii. $g(f)$ is even.
Solution: True.
$g(f(-x)) = g(f(x))$

iv. $g + f$ is even.
Solution: False. For example x and x^2.

(c) Let f and g be increasing. Then

i. $f + g$ is increasing.
Solution: True. Let $s < t$. Then $(f + g)(s) = f(s) + g(s) \leq f(t) + g(t)$.

ii. fg is increasing.
Solution: False. For example x and x.

iii. $f(g)$ is increasing.
Solution: True. Let $s < t$. Then $g(s) \leq g(t)$. Hence $f(g(s)) \leq f(g(t))$.

(d) Let f be even.

i. If f is increasing on $(0, \infty)$, then f is increasing also on $(-\infty, 0)$.
Solution: False. For example x^2.

ii. If f is convex on $(0, \infty)$, then f is convex also on $(-\infty, 0)$.
Solution: True.
Let us use the Lemma. Since f is convex on $(0, \infty)$, then for $0 < s < t < u$ we have

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(t)}{u - t}$$

Because f is even, for $-u < -t < -s < 0$ we then have

$$\frac{f(-u) - f(-t)}{-u + t} = \frac{f(u) - f(t)}{-(u - t)} \leq -\frac{f(t) - f(s)}{t - s} = \frac{f(-t) - f(-s)}{-t + s}$$