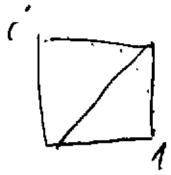
Mocniny a odmocniny

(a) $\underline{(1+i)^6}$



$$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= e^{\ln\sqrt{2} + \frac{\pi}{4}i}$$

$$(\quad)^6 = e^{\ln\sqrt{2}^6 + \frac{3}{2}\pi i} = 8 \cdot e^{\frac{3}{2}\pi i} = 8\cdot(0 - i) = \underline{\underline{-8i}}$$

(b) $\underline{(5\sqrt{3} - 5i)^7}$

$$|z| = \sqrt{25\cdot 3 + 25} = 10$$

$$z = 10\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 10\left(e^{\frac{\pi}{6}i}\right)$$

$$z^7 = 10^7 e^{\frac{7}{6}\pi} = 10^7\left(\cos\frac{7}{6}\pi + i\sin\frac{7}{6}\pi\right) =$$

$$= \underline{\underline{10^7\left(-\frac{\sqrt{3}}{2}\right) - 10^7\left(\frac{1}{2}\right)i}}$$

odmocniny

(a) $\underline{\sqrt[2]{4}} = \left(4 \cdot e^{2k\pi i}\right)^{1/2} = 2 \cdot e^{k\pi i} = \underline{2\left(\cos 0 + i \sin 0\right) = 2}$

$= \underline{2\left(\cos \pi + i \sin \pi\right) = -2}$

(b) $\underline{\sqrt[5]{32}} = \left(32\, e^{2k\pi i}\right)^{1/5} = \underline{2 \cdot e^{\frac{2}{5}k\pi i}}$

$$\underline{e^{1+\pi i}} = e^1 e^{\pi i} = e(\cos\pi + i\sin\pi) = \underline{-e}$$

$$\underline{e^{2+\frac{\pi}{2}i}} = e^2 e^{\frac{\pi}{2}i} = \underline{i e^2}$$

$$\underline{e^{\frac{1}{2}\ln 2 + \frac{\pi}{4}i}} = e^{\ln\sqrt{2}}\, e^{\frac{\pi}{4}i} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) =$$

$$= \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \underline{1+i}$$

3! e nei 1 [illegible]

$f(z) = z + 1 - i$

$$\begin{cases} f_1 = x + 1 \\ f_2 = y - 1 \end{cases}$$ posunutí $\qquad D_f = H_f = \mathbb{C}$

$f(z) = \dfrac{i-1}{\sqrt{2}} \cdot z$

$$f(x+iy) = \frac{1}{\sqrt{2}}(i-1)(x+iy) = \frac{1}{\sqrt{2}}(-x-y+i(x-y))$$

$$\left| \begin{array}{l} f_1 = -\frac{1}{\sqrt{2}}(x+y) \\ f_2 = \frac{1}{\sqrt{2}}(x-y) \end{array} \right.$$

Násobení číslem $\dfrac{i-1}{\sqrt{2}}$ je otočení o $\frac{3}{4}\pi$,

neb $\left|\dfrac{i-1}{\sqrt{2}}\right| = 1$, tak nezvětšujeme

$H_f = \mathbb{C}$

$f(z) = e^{i\alpha} \cdot z \qquad \alpha \in [0, 2\pi]$

$$f(z) = (\cos\alpha + i\sin\alpha)(x+iy) = \underbrace{x\cos\alpha - y\sin\alpha}_{f_1} + i\underbrace{(x\sin\alpha + y\cos\alpha)}_{f_2}$$

$$f(z) = e^{i\alpha} \cdot z = re^{i\varphi} e^{i\alpha} = re^{i(\varphi+\alpha)}$$

$\rightarrow$ otočení o α

$f(z) = \bar{z}$

$f_1 = x \quad f_2 = -y$

souměrnost kol osy $\mathbb{R}$ čísel

$f(z) = \dfrac{z}{|z|}$

znormování z

(1) Urči reálnou a imaginární část fce $f(z)$

(a) $\underline{f(z) = (z+i)^2}$

$f(a+bi) = (a+bi+i)^2 = (a+i(b+1))^2 =$

$= a^2 - 1(b+1)^2 + 2ai(b+1)$

$\underline{f_1 = a^2 - b^2 - 1 - 2b}$

$\underline{f_2 = 2ab + 2a}$

(b) $\underline{f(z) = e^{-iz}}$

$f(a+bi) = e^{-i(a+bi)} = e^{-ia+b} = e^{b}(\cos(-a) + i\sin(-a))$

$= e^{b}(\cos(a) + i\sin(a))$

$\underline{f_1 = e^{b}\cos a}$

$\underline{f_2 = e^{b}\sin a}$

$(z+i)^2 =$ posunutí

$w^2 = \rho(\cos\varphi + i\sin\varphi) =$

$w^2 = (e^{r+i\varphi})^2 = e^{2r}e^{2i\varphi} = \rho^2(\cos 2\varphi + i\sin 2\varphi)$

otočení o úhel

(1) Spočítte derivaci fcí

(a) $\underline{\operatorname{Re} z}$

$$f(x+yi) = \underbrace{x}_{f_1} + \underbrace{0i}_{f_2}$$

$$\frac{\partial f_1}{\partial x} \stackrel{?}{=} \frac{\partial f_2}{\partial y} \qquad \frac{\partial f_2}{\partial x} \stackrel{?}{=} -\frac{\partial f_1}{\partial y}$$

$$\frac{x}{\partial x} \stackrel{?}{=} \frac{0}{\partial y} \qquad 0 = -0$$

$$1 \neq 0$$

nenastane v žádném bodě $\rightarrow$ fce <u>není holomorfní</u>

(b) $\underline{|z|}$

$$f(x+iy) = \underbrace{\sqrt{x^2+y^2}}_{f_1} + \underbrace{0i}_{f_2}$$

$$\frac{\partial f_1}{\partial x} = \frac{1}{\cancel{2}}\frac{\cancel{2}x}{\sqrt{x^2+y^2}} \stackrel{?}{=} \frac{\partial f_2}{\partial y} = 0$$

pro $x = 0$

$$\frac{\partial f_2}{\partial x} = 0 \stackrel{?}{=} -\frac{\partial f_1}{\partial y} = \frac{-y}{\sqrt{x^2+y^2}}$$

jen pro $y = 0$

$\Rightarrow$ $(x,y) = (0,0)$ <u>ale tam fce není definovaná</u>

$\rightarrow$ neplatí v žádném bodě

(c) $f(z) = \frac{1}{z}$

$$f(x+iy) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$$f_1 = \frac{x}{x^2+y^2} \qquad f_2 = \frac{-y}{x^2+y^2}$$

$$\frac{\partial f_1}{\partial x} = \frac{x^2+y^2 - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(y^2+x^2)^2}$$

$$\frac{\partial f_2}{\partial y} = -\frac{x^2+y^2 - y\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(y^2+x^2)^2}$$

$$\frac{\partial f_1}{\partial y} = \frac{-x\cdot 2y}{(x^2+y^2)^2}$$

$$-\frac{\partial f_2}{\partial x} = -\frac{y\cdot 2x}{(x^2+y^2)^2}$$

(CR) platí pro $\forall z \neq (0,0)$

a $f'(z) = \frac{\partial f}{\partial x} =$

$$= \frac{\partial f_1}{\partial x} + i\frac{\partial f_2}{\partial x} = \frac{y^2-x^2+i\,2xy}{(x^2+y^2)^2} = \frac{1}{z^2}$$

ověřte

fce je holomorfní pro $z \neq (0,0)$

(d) $f(z) = z \cdot \operatorname{Re}(z)$

$$f(x+iy) = (x+iy)\cdot x = \underbrace{x^2}_{f_1} + i\,\underbrace{xy}_{f_2}$$

$$\frac{\partial f_1}{\partial x} = 2x \neq \frac{\partial f_2}{\partial y} = x$$

$$-\frac{\partial f_1}{\partial y} = -0 \neq \frac{\partial f_2}{\partial x} = y$$

není holomorfní

(e) $f(z) = \cos z = \cos(x+iy)$

$$f(z) = \cos z = \frac{1}{2}e^{iz} + \frac{1}{2}e^{-iz} = \frac{1}{2}e^{ix-y} + \frac{1}{2}e^{-ix+y}$$

$$= \frac{1}{2}e^{-y}(\cos x + i\sin x) + \frac{1}{2}e^{y}(\cos(-x) + i\sin(-x))$$

$$= \frac{1}{2}e^{-y}\cos x + \frac{1}{2}e^{y}\cos x + i(\sin x)\left(+\frac{1}{2}e^{-y} - \frac{1}{2}e^{y}\right)$$

$$= \underbrace{\cos x\left(\frac{1}{2}e^{-y} + \frac{1}{2}e^{y}\right)}_{f_1} + i\,\underbrace{\sin x\left(+\frac{1}{2}e^{-y} - \frac{1}{2}e^{y}\right)}_{f_2}$$

$$\frac{\partial f_1}{\partial x} = -\sin x\left(\frac{1}{2}e^{-y} + \frac{1}{2}e^{y}\right)$$

$$\frac{\partial f_2}{\partial y} = \sin x\left(-\frac{1}{2}e^{-y} - \frac{1}{2}e^{y}\right)$$

$\Rightarrow$ fce je holomorfní

$$\frac{\partial f_1}{\partial y} = \frac{1}{2}\cos x\left(-e^{-y} + e^{y}\right)$$

$$-\frac{\partial f_2}{\partial x} = -\frac{1}{2}\cos x\left(e^{-y} - e^{y}\right)$$

$$\frac{\partial f}{\partial x} = -\sin x\left(\frac{1}{2}e^{-y} + \frac{1}{2}e^{y}\right) + i\cos x\left(\frac{1}{2}e^{-y} - \frac{1}{2}e^{y}\right)$$

Kontrola

$$-\sin x = -\frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = -\frac{1}{2i}e^{-y}(\cos x + i\sin x) + \frac{1}{2i}e^{y}(\cos x - i\sin x)$$

$$= \frac{1}{2}\sin x\left(e^{-y} + e^{y}\right) + i\,\frac{1}{2}\cos x\left(e^{-y} - e^{y}\right)$$