(1) Zjistěte obor konvergence

$$\boxed{\sum_{n=-\infty}^{\infty} 2^{-|n|} (z-1)^n}$$

(a) $\displaystyle\sum_{n=0}^{\infty} 2^{-n} (z-1)^n$

$$k = \limsup_{n\to\infty} \frac{1}{\sqrt[n]{2^{-n}}} = 2$$

na hranici $\partial U(1,2)$

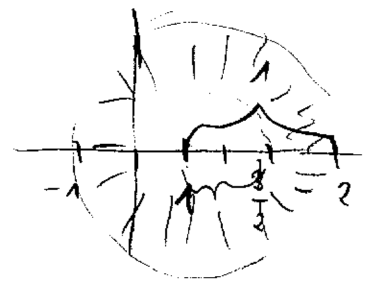
nekonvergují v žádném bodě, neb

$$2^{-n} |z-1|^n = 1 \quad \text{a tedy} \not\to 0$$

$$\sum_{n=-\infty}^{-1} 2^{-|n|} (z-1)^n = \sum_{m=1}^{\infty} 2^{-m} \frac{1}{(z-1)^m}$$

substituce $u = \frac{1}{1-z}$

$$\sum_{m=1}^{\infty} 2^{-m} u^m$$



poloměr konvergence = 2

hl. část tedy konvergují pro $\left|\frac{1}{z-1}\right| < 2$ tj. $|z-1| > \frac{1}{2}$

na hranici opět nekonvergují neb

$$2^{-m} \frac{1}{\frac{1}{2^m}} = 1 \quad \text{nejde k } 0$$

Obor konvergence

$$\boxed{\sum_{n=-\infty}^{-1} 2^n z^n + \sum_{n=0}^{\infty} 3^n z^n} = \sum_{n=1}^{\infty} \frac{1}{(2z)^n} + \sum_{n=0}^{\infty} (3z)^n$$

konverguje když z.č. menší než 1, tedy

$$\left|\frac{1}{2z}\right| < 1 \quad \Rightarrow \quad |z| > \frac{1}{2}$$

$$|3z| < 1 \quad \Rightarrow \quad |z| < \frac{1}{3}$$

obor konvergence je $\underline{\underline{\emptyset}}$

$$\boxed{\sum_{n=-\infty}^{\infty} \frac{1}{2^{n^2}} (z-1)^n}$$

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n^2}}$

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{\frac{1}{2^{n^2}}} = \limsup \frac{1}{\sqrt[n]{2^{n\cdot n}}} =$$

$$= \limsup \frac{1}{2^n} = 0$$

$\Rightarrow R = \infty$ reg. část konverguje na $\underline{\underline{\mathbb{C}}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n^2}} \cdot \frac{1}{(z-1)^n} = \sum \frac{1}{2^{n^2}} u^n$ kde $u = \dfrac{1}{z-1}$

$$\frac{1}{R} = \limsup \sqrt[n]{\frac{1}{2^{n^2}}} = 0$$

$\Rightarrow R = \infty$ hl. část konverguje $\forall\, u \in \mathbb{C}$

$\Rightarrow \forall\ \frac{1}{z-1} \in \mathbb{C} \Rightarrow z \neq 1$

(c) Dohromady konverguje na $P(\underline{\underline{1, 0, \infty}})$

$$\boxed{\sum_{n=-\infty}^{\infty} \frac{1}{3^n+1} (z-i)^n}$$

(a)

$$\frac{1}{R} = \lim_{n\to\infty} \frac{\frac{1}{3^{n+1}+1}}{\frac{1}{3^n+1}} = \lim \frac{3^n+1}{3^{n+1}+1} = \lim \frac{1+3^{-n}}{3+3^{-n}} = \frac{1}{3}$$

tep. č. konvergují tedy pro $|z-i| < 3$

(b) $$\sum_{n=1}^{\infty} \frac{1}{3^{-n}+1} \frac{1}{(z-i)^n} = \sum \frac{1}{3^{-n}+1} u^n \qquad u = \frac{1}{z-i}$$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{\frac{1}{3^{-n-1}+1}}{\frac{1}{3^{-n}+1}} = 1$$

konv. pro $|u| < 1 \Rightarrow |z-i| > 1$

(c) celkem $\underline{\underline{P(i, 1, 3)}}$

$$\boxed{f(z) = \frac{1}{1+z}} \qquad z_0 = 0$$

(a) $|z| < 1$

$$f(z) = \frac{1}{1+z} = \frac{1}{1-(-z)} = \sum (-z)^n = \sum_{n=0}^{\infty} (-1)^n z^n$$

(b) $|z| > 1$ $\qquad \left|\frac{1}{z}\right| < 1$

$$f(z) = \frac{1}{1+z} = \frac{\frac{1}{z}}{1+\frac{1}{z}} = \frac{\frac{1}{z}}{1-\left(-\frac{1}{z}\right)} = \sum \frac{1}{z}\left(-\frac{1}{z}\right)^n = \sum_{n=0} \frac{(-1)^n}{z^{n+1}}$$

$$\boxed{f(z) = \frac{1}{1+z}} \quad \boxed{z_0 = -1}$$

$$\frac{1}{1+z} = \frac{1}{z-(-1)} \qquad f(z) = [z-(-1)]^{-1}$$

$$\boxed{f(z) = \frac{1}{1+z}} \quad \boxed{z_0 = 1}$$

$$f(z) = \frac{1}{1+z} = \frac{1}{1+(z-1)+1} = \frac{1}{2+(z-1)} = \frac{1}{2\left(1+\frac{z-1}{2}\right)}$$

pro $\left|\frac{z-1}{2}\right| < 1$

$$\frac{1}{1+z} = \frac{1}{2}\frac{1}{1-\left(-\frac{z-1}{2}\right)} = \sum \frac{1}{2}\left(-\frac{z-1}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n \frac{(z-1)^n}{2^{n+1}}$$

pro $\left|\frac{z-1}{2}\right| > 1$

$$\frac{1}{1+z} = \frac{1}{2+(z-1)} = \frac{\frac{1}{z-1}}{1+\frac{2}{z-1}} = \sum_{n=0}^{\infty} \frac{1}{z-1}\left(-\frac{2}{z-1}\right)^n = \sum_{n=0}^{\infty}(-1)^n \frac{2^n}{(z-1)^{n+1}}$$

$$\boxed{f(z) = \frac{1}{(z-2)(z-3)}} \qquad z_0 = 0$$

Rozklad na parc. zlomky

$$\frac{1}{z-3} - \frac{1}{z-2}$$

$$\frac{1}{z-3} = -\frac{1}{3}\frac{1}{1-\frac{z}{3}} = -\sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}} \qquad |z| < 3$$

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=-\infty}^{-1} 2^{-n-1} z^n \qquad |z| > 2$$

Proč takhle

$\frac{1}{z-3}$ je holomorfní na $|z|<3$

a na $|z|>3$

$\frac{1}{z-2}$ $|z|<2$

a na $|z|>2$

mezikruží: $3 > |z| > 2$

Dohromady $$-\sum_{n=-\infty}^{-1} 2^{-n-1} z^n - \sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}}$$

Rozvinte v ∞

$$\boxed{f(z) = z^2 e^{\frac{1}{z}}}$$

$$z^2 e^{\frac{1}{z}} = z^2 \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} = z^2 + z + \sum_{n=2}^{\infty} \frac{1}{n!\, z^{n-2}}$$

$$= z^2 + z + \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \frac{1}{z^n}$$

$$\boxed{f(z) = \frac{1}{(z+1)(z+2)}} \qquad z_0 = -2$$

$$f(z) = \frac{1}{z+1} - \frac{1}{z+2}$$

(a) $|z+2| < 1$

$$\frac{1}{z+1} = \frac{-1}{1-(z+2)} = -\sum_{n=0}^{\infty} (z+2)^n$$

$$\frac{1}{z+2} = \frac{1}{z+2}$$

celkem

$$\underline{\underline{-\sum_{k=-1}^{\infty} (z+2)^k}}$$

(b) $|z+2| > 1$

$$\frac{1}{z+1} = \frac{1}{z+2-1} = \frac{\frac{1}{z+2}}{1-\frac{1}{z+2}} =$$

$$= \frac{1}{z+2} \sum_{n=0}^{\infty} \frac{1}{(z+2)^n}$$

celkem $\quad \underline{\underline{\sum_{n=2}^{\infty} \frac{1}{(z+2)^n}}}$

$$\boxed{f(z) = \frac{1}{(z+1)(z+2)}} \qquad z_0 = -1$$

$$f(z) = \frac{1}{z+1} - \frac{1}{z+2}$$

(a) $0 < |z+1| < 1$

$$\frac{1}{z+2} = \frac{1}{1+\underbrace{z+1}_{z-(-1)}} = \frac{1}{1-(-(z+1))} = \sum_{n=0}^{\infty} (-1)^n (z+1)^n =$$

$$\frac{1}{z+1} = (z-(-1))^{-1}$$

dohromady $\quad f(z) = \sum_{k=-1}^{\infty} (-1)^{k+1} (z+1)^k$

(b) $|z+1| > 1$

$$\frac{1}{z+2} = \frac{1}{z+1+1} = \frac{\frac{1}{z+1}}{1+\frac{1}{z+1}} = \frac{\frac{1}{z+1}}{1-\left(\frac{-1}{z+1}\right)} =$$

$$= \frac{1}{z+1} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(z+1)^n} = \sum_{n=-1}^{-\infty} (-1)^{n+1} (z+1)^n.$$

dohromady

$$f(z) = \sum_{n=2}^{\infty} \frac{(-1)^n}{(z+1)^n}$$

$$\boxed{f(z) = \frac{1}{(z+1)(z+2)}} \qquad \underline{z_0 = 1}$$

nutno tam dostat výraz $(z-1)$

$$f(z) = \frac{1}{z+1} - \frac{1}{z+2}$$

(a) $$\frac{1}{z+1} = \frac{1}{2 + z-1} = \frac{1}{2}\,\frac{1}{1 - \left(-\frac{z-1}{2}\right)} =$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (z-1)^n \qquad \text{pro } \left|\frac{z-1}{2}\right| < 1$$

$$-\frac{1}{z+2} = -\frac{1}{3+z-1} = -\frac{1}{3}\,\frac{1}{1-\left(-\frac{z-1}{3}\right)} =$$

$$= -\frac{1}{3}\sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)^n (z-1)^n \qquad \text{pro } \left|\frac{z-1}{3}\right| < 1$$

celkem $$\underline{\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}}\right)(z-1)^n} \quad \text{na } \underline{|z-1| < 2}$$

(b) $$\frac{1}{z+1} = \frac{\frac{1}{z-1}}{1 - \frac{-2}{z-1}} = \frac{1}{z-1}\sum_{n=0}^{\infty}(-2)^n \frac{1}{(z-1)^n} =$$

$$= \sum \frac{-2^{n-1}}{(z-1)^n} \qquad u = \frac{-2}{z-1}$$

na $|u| < 1 \Rightarrow |z-1| > 2$

celkem
$$\underline{\sum_{n=-1}^{-\infty} (-2)^{-(n+1)} (z-1)^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (z-1)^n}$$

na $\underline{2 < |z-1| < 3}$

(c) $$\frac{1}{z+2} = \frac{1}{z-1+3} = \frac{\frac{1}{z-1}}{1 - \frac{-3}{z-1}} = \frac{1}{z-1}\sum_{n=0}^{\infty}(-3)^n \frac{1}{(z-1)^n} \qquad \text{na } |z-1| > 3$$

celkem $$\underline{\sum_{n=-1}^{-\infty} (-1)^{n+1}\left[2^{-(n+1)} - 3^{-(n+1)}\right](z-1)^n} \quad \text{na } \underline{|z-1| > 3}$$

$$\boxed{f(z) = \frac{e^z}{z(z^2+1)}}$$

najdi 1. 4 členy

na $0 < z < 1$

Ze zadání máme, že $z_0 = 0$

(a) víme: $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$

(b) spočteme

$$\frac{1}{z(z^2+1)} = \frac{1}{z}\,\frac{1}{1+z^2} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n z^{2n} =$$

$$= \sum_{n=0}^{\infty}(-1)^n z^{2n-1} = \frac{1}{z} - z + z^3 - z^5 + \dots$$

(c) určeme

$$f(z) = \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots\right)\left(\frac{1}{z} - z + z^3 - z^5 + \dots\right)$$

$$= \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2$$