Př.

$f(z) = \frac{1+z}{z}$ má v ∞ odst sing
lim $= 1$

$f(z) = \frac{1}{z}$ pól v 0

$f(z) = z^2$ pól v ∞

e^z podst. sing v ∞ (příklad $z_n = n$
$z_n' = -n$

$e^{1/z}$ v 0

v ∞ : obvykle zkoumáme v 0 fci $f\left(\frac{1}{z}\right)$

: rozvíjíme na fopliži kruhu

výpočet res v ∞ - je-li ∞ izol. sing. fce f, pak

$$\operatorname{res}_{\infty} f = \operatorname{res}_0 g$$

$$g(\xi) = -\frac{f(1/\xi)}{\xi^2}$$

∞ pól nás. k :

$$\operatorname{res}_{\infty} f = (-1)^k \frac{1}{(k+1)!} \lim_{z\to\infty} \left[z^{k+2} f^{(k+1)}(z) \right]$$

(1) $\boxed{I = \int_C \frac{dz}{z(z-1)^2}}$ $\qquad C: \ |z+1-i| = 2$

- 2 sing. body $0, 1$.
- bod 1 leží mimo (nezajímá nás)
  0 je pól 1. řádu uvnitř té kružnice
- $\operatorname{res}_0 f = \lim_{z \to 0} \frac{1}{(z-1)^2} = 1$

$\Rightarrow \quad I = \underline{\underline{2\pi i \cdot 1}}$

(3) $\boxed{I = \int_C \frac{dz}{z(z-1)^2}}$ $\qquad C: \ |2z+3| = 2$

ani bod 0, ani 1 neleží uvnitř dané kružnice

$\Rightarrow$ z Cauch. věty

$\underline{\underline{I = 0}}$

(2) $\boxed{I = \int_C \frac{dz}{z(z-1)^2}}$ $\qquad C: \ |2z-3| = 2$

sing. body : $0, 1$

bod 1 leží uvnitř, 0 mimo

1 je pól nás. 2 $\Rightarrow$

$\operatorname{res}_1 f = \lim_{z \to 1} \frac{1}{1!}\left[(z-1)^2 \frac{1}{(z-1)^2 z}\right]' = \lim_{z \to 1} \frac{-1}{z^2} = -1$

$\underline{\underline{I = -2\pi i}}$

$$\boxed{I = \oint_C \frac{dz}{z(z-1)^2}}$$

$C: \ x^2 + 2y^2 = 2$

elipsa, střed v $z_0 = 0$, ohnisko na $\mathbb{R}$ ose, poloosy

$a = \sqrt{2}, \quad b = 1$

Oba sing. body jsou uvnitř $\Rightarrow$

$$I = 2\pi i \left(\operatorname{res}_0 f + \operatorname{res}_1 f \right) = 2\pi i \, (1 - 1) = \underline{\underline{0}}$$

$$\boxed{I = \oint_C \frac{z}{z^2+4}\, dz}$$

$C: \ |z - i| = 4$

sing: $z_1 = 2i$

$z_2 = -2i$

$\rightarrow$ póly nás. 1, oba uvnitř

$$\operatorname{res}_{2i} f = \lim_{z \to 2i} \frac{(z-2i)z}{(z-2i)(z+2i)} = \frac{1}{2}$$

$$\operatorname{res}_{-2i} f = \lim_{z \to -2i} \frac{(z+2i)z}{(z-2i)(z+2i)} = \frac{1}{2}$$

$$I = 2\pi i \left(\frac{1}{2} + \frac{1}{2} \right) = \underline{\underline{2\pi i}}$$

$$\boxed{I = \oint_c \frac{z^2}{z^3+8}} \qquad c: |z+1+i| = 3$$

sing: $z_1 = 1+i\sqrt{3}$ → vně

$z_2 = -2$

$z_3 = 1-i\sqrt{3}$ } póly 1. ř. uvnitř

$$I = \left(\operatorname{res}_{-2} f + \operatorname{res}_{1-i\sqrt{3}} f\right) 2\pi i = 2\pi i \left(\frac{1}{3} + \frac{1}{3}\right) = \underline{\underline{\frac{4}{3}\pi i}}$$

$$\boxed{I = \oint_c \frac{dz}{(z^2-1)(z^3-1)}} \qquad c: |2z-3| = 3$$

sing: $z_1 = 1$ → uvnitř, pól 2. řádu

$z_2 = -1$

$z_3 = \frac{-1+i\sqrt{3}}{2}$

$z_4 = \frac{-1-i\sqrt{3}}{2}$ } vně

$$\operatorname{res}_1 f = \lim_{z\to 1} \left(\frac{(z-1)^2}{(z-1)^2(z+1)(z^2+z+1)}\right)' = \lim_{z\to 1} \frac{1}{z^3+2z^2+2z+1} =$$

$$= -\lim_{z\to 1} \frac{3z^2+4z+2}{(z^3+2z^2+2z+1)^2} = -\frac{9}{6^2} = -\frac{1}{4}$$

$$I = 2\pi i\left(-\frac{1}{4}\right) = \underline{\underline{-\frac{\pi i}{2}}}$$

$$\boxed{I = \int_{\mathcal{C}} \frac{dz}{z^{10}+1}}$$

$\mathcal{C}$ je kl. orientovaná kružnice se středem 0 a pol. 2.

Uvnitř kružnice (0, 2) má $\frac{1}{z^{10}+1}$ 10 pólů

(řešení rce $z^{10} = -1$) rozložených po jednotkové kružnici.

Počítání = otrava

ale $\sum_{i=1}^{10} \operatorname{res}_{a_i} f = \operatorname{res}_\infty f$

$\Rightarrow$ počítáme $\operatorname{res}_\infty f$:

$$\frac{1}{z^{10}+1} = \frac{1}{z^{10}} \cdot \frac{1}{1+z^{-10}} = \frac{1}{z^{10}} \sum_{n=0}^{\infty} (-1)^n z^{-10n} =$$

$$= \sum_{n=1}^{\infty} (-1)^n z^{-10n} \qquad a_{-1} = 0$$

$\Rightarrow \underline{I} = -2\pi i \operatorname{res}_\infty f = 0$

$$\boxed{f(z) = e^{1/z}} \qquad z \in \mathbb{C} \setminus \{0\}$$

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!} \qquad w \in \mathbb{C}$$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^n} \qquad z \in \mathbb{C} \setminus \{0\}$$

proto $\quad \operatorname{res}_0 f = -\operatorname{res}_\infty f = 1$

Razvitje v ∞

$$\boxed{f(z) = z^2 e^{\frac{1}{z}}}$$

$$z^2 e^{\frac{1}{z}} = z^2 \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} = z^2 + z + \sum_{n=2}^{\infty} \frac{1}{n!\, z^{n-2}}$$

$$= z^2 + z + \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \frac{1}{z^n}$$

$$\boxed{\frac{1}{z(z-1)}} \qquad \boxed{z=0}$$

$$\frac{1}{z(z-1)} = \frac{1}{z} + \frac{1}{z-1}$$

(a) $|z| < 1$

$$\frac{1}{z-1} = \sum_{n=0}^{\infty} z^n$$

ahem $\boxed{\sum_{n=-1}^{\infty} z^n}$ $\qquad a_{-1} = 1$

(b) $\boxed{z=1}$

$|z-1| < 1$

$$\frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{n=0}^{\infty} (1-z)^n = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

tedy pro $0 < |z-1| < 1$

$$\frac{1}{z(1-z)} = \sum_{n=-1}^{\infty} (-1)^n (z-1)^n \qquad a_{-1} = 1$$

(c) $\boxed{z=\infty}$

pro $|z| > 1$

$$\frac{1}{1-z} = -\frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = -\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = -\sum_{n=1}^{\infty} \frac{1}{z^n}$$

ahem

$$\frac{1}{z(1-z)} = -\sum_{n=2}^{\infty} \frac{1}{z^n} \qquad -a_{-1} = 0$$

$$\boxed{f(z) = \ln \frac{z+i}{z-i} \quad \text{v} \quad \infty}$$

$$f'(z) = \frac{z-i}{z+i} \cdot \frac{(z-i)-(z+i)}{(z-i)^2} = \frac{\cancel{(z-i)}(-2i)}{(z+i)(z-i)} = -2i \frac{1}{z^2 - i^2} = \frac{-2i}{z^2+1}$$

$$f'(z) = -2i \frac{1}{1-(-z^2)} = -2i \sum_{n=0}^{\infty} (-z^2)^n$$

na doplněk

$$f'(z) = -2i \frac{\frac{1}{z^2}}{1 + \frac{1}{z^2}} = -2i \frac{1}{z^2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{2n}} =$$

$$= -2i \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{2n+2}} \qquad \text{na} \quad \left|\frac{1}{z^2}\right| < 1$$

zintegrovat obě strany

$$1 < |z^2|$$

$$f(z) = -2i \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} \frac{1}{z^{(2n+1)}} + k$$

$$1 < |z||z|$$

$$1 < |z|^2$$

$$1 < |z|$$

$$\boxed{\frac{z+1}{z^2+2z+2}}$$

$z_1 = -1+i$

$z_2 = -1-i$ 1. Fádu

$$\operatorname{res}_{-1+i} f = \lim_{z \to -1+i} \frac{z+1}{z+1+i} = \frac{\cancel{-1}+i+\cancel{1}}{\cancel{-1}+i+\cancel{1}+i} = \frac{i}{2i} = \underline{\underline{\frac{1}{2}}}$$

$$\operatorname{res}_{-1-i} f = \lim_{z \to -1-i} \frac{z+1}{z+1-i} = \frac{\cancel{-1}-i+\cancel{1}}{\cancel{-1}-i+\cancel{1}-i} = \frac{-i}{-2i} = \underline{\underline{\frac{1}{2}}}$$

$$\underline{\underline{\operatorname{res}_{\infty} = -1}}$$

$$\boxed{\frac{e^z}{z^2(z^2+9)}} = \frac{e^z \rightarrow \text{holom.}}{z^2(z+3i)(z-3i)}$$

(a) res v bodě 0 - 2. nás. pól

$$\underline{\text{res}_0 f} = \lim_{z\to 0} \frac{1}{(2-1)!}\left[(z-0)^2 f(z)\right]^{(2-1)} \quad \text{dc}$$

$$= \lim_{z\to 0}\left(\frac{e^z}{(z^2+9)}\right)' = \lim_{z\to 0}\frac{e^z(z^2+9) - e^z 2z}{(z^2+9)^2}$$

$$= \frac{1\cdot 9 - 0}{9^2} = \underline{\underline{\frac{1}{9}}}$$

(b) bod $-3i$, $3i$ → 1 duchý pól

$$\underline{\text{res}_{3i} f} = \lim_{z\to 3i}\frac{1}{(1-1)!}\left[(z-3i)^1\frac{e^z}{z^2(z+3i)(z-3i)}\right]^{(1-1)} =$$

$$= \lim_{z\to 3i}\frac{e^z}{z^2(z+3i)} = \frac{1}{-1\cdot 9\cdot 6i}\cdot e^{3i} =$$

$$= \frac{1}{54}i(\cos 3 + i\sin 3) = \underline{\frac{1}{54}(-\sin 3 + i\cos 3)}$$

(c) bod $3i$ 1 duchý pól

$$\text{res}_{-3i} f = \lim_{z\to -3i}\frac{1}{(1-1)!}\left[(z+3i)^1\frac{e^z}{z^2(z+3i)(z-3i)}\right]^{(1-1)} =$$

$$= \lim_{z\to -3i}\frac{e^z}{z^2(z-3i)} = \frac{e^{-3i}}{-9(-6i)} = -\frac{1}{54}i\left(\cos -3 + i\sin(-3)\right)$$

$$= \underline{-\frac{1}{54}(\sin 3 + i\cos 3)}$$

(d) v $\underline{\infty}$

$$\sum_{a \in A \cup \{\infty\}} \operatorname{res}_a f = 0$$

tedy $\quad \operatorname{res}_0 + \operatorname{res}_{3i} + \operatorname{res}_{-3i} + \operatorname{res}_{\infty} = 0$

$$\Rightarrow \quad \operatorname{res}_{\infty} f = -\left(\frac{1}{9} + \frac{1}{54}\left(-\sin 3 + i\cos 3\right) - \frac{1}{54}\left(\sin 3 + i\cos 3\right)\right) =$$

$$= \underline{-\frac{1}{9} + \frac{1}{27}\sin 3}$$