2.1 Úlohy s pevnými koncovými body

Příklad 2.1 Nalezněte extremálu funkcionálu

$$J(x(t)) = \int_0^1 \dot{x}(t) \cdot t + [\dot{x}(t)]^2 \, dt$$

s koncovými podmínkami extremály $x(0) = 0$ a $x(1) = 2$.

Řešení: Nejprve určíme podle (2) Eulerovu - Lagrangeovu rovnici

$$\frac{\partial g}{\partial x} = 0, \qquad \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} = \frac{d}{dt}\big(t + 2\dot{x}(t)\big) = 1 + 2\ddot{x}(t) \qquad \Rightarrow \qquad 0 = -1 - 2\ddot{x}(t)\,.$$

Nyní provedeme integraci

$$\frac{d^2x(t)}{dt^2} = -\frac{1}{2} \quad \longrightarrow \quad \frac{dx(t)}{dt} = -\frac{1}{2}t + a \quad \longrightarrow \quad x(t) = -\frac{1}{4}t^2 + a\,t + b.$$

Koeficienty a a b určíme z koncových podmínek extremály.

$$\begin{array}{lclcl} x(0) = 0 & \Rightarrow & 0 = b & & \\ x(1) = 2 & \Rightarrow & 2 = -\frac{1}{4} \cdot 1 + a & \Rightarrow & a = \frac{9}{4} \end{array}$$

Extremála, pro kterou zadaný funkcionál $J\big(x(t)\big)$ nabývá extrému, tedy je

$$x(t) = -\frac{1}{4}t^2 + \frac{9}{4}t.$$

□

Příklad 2.2 Nalezněte funkci $x \in C^2(0,1)$, pro kterou nabývá funkcionál

$$J(x(t)) = \int_0^{\pi/2} [\dot{x}(t)]^2 - [x(t)]^2 \, dt$$

extrému za podmínky $x(0) = 0$ a $x(\pi/2) = 1$.

Řešení: Eulerova - Lagrangeova rovnice je

$$0 = 2x(t) + 2\ddot{x}(t).$$

Obecným řešením této rovnice je funkce

$$x(t) = c_1 \sin t + c_2 \cos t.$$

Integrační konstanty c_1, c_2 určíme z okrajových podmínek.

$$\begin{aligned} x(0) = 0 \quad &\Rightarrow \quad 0 = c_1 \sin 0 + c_2 \cos 0 = c_2 \\ x(\pi/2) = 1 \quad &\Rightarrow \quad 1 = c_1 \sin(\pi/2) = c_1 \end{aligned}$$

Extremála, pro kterou zadaný funkcionál $J(x(t))$ nabývá extrému, tedy je

$$x(t) = \sin t.$$

□

(i) $\int_0^1 1 + (1+x^2)(y')^2 \, dx \qquad y(0)=0 \qquad y(1) = \frac{\pi}{4}$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$0 - \frac{d}{dx}\left((1+x^2)\,2y'\right) = 0$$

$$-\ 2x \cdot 2y' - (1+x^2)\,2y'' = 0$$

$$\cancel{2}y''(1+x^2) + 2xy' = 0$$

$$\frac{y''}{y'} = \frac{-2x}{1+x^2}$$

$y' = z$

$$\ln z = -\ln(1+x^2) + c$$

$$z = c \cdot \frac{1}{1+x^2}$$

$$y' = c \cdot \frac{1}{1+x^2}$$

$$y = k_1 \cdot \arctan x + k_2$$

$y(0) = 0 = k_1 \cdot 0 + k_2$

$y(1) = \frac{\pi}{4} = k_1 \cdot \arctan 1 = \frac{\pi}{4} \cdot k_1$

$$\boxed{y = \arctan x}$$

(ii) $\int_0^1 (y')^2 - 6x^2 y + y^3 y' \, dx \qquad y(0) = 1 \qquad y(1) = 2$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$-6x^2 + 3y^2 y' - \frac{d}{dx}\left(2y' + y^3\right) = 0$$

$$-6x^2 + \cancel{3y^2 y'} - 2y'' - \cancel{3y^2 \cdot y'} = 0$$

$$2y'' + 6x^2 = 0$$

$$y'' = -3x^2$$

$$y' = -x^3 + k_1$$

$$y = -\frac{x^4}{4} + k_1 x + k_2$$

$y(0) = 1 = k_2 \qquad y(1) = -\frac{1}{4} + k_1 + 1 = 2$

$k_1 = \frac{5}{4}$

$$\boxed{y = -\frac{x^4}{4} + \frac{5}{4}x + 1}$$

(iii) $\int_0^1 (y')^2 - 2xy - 3y'y^2 \, dx \qquad y(0) = 0 \qquad y(1) = -1$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$-2x - 3y'2y - \frac{d}{dx}\left(2y' - 3y^2\right) = 0$$

$$+2x + \cancel{3y'2y} + 2y'' - \cancel{6yy'} = 0$$

$$y'' = -x$$

$$y' = -\frac{x^2}{2} + k_1$$

$$y = -\frac{x^3}{6} + k_1 x + k_2$$

$$0 = y(0) = k_2$$

$$-1 = y(1) = -\frac{1}{6} + k_1 \qquad k_1 = -\frac{5}{6}$$

$$\boxed{y = -\frac{x^3}{6} - \frac{5}{6}x}$$

$$(iv) \quad \int_1^2 \underbrace{x^3 (y')^2}_{f} \, dx \qquad y(1) = 5 \quad y(2) = 2$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$0 - \frac{d}{dx}\left(x^3 \, 2y'\right) = 0$$

$$3x^2 2y' + x^3 2y'' = 0$$

$$6y' + x 2y'' = 0$$

$$y'' = -3\frac{y'}{x}$$

$$\frac{y''}{y'} = -\frac{3}{x} \qquad z = y'$$

$$\frac{z'}{z} = -\frac{3}{x}$$

$$\ln z = -3 \ln x + c$$

$$z = c x^{-3}$$

$$y' = c x^{-3}$$

$$y = -\frac{c}{2} x^{-2} + k$$

$$y(1) = -\frac{c}{2} + k = 5$$

$$y(2) = -\frac{c}{2}\frac{1}{4} + k = 2$$

$$-\frac{c}{8} + 5 + \frac{c}{2} = 2$$

$$\frac{3}{8}c = -3$$

$$c = -8$$

$$k = 5 + \frac{-8}{2} = 1$$

$$\boxed{y = 4x^{-2} + 1}$$

(v) $\displaystyle\int_0^{\pi} 4(y')^2 + 2yy' - y^2 \, dx$ $\qquad y(0) = 2$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y(\pi) = 0$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$2y' - 2y - \frac{d}{dx}\left(8y' + 2y\right) = 0$$

$$2y' - 2y - 8y'' - 2y' = 0$$

$$8y'' + 2y = 0$$

$$4y'' + y = 0$$

$$4\lambda^2 + 1 = 0$$

$$\lambda^2 = -\frac{1}{4}$$