# The Non-Linear Field Theories of Mechanics.

By

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## A. Introduction[1].

**1. Purpose of the non-linear theories.** Matter is commonly found in the form of materials. Analytical mechanics turned its back upon this fact, creating the centrally useful but abstract concepts of the mass point and the rigid body, in which matter manifests itself only through its inertia, independent of its constitution; "modern" physics likewise turns its back, since it concerns solely the small particles of matter, declining to face the problem of how a specimen made up of such particles will behave in the typical circumstances in which we meet it. Materials, however, continue to furnish the masses of matter we see and use from day to day: air, water, earth, flesh, wood, stone, steel, concrete, glass, rubber, ... All are *deformable*. A theory aiming to describe their mechanical behavior must take heed of their deformability and represent the definite principles it obeys.

The rational mechanics of materials was begun by JAMES BERNOULLI, illustrated with brilliant examples by EULER, and lifted to generality by CAUCHY. The work of these mathematicians divided the subject into two parts. First, there are the *general principles*, common to all media. A mathematical structure is necessary for describing deformation and flow. Within this structure, certain physical laws governing the motion of all finite masses are stated. These laws, expressed nowadays as integral equations of balance, or "conservation laws", are equivalent either to *field equations* or to *jump conditions*, depending on whether smooth or discontinuous circumstances are relevant. Specifically, the axioms of

[1] *Acknowledgment.* This treatise, while it covers the entire domain indicated by its title, emphasizes the reorganization of classical mechanics by NOLL and his associates. He laid down the outline followed here and wrote the first drafts of most sections in Chaps. B, C, and E and of a few in Chap. D. Among the places where he has given new results not published elsewhere, shorter proofs, or major simplifications of older ideas may be mentioned Sects. 22, 27, 30, 34, 44, 46, 52, 63, 64, 65, 68b, 83, 99, 100, 101, 106, 107, 112, 117, 119, 121, and 122. The larger part of the text was written by TRUESDELL, who also took the major share in searching the literature. While NOLL revised many of the sections drafted by TRUESDELL, it is the latter who prepared the final text and must take responsibility for such oversights, crudities, and errors as may remain.

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continuum physics assert the balance or conservation of *mass, linear momentum, moment of momentum, energy, electric charge,* and *magnetic flux.* There is a seventh law, a principle of *irreversibility,* expressed in terms of the *entropy,* but the true form of this law, in the generality we keep here, is not yet known[1]. The reader of this treatise is presumed to be familiar with these piers of continuum mechanics and also to have some competence in the classical linear or infinitesimal theories; such a reader will be able to follow our analysis, which we have attempted to keep self-contained. However, a detailed modern exposition of the general principles, *The Classical Field Theories,* by C. TRUESDELL and R. TOUPIN, with an Appendix on *Invariants* by J. L. ERICKSEN, has been published in Vol. III/1 of this Encyclopedia. Frequent references to sections and equations of that work, indicated by the prefix "CFT", are given so as to provide a helping hand at need; they do not imply that the reader of this treatise is expected to have read CFT or to keep it by him.

The general physical laws in themselves do not suffice to determine the deformation or motion of a body subject to given loading. Before a determinate problem can be formulated, it is usually necessary to specify the *material* of which the body is made. In the program of continuum mechanics, such specification is stated by *constitutive equations,* which relate the stress tensor and the heat-flux vector to the motion. For example, the classical theory of elasticity rests upon the assumption that the stress tensor at a point depends linearly on the changes of length and mutual angle suffered by elements at that point, reckoned from their configurations in a state where the external and internal forces vanish, while the classical theory of viscosity is based on the assumption that the stress tensor depends linearly on the instantaneous rates of change of length and mutual angle. These statements cannot be universal laws of nature, since they contradict one another. Rather, they are *definitions of ideal materials.* The former expresses in words the constitutive equation that defines a *linearly and infinitesimally elastic material*; the latter, a *linearly viscous fluid.* Each is consistent, at least to within certain restrictions, with the general principles of continuum mechanics, but in no way a consequence of them. There is no reason *a priori* why either should ever be physically valid, but it is an empirical fact, established by more than a century of test and comparison, that each does indeed represent much of the mechanical behavior of many natural substances of the most various origin, distribution, touch, color, sound, taste, smell, and molecular constitution. Neither represents all the attributes, or suffices even to predict all the mechanical behavior, of any one natural material. No natural body is perfectly elastic, or perfectly fluid, any more than any is perfectly rigid or perfectly incompressible. These trite observations do not lessen the worth of the two particular constitutive equations just mentioned. That worth is twofold: First, each represents in ideal form an *aspect,* and a different one, of the mechanical behavior of nearly all natural materials, and, second, each does predict with considerable though sometimes not sufficient accuracy the observed response of *many different natural materials in certain restricted situations.*

Pedantry and sectarianism aside, the aim of theoretical physics is to construct mathematical models such as to enable us, from use of knowledge gathered in a few observations, to predict by logical processes the outcomes in many other circumstances. Any logically sound theory satisfying this condition is a good theory, whether or not it be derived from "ultimate" or "fundamental" truth. It is as ridiculous to deride continuum physics because it is not obtained from

[1] Note added in proof: Major progress toward finding this law has been made by COLEMAN in work described in Sect. 96 bis.

nuclear physics as it would be to reproach it with lack of foundation in the Bible. The conceptual success of the classical linear or infinitesimal field theories is perhaps the broadest we know in science: In terms of them we face, "explain", and in varying amount control, our daily environment: winds and tides, earthquakes and sounds, structures and mechanisms, sailing and flying, heat and light.

There remain, however, simple mechanical phenomena that are clearly outside the ranges of the infinitesimal theory of elasticity and of the linear theory of viscosity. For example, a rod of steel or rubber if twisted severely will lengthen in proportion to the square of the twist, and a paint or polymer in a rotating cup will climb up an axial rod. Moreover, the finite but discrete memory of the elastic material and the infinitesimal memory of the viscous fluid are obviously idealized limiting cases of the various kinds of cumulative memories that natural materials show in fast or slow or repeated loading or unloading, leading to the phenomena of creep, plastic flow, strain hardening, stress relaxation, fatigue, and failure.

**2. Method and program of the non-linear theories.** The *non-linear field theories* also rest upon constitutive equations defining ideal materials, but ideal materials more elaborate and various in their possible responses. Of course the aim is to represent and predict more accurately the behavior of natural materials, and in particular to bring within the range of theory the effects mentioned above, typical in nature but altogether wanting in the classical linear or infinitesimal theories.

Insofar as a constitutive equation, relating the stress tensor to the present and past motion, is laid down as defining an ideal material and is made the starting point for precise mathematical treatment, the methods of the linear and non-linear theories are the same, both in general terms and in respect to particular solutions yielding predictions to be compared with the results of experiment in certain definite tests, but in other ways they differ.

*α) Physical range.* When two different natural materials are brought out of the range in which their responses are approximately linearly elastic or linearly viscous, there is no reason to expect their mechanical behaviors to persist in being similar. Rubber, glass, and steel are all linearly elastic in small strain, but their several responses to large strain or to repeated strain differ from one another. It is easy to see mathematically that infinitely many non-linear constitutive equations, differing not only in quantity but also in quality, may have a common linear first approximation. Thus, both from theory and from physical experience, there is no reason to expect any one non-linear theory to apply properly to so large a variety of natural substances as do the classical linear or infinitesimal theories. Rather, each non-linear theory is designed *to predict more completely the behavior of a narrower class of natural materials.*

*β) Mathematical generality.* Because of the physical diversity just mentioned, it becomes wasteful to deal with special non-linear theories unless unavoidably necessary. To the extent that several theories may be treated simultaneously, they surely ought to be. The *maximum mathematical generality* consistent with concrete, definite physical interpretation is sought. The place held by material constants in the classical theories is taken over by material functions or functionals. It often turns out that simplicity follows also when a situation is stripped of the incidentals due to specialization. For example, the general theory of waves in elastic materials is not less definite but is physically easier to understand as well as mathematically easier to derive than is the second-order approximation to it, or any theory resulting from quadratic stress-strain relations.

*γ) Experiential basis*[1]. While laymen and philosophers of science often believe, contend, or at least hope, that physical theories are directly inferred from experiments, anyone who has faced the problem of discovering a good constitutive equation or anyone who has sought and found the historical origin of the successful field theories knows how childish is such a prejudice. The task of the theorist is to bring order into the chaos of the phenomena of nature, to invent a language by which a class of these phenomena can be described efficiently and simply. Here is the place for "intuition", and here the old preconception, common among natural philosophers, that nature is simple and elegant, has led to many great successes. Of course, physical theory must be based on experience, but experiment comes after, rather than before, theory. Without theoretical concepts one would neither know what experiments to perform nor be able to interpret their outcome.

*δ) Mathematical method.* The structure of space and time appropriate to classical mechanics requires that certain *principles of invariance* be laid down. Alongside principles of invariance must be set up *principles of determinism*, asserting which phenomena are to be interconnected, and to what extent. In more popular but somewhat misleading terms, "causes" are to be related to "effects". Principles of these two kinds form the basis for the construction of constitutive equations. General properties of materials such as isotropy and fluidity are related to certain properties of invariance of the defining constitutive equations.

*ε) Product.* After suitably invariant principles of determinism are established, we are in a position to specialize intelligently if need be, but in some cases no further assumptions are wanted to get *definite solutions* corresponding to physically important circumstances. In addition to such solutions, absolutely necessary for connecting theory with experience and experiment, we often seek also *general theorems* giving a picture of the kind of physical response that is represented and serving also to interconnect various theories.

The physical phenomena these theories attempt to describe, while in part newly discovered, are mainly familiar. The reader who thinks that one has only to do experiments in order to know how materials behave and what is the correct theory to describe them would do well to consult a paper by BARUS[2], published in 1888. Most of the effects BARUS considered had been known for fifty to a hundred years, and he showed himself familiar with an already abundant growth of mathematical theories. That he interpreted his own sequences of experiments as confirming MAXWELL's theory of visco-elasticity has not put an end either to further experiments reaching different conclusions or to the creation of other theories, even for the restricted circumstances he considered. If the basic problem were essentially experimental, surely two hundred years of experiment could have been expected to bring better understanding of the mechanics of materials than in fact is had today.

This and many other examples have caused us to write the following treatise with an intent different from that customary in works on plasticity, rheology, strength of materials, etc. We do not attempt to fit theory to data, or to apply the results of experiment so as to confirm one theory and controvert another. Rather, just as the geometrical figure, the rigid body, and the perfect fluid afford simple, natural, and immediate *mathematical models* for some aspects of everyday experience, models whose relevance or application to each particular physical

[1] For further remarks on methods of formulating constitutive equations, see Sects. 293 and 3 of CFT.

[2] BARUS [1888, *1*].

situation must be determined by the user, we strive to find a rational ingress to more complex mechanical phenomena by setting up clear and plausible theories of material behavior, embodying various aspects of long experience with natural materials.

**3. Structure theories and continuum theories**[1]. Widespread is the misconception that those who formulate continuum theories believe matter "really is" continuous, denying the existence of molecules. This is not so. Continuum physics presumes *nothing* regarding the structure of matter. It confines itself to relations among gross[2] phenomena, neglecting the structure of the material on a smaller scale. Whether the continuum approach is justified, in any particular case, is a matter, not for the philosophy or methodology of science, but for *experimental test*. In order to test a theory intelligently, one must first find out what it predicts. Few of the current critics of continuum mechanics have taken so much trouble[3].

Continuum physics stands in no contradiction with structural theories, since the equations expressing its general principles may be identified with equations of exactly the same form in sufficiently general statistical mechanics[4]. If this identification is just, the variables that are basic in continuum mechanics may be regarded as averages or expected values of molecular actions.

It would be wrong, however, to infer that quantities occurring in continuum mechanics *must* be interpreted as certain particular averages. Long experience with molecular theories shows that quantities such as stress and heat flux are quite insensitive to molecular structure: Very different, apparently almost contradictory hypotheses of structure and definitions of gross variables based upon them, lead to the same equations for continua[5]. Over half a century ago, when molecular theories were simpler than they are today, POINCARÉ[6] wrote, "In most questions the analyst assumes, at the beginning of his calculations,

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[1] Other remarks on this subject are given in Sect. 1 of CFT.

[2] The word "macro*scopic*" is often used but is misleading because the scale of the phenomena has nothing to do with whether or not they can be seen (σκοπεῖν). "Molar", the old antithesis to "molecular", is also a fit term to the extent that only massy materials are considered.

[3] What it is surely to be hoped is the high-water mark of logical confusion and bastard language has been reached in recent studies of the aerodynamics of rarefied gases, where the term "non-continuum flow" often refers to anything asserted to be incompatible with the Navier-Stokes equations. Even the better-informed authors in this field usually decide by *ex cathedra* pronouncement based on particular molecular concepts, rather than by experimental test, when continuum mechanics is to be regarded as applicable and when it is not.

[4] Cf. the recent work of DAHLER and SCRIVEN [1963, *19*] on the statistical mechanics of systems with structure: "Both approaches, continuum and statistical, yield the same macroscopic behaviour, regardless of the nature of the molecules and submolecular particles of which the physical system is composed."

[5] The structural theories of NAVIER are no longer considered correct by physicists, but the equations of linear viscosity and linear elasticity he derived from them have been confirmed by experiment and experience for a vast range of substances and circumstances. MAXWELL derived the Navier-Stokes equations from his kinetic theory by using, along with certain hypotheses, a definition of stress as being entirely an effect of transfer of molecular momentum, but experience shows the Navier-Stokes equations to be valid for many flows of many liquids, in which no one considers transfer of momentum as the main molecular explanation for stress. In recent work on the general theory of ensembles in phase space, different definitions of stress and heat flux as phase averages lead to identical field equations for them. Examples could be multiplied indefinitely.

Cf. TRUESDELL [1950, *16*, § 1]: "History teaches us that the conjectures of natural philosophers, though often positively proclaimed as natural laws, are subject to unforeseen revisions. Molecular hypotheses have come and gone, but the phenomenological equations of D'ALEMBERT, EULER, and CAUCHY remain exact as at the day of their discovery, exempt from fashion."

[6] POINCARÉ [1905, *2*, Ch. IX].

either that matter is continuous, or the reverse, that it is formed of atoms. In either case, his results would have been the same. On the atomic supposition he has a little more difficulty in obtaining them — that is all. If, then, experiment confirms his conclusions, will he suppose that he has proved, for example, the real existence of atoms?" While the logical basis of POINCARÉ'S statements remains firm, the evidence has changed. The reader of this treatise is not asked to question the "real" existence of atoms or subatomic particles. His attention is directed to phenomena where differences among such particles, as well as the details of their behavior, are unimportant. However, we cannot give him assurance that quantum mechanics or other theories of modern physics yield the same results. Any claim of this kind must await such time as physicists turn back to gross phenomena and demonstrate that their theories do in fact predict them, not merely "in principle" but also in terms accessible to calculation and experiment.

The relative position of statistical theories, engineering experiment, and the rational mechanics of continua was surveyed as follows by v. MISES[1] in 1930:

„Lassen Sie mich diesen kurzen Andeutungen zwei Schlußbemerkungen anfügen. Die eine wird nahegelegt durch die Weiterbildung, die die Grundlagen der Mechanik in jüngster Zeit auf Seiten der Physiker gefunden haben. Man wird nicht vermuten, daß die Modifikationen, zu denen die Relativitätstheorie oder die Wellenmechanik führen, für die hier von mir behandelten Probleme von Bedeutung sein könnten. Aber es steht anders mit der *Statistik*. Es ist denkbar, daß eine einigermaßen befriedigende Darstellung der typischen Erscheinungen an festen Körpern im Rahmen der Differentialgleichungs-Physik gar nicht möglich ist, daß es keine Ansätze gibt, die in Erweiterung oder Zusammenfassung der bisherigen das Charakteristische der bleibenden Formänderungsvorgänge richtig wiedergeben. In der Hydromechanik der turbulenten Bewegungen scheint es ja schon festzustehen, daß der statistische Grundzug der Erscheinung schon beim ersten Ansatz einer brauchbaren Theorie berücksichtigt werden muß. Fragt man aber, ob wir von der „statistischen Mechanik" her Hilfe für unsere Aufgaben erwarten dürfen, so sieht es damit wohl schlecht aus. Es zeigt sich ja umgekehrt, daß dort, wo man ganz unzweifelhaft mit statistischem Material zu tun hat, etwa in der Mechanik der Kolloide, das Beste, was überhaupt erreichbar ist, durch Herübernahme von Ansätzen aus der Mechanik der Kontinua gewonnen wird. — Die zweite Bemerkung kehrt an den Ausgangspunkt meines Berichtes zurück, das Verhältnis der Technik zu den Bestrebungen der rationellen Mechanik nach Aufklärung des mechanischen Verhaltens der wirklich beobachtbaren Körper. Kein Zweifel: Die Aufgaben der Materialprüfung drängen nach wenigstens vorläufigen praktischen Lösungen, und sie sucht sie in Ansätzen der hier beschriebenen Art, aber ohne rationelle Grundlage, meist ohne Kenntnis des Vorhandenen, unter ständig zunehmender Verwirrung der Begriffe und Bezeichnungen. Zahllose Aufsätze, die an neue experimentelle Feststellungen anknüpfen, suchen immer von neuem die Grundbegriffe zu definieren, Meßverfahren, ja Maßeinheiten für die Stoffeigenschaften einzuführen — es gibt wenigstens ein halbes Dutzend verschiedener Plastizitätsmesser — ohne jede vernünftige theoretische Grundlegung. Von amerikanischer Seite ist schon der Vorschlag gemacht worden, zur Klärung der Verhältnisse einen Ausschuß von Fachleuten einzusetzen. Ich glaube, daß wir den Fortschritt zunächst auf andere Weise suchen müssen: durch sorgfältige Beachtung der logischen Grundlagen der Theorie und der bisherigen mathematischen Ansätze, deren Ausgestaltung *allein dazu führen kann, die experimentelle Forschung in geordnete und fruchtbare Bahnen zu leiten.*"

Since 1930, the data on which v. MISES based this summary have been replaced by other, more compelling facts. While intensive and fruitful work has been carried out both in statistical theories of transport processes and in experiment on materials, on a scale overshadowing all past efforts, the reader of this treatise will see that the rational mechanics of continua has grown in even greater measure.

It should not be thought that the results of the continuum approach are necessarily either less or more accurate than those from a structural approach. The two approaches are *different*, and they have different uses.

First, a structure theory implies more information about *a given material*, and hence less information about *a class of materials*. The dependence of viscosity on

[1] v. MISES [1930, *2*].

temperature in a gas, for example, is predicted by the classical kinetic theory of moderately dense gases, while in a continuum theory it is left arbitrary. For each different law of intermolecular force, the result is different, and for more complicated models it is not yet known. A continuum theory, less definite in this regard, may apply more broadly. The added information of the structural theory may be unnecessary and even irrelevant. To take an extreme example, a full structural specification implies, in principle, all physical properties. From the structure it ought to be possible to derive, among all the rest, the smell and color of the material. A specification so minute will obviously carry with it extreme mathematical complexity, irrelevant to mechanical questions regarding finite bodies.

Second, structural specification necessarily presents *all the attributes* of a material simultaneously, while in continuum physics we may easily separate for special study *an aspect* of natural behavior. For example, the classical kinetic theory of monatomic gases implies a special constitutive equation of extremely elaborate type, allowing all sorts of thermo-mechanical interactions, with definite numerical coefficients depending on the molecular model. For a natural gas really believed to correspond to this theory, these complexities are sometimes relevant, and the theory is of course a good one. On the other hand, it is a highly special one, offering no possibility of accounting for many simple phenomena daily observed in fluids. For example, it does not allow for a shear viscosity dependent on density as well as temperature, or for a non-zero bulk viscosity, both of which are easily handled in classical fluid dynamics. Doubtless it is true that natural fluids for which such viscosities are significant have a complicated molecular structure, but this does not lessen the need for theories that enable us to predict their response in mechanical situations, perhaps long before their structure is determined.

Third, a continuum theory may obtain by *a more efficient process* results shown to be true also according to certain molecular theories. For example, a simple continuum argument suggests the plausibility of the Mooney theory of rubber, which was later shown to follow also from a sufficiently accurate and general theory of long-chain molecules. A more subtle but more important possibility comes from the general principles of physics. For example, certain requirements of invariance and laws of conservation may be applied directly to the continuum, rendering unnecessary the repeated consideration of consequences of these same principles in divers special molecular models, so that the continuum method may enable us to derive directly, once and for all, results common to many different structural theories. In this way we may separate properties that are truly sensitive to a particular molecular structure from those that are necessary consequences of more general laws of nature or more general principles of division of natural phenomena.

Fourth, the information needed to apply a continuum theory in an experimental context is *accessible to direct measurement*, while that for a structural theory usually is not. For example, in the classical infinitesimal theory of isotropic elasticity it is shown that data measured in simple shear and simple extension are sufficient to determine all the mechanical response of the material. Both the taking of the data and the test of the assertion are put in terms of the kind of measurement for which the theory is intended. The non-linear theories show this same accessibility, though in more complex form.

In summary, then, continuum physics serves to *correlate the results of measurements* on materials and to isolate *aspects* of their response. It neither conflicts with structural theories nor is rendered unnecessary by them.

The foregoing observations refer to those structural theories in which the presumed structure is intended to represent the molecules or smaller particles of natural materials. In regard to the mechanismomorphic structures imagined by the rheologists, we can do no better than quote some remarks of COLEMAN and NOLL[1] in a more special context:

"It is often claimed that the theory of infinitesimal viscoelasticity can be derived from an assumption that on a microscopic level matter can be regarded as composed of 'linear viscous elements' (also called 'dashpots') and 'linear elastic elements' (called 'springs') connected together in intricate 'networks'....

"We feel that the physicist's confidence in the usefulness of the theory of infinitesimal viscoelasticity does not stem from a belief that the materials to which the theory is applied are really composed of microscopic networks of springs and dashpots, but comes rather from other considerations. First, there is the observation that the theory works for many real materials. But second, and perhaps more important ..., is the fact that the theory looks plausible because it seems to be a mathematization of little more than certain intuitive prejudices about smoothness in macroscopic phenomena."

**4. General lines of past research on the field theories of mechanics.** While, reflecting the stature of the researchers themselves, the early researches on the foundations of continuum mechanics did not show any preference for linear theories, with the rise of science as a numerous profession in the nineteenth century it was quickly seen that linearity lends itself to volume of publication. The linear theories of heat conduction, attraction, elasticity, and viscosity, along with the linear mathematical techniques that could be applied in them, were developed so intensively and exclusively that in the minds of many scientists down to the present day they are synonymous with the mechanics of continuous media. It would be no great exaggeration to say that in the community of physicists, mathematicians, and engineers, less was known about the true principles of continuum mechanics in 1945 than in 1895.

Blame for this neglect of more fundamental study may be laid to two contradictory misconceptions: First, that the classical linear or infinitesimal theories account for everything known about natural materials, and, second, that these two theories are merely crude "empirical" fits to data. The second is still common among physicists, many of whom believe that only a molecular-statistical theory of the structure of materials can lead to understanding of their behavior. The prevalence of the former among engineers seems to have grown rather from a rigid training which deliberately confined itself to linearly biased experimental tests and deliberately described every phenomenon in nature, no matter how ineptly, in terms of the concepts of the linear theories.

Of course, at all times there have been a few scientists who thought more deeply or at least more broadly in regard to theories of materials. Various doctrines of *plasticity* arose in the latter part of the last century and have been cultivated diffusely in this. These theories have always been closely bound in motive, if often not in outcome, to engineering needs and have proliferated at once in detailed approximate solutions of boundary-value problems. Their mechanical foundation is insecure to the present day, and they do not furnish representative examples in the program of continuum physics. Similarly, the group of older studies called *rheology* is atypical in its nearly exclusive limitation to one-dimensional response, to a particular cycle of material tests, and to models suggested by networks of springs and dashpots.

While only very few scientists between 1845 and 1945 studied the foundations of continuum mechanics, among them were some of the most distinguished savants of the period: ST. VENANT, STOKES, KIRCHHOFF, KELVIN, BOUSSINESQ, GIBBS, DUHEM, and HADAMARD. Although phenomena of viscosity and plasticity were not altogether neglected, the main effort and main success came in the theory of

[1] COLEMAN and NOLL [1961, 7, § 1].

finite elastic strain. The success, however, was but small. When the brothers COSSERAT published their definitive exposition[1] in 1896, its 116 pages contained little more than a derivation of various forms of the general equations. Beyond the laws of wave propagation and the great theorem on elastic stability obtained shortly afterward by HADAMARD, no concrete progress was made in the finite theory for the following fifty years. Not only did the want of concepts such as to suggest a simple notation lay a burden of page-long formulae on the dragging steps of writer or reader, but also there was no evidence of a program of research. Linear thinking, leading to easy solutions for whole classes of boundary-value problems, obviously would not do, but nothing was suggested to take its place, except, perhaps, the dismaying prospect of creeping from stage to stage in a perturbation process.

In that period, however, many papers on the subject were published. When not essentially repetitions of earlier studies, these concerned special theories or approximations, most of which have later turned out to be unnecessary in the cases when they are justified. Knowledge of the true principles of the general theory seems to have diminished except in Italy, where it was kept alive by the teaching and writing of SIGNORINI.

A new period was opened by papers of REINER[2] and RIVLIN[3]. The former was the first to suggest any *general approach* or *unifying principle* for non-linear constitutive equations[4]; the latter was the first to obtain *concrete, exact solutions* to specific problems of physical interest in non-linear theories where the response is specified in terms of *arbitrary functions* of the deformation. Both considered not only finitely strained elastic materials but also non-linearly viscous fluids. RIVLIN was the first to see the far-reaching simplification effected in a non-linear theory by assuming the material to be incompressible.

In 1952 was published a detailed exposition, *The Mechanical Foundations of Elasticity and Fluid Dynamics*[5], in which both the old and the new trends were summarized. On the one hand, the numerous special or approximate theories were set in place upon a general frame and related to each other insofar as possible, especially so as to make clear the arbitrary and unsupported physical assumptions and the insufficient if not faulty mathematical processes by which they had been inferred. On the other, the concrete and trenchant gains won by the new approaches were presented in full and with emphasis.

A summary[6] of the researches of 1945—1952, referring especially to problems of flow, has stated:

"By 1949 it could be said fairly that all work on the foundations of rheology done before 1945 had been rendered obsolete. The phenomenon of normal stresses had been shown to be of second order, while departures from the classically assumed linear relation between shearing tractions and rates of shearing are of third order in the rates. The old viscometers, designed without a thought of normal stresses, had fixed opaque walls to help the experimenter overlook the most interesting effect in the apparatus or to prevent his measuring the forces supplied so as to negate it. By theory, the phenomenon of normal stresses was straight-

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[1] E. and F. COSSERAT [1896, *1*]. Essentially the same material, but expressed in tensor notation, is contained in the widely read paper of MURNAGHAN [1937, *2*].

[2] REINER [1945, *3*] [1948, *9*].

[3] RIVLIN [1948, *12*, *14*] [1949, *15*, *16*, *17*, *18*].

[4] Considerations of invariance had occurred earlier, notably in the work of POISSON and CAUCHY, but always in rather special contexts. Cf. Sect. 19A.

[5] TRUESDELL [1952, *20*] [1953, *25*]. A corrected reprint has been announced as a volume in the International Science Review Series, Gordon & Breach, N.Y.

[6] TRUESDELL [1960, *58*, pp. 13, 15].

away seen to be a universal one, to be expected according to all but very special kinds of non-linear theories. Of course a result so universally to be expected must have been occurring for a long time in nature, and it was quickly seen that many familiar effects, such as the tendency of paints to agglomerate upon stirring mechanisms, as well as some carefully concealed mysteries of the artificial fiber industry, are examples of it, though a century of linear thinking in physics had blinded theorists to the possibility that simple mechanics, rather than chemistry, is all that is needed in explanation. ...

"While ... [this research] gained a number of theoretical predictions of remarkable completeness, these are the least of what it gave us. Next is the fact that, with little exaggeration, *there are no one-dimensional problems*: A situation which is one-dimensional in a linear theory is automatically two-dimensional or three-dimensional in any reasonable non-linear theory. More important is the *independence in theory* which resulted from the realization that any sort of admissible non-linearity would yield the correct general kind of behavior, and that to account for the phenomena, far from being difficult, was all too easy[1]. Of a theory, we learned that both less and more had to be expected. To calculate the creep in a buckled elliptical column with a square hole in it is too much until the response of materials shall be better understood than it is today; to be satisfied with a normal stress of the right sign and order, with an adjustable coefficient, is too little until the response of the *same* material in a *variety* of situations is determined and correlated, with no material constants or functions altered in the process. What is needed is a theory of theories."

Since 1952, it cannot be said that the older type of work has ceased; rather, in the common exuberance of modern publication, an easy place is found not only for continued search of avenues known to be sterile, but also for frequent rediscovery of special theories included and criticized in *The Mechanical Foundations*, and of special cases of solutions presented there in explicit generality. Beyond this, and heedless of it, a small school of younger scientists, of backgrounds and trainings as various as mathematics, physics, chemistry, and engineering, has developed the newer approaches. Not only have major results been obtained in the classical general theory of finite elastic strain, to the point that there is now a technology of the subject, but also success beyond any fair expectation has been met in a very general theory of non-linear viscosity and relaxation. A great range of the mechanical behavior of materials previously considered intractable if not mysterious has been brought within the control of simple, precise, and explicit mathematical theory. Just a little earlier, relevant experiments had begun on a material which lends itself particularly well to measurements of the effects of large deformation and flow: polyisobutylene. It should not be thought, however, that the theories apply only to high polymers. The non-linear effects are typical of mechanics, and there is reason to think they occur in nearly all materials — for example, in air and in metals — but generally their variety is so great that it is difficult to separate one from another. High polymers are distinguished not so much for the existence as for the simplicity of the non-linear effects they exhibit. The new researches on the general theories, preceded by the classical foundation established in the last century, form the subject of the present treatise.

Of the several kinds of attack to which the new continuum-mechanics has been subject, only two deserve notice, because only these have some basis in truth. First, some scientists of the "practical" kind presume that pages full of tensors and arbitrary functions or functionals

[1] Detailed substantiation is given by our analysis of the Poynting effect in Sect. 54 and our presentation of normal-stress effects in Sects. 106–115.

can never yield results specific enough to apply to the real world. Second, the analyst who has been taught that everything begins with existence and uniqueness theorems may reject as being only "physics" or "engineering" anything that does not consist solely of convergence proofs and estimates. We hope that critics of the former kind will notice in our text the multitude of exact or approximate solutions of specific problems for elastic materials and for simple fluids as well as certain explicit calculations for more general materials, with results fit for comparison with measurements; while this treatise is purely mathematical in content, we have included by way of an existence proof some tables and graphs of data on experiments done expressly in response to the analyses here summarized. We hope that critics of the latter kind will notice page after page of definite theorems and strict proofs and will allow that mathematics is not confined to any rigid pattern; in particular, we hope that this treatise will be admitted in evidence that mathematics enables us to correlate information available on various aspects of a class of physical theories even when that information is too imperfect to lay down a "well set problem" in the style of the common theories of the last century. Finally, we trust that those who regard as essential to modern science the expense and labor of numerical computation on large machines will easily find for themselves a thousand points in our subject where such a taste can be gratified at any time.

**5. The nature of this treatise.** In 1955 it was planned to contribute to this Encyclopedia two articles that would in effect bring *The Mechanical Foundations* up to date and complete it by a correspondingly detailed presentation of aspects of the foundations of continuum mechanics omitted from it. The former part of the project, concerning the general principles of continuum physics, has been finished and printed as *The Classical Field Theories* (CFT) in Vol. III/1. The latter part has had to be modified[1].

In the first place, the flow of important publication on the basic principles of non-linear theories and on experiment in connection with them has increased tenfold: Scarcely a month passes unmarked by a major paper. What follows here has been not only rewritten but also several times reorganized so as to incorporate researches published after we had begun — in some cases, researches growing straight from the difficulties we ourselves encountered in the writing. Second, the special or approximate theories of elasticity or viscosity, to explaining and interrelating which a considerable part of *The Mechanical Foundations* was devoted, have lost their value because of the greater efficiency and enlightenment the more general methods have since been shown to offer. Third, the theories usually named "plasticity" remain in essentially the same state as they were in 1952, when they were intentionally omitted from *The Mechanical Foundations*[2].

For these reasons, the present treatise is of lesser scope than was originally planned. First, although we have taken pains to include a new and general foundation for the continuum theory of dislocations, we have not felt able to do more in regard to the usual theories of "plasticity" than to refer the reader to the standard treatises, e.g. to the article by FREUDENTHAL and GEIRINGER in Vol. VI of this Encyclopedia. Second, we have omitted most of the special theories of elasticity and viscosity, for them referring the reader to *The Mechanical Foundations*[3].

Work in this field is often criticized for opaque formalism. Some of those not expert in the subject have implied that the specialists attempt to make it seem more difficult than it is. In the original development of any science, the easiest way is often missed, and the lack of a pre-organized common experience and vocabulary, often called "intuition" by those whose concern is paedagogy or professional amity rather than discovery, makes the path of the creator hard to follow. In writing the treatise we present here, earnest and conscious effort has been put out to render the subject simple, easy, and beautiful, which we believe it is, increasingly with the repeated reconsideration of the groundwork and the major results which have appeared in the last decade. On the other hand, we have not followed the lead of some experts in other fields who have lightly entered

[1] In the mean time a general exposition of the field has been published by ERINGEN [1962, *18*] and reviewed by PIPKIN [1964, *67*].

[2] Recently GREEN and NAGHDI [1965, *18*] have proposed a rational theory of finitely deformed plastic materials, but they adopt a yield condition as in the older literature.

[3] TRUESDELL [1952, *20*, §§ 48–54, 60, 81–82] [1953, *25*].

this with too hasty expositions that by their slips and gaps prosper in making the subject appear to their unwary readers as being simpler and easier (though less beautiful) than in fact the physical behavior of materials in large and rapid deformation can be.

Instead of completeness, we have attempted to achieve *permanence*. As the main subjects of this treatise we have selected those researches that formulate and solve *once and for all* certain clear, definite, and broad conceptual and mathematical problems of non-linear continuum mechanics. We not only hope but also believe that the major part of the contents is not controversial or conjectural, representing instead unquestionable conquests that will become and remain standard in the subject. After the classic researches done before 1902, nearly everything in this treatise was first published, at least in the form here given to it, after 1952. We do not pretend, however, to be exhaustive[1] even for the most recent work or for citation of it, since we have subordinated detail to importance, and, above all, to *clarity and finality*.

Our citations refer either to the original sources or to works containing related developments not given in this treatise. Thus, since scant service would be done any reader by directing him to the numerous textbooks and paedagogic "introductions", we follow the precedent of *The Classical Field Theories*, criticized by one reviewer for preferring very old or very new references.

Properly, our title should have indicated restriction to classical mechanics, for relativistic field theories lie outside our scope. Since, however, the term "classical" suggests to many a domain long mastered — indeed, one reviewer criticized *The Classical Field Theories* for including material he did not already know — that word seems inappropriate in the title of a treatise devoted mainly to very recent work. Specifically, we consider *the mechanical response of materials in three-dimensional Euclidean space*. While often the dimension 3 can be replaced effortlessly by $n$, the main conceptual structure is closely bound to Euclidean geometry. Relativistic generalization has required major changes in views and details which were not yet known when this treatise was planned[2].

This treatise is written, not for the beginner, but for the specialist in mechanics who wishes to gain quickly and efficiently the solid and complete foundation necessary to do theoretical research, either in applications or in further study of the groundwork, in non-linear continuum mechanics. We use the term *non-linear* in the sense of material response, not of mathematical analysis[3].

Accordingly, after an introductory chapter fixing notations and listing a number of mathematical theorems for use in the sequel, this treatise is divided into three major parts, as follows.

Chapter C presents a *general approach*, based upon principles of *determinism*, *local action*, and *material frame-indifference*, to the mechanical properties of materials. For the special case of a *simple material*, in which the stress at a particle is determined by the cumulative history of the deformation gradient at that particle,

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[1] It seems necessary to add, however, that when we merely cite and describe a work, without presenting its contents in detail, we do so merely to help the reader find his way in the literature, implying *neither endorsement nor criticism*.

[2] A bibliography of older work on relativistic theories of materials, mainly fluids, is given at the end of CFT. BRESSAN [1963, *13*, *14*, *16*] and BRAGG [1965, *3*] are the first authors to consider correctly finite deformations and accumulative effects in relativity.

[3] The classical theory of viscous incompressible fluids, for example, is governed by non-linear partial differential equations, but we do not include it here since its defining constitutive equation is a linear one. In fact, since the acceleration is a non-linear function of velocity and velocity-gradient, all theories of the motion of continua are non-linear in the spatial description, so the analytical distinction is an empty one except in regard to methods of approximation.

all three fundamental principles may be expressed in a final and explicit mathematical form. Qualities distinguishing one kind of material from another are then defined by invariant properties of the response functionals; the terms "materially uniform", "homogeneous", "solid", "fluid", and "isotropic" are made precise in terms of mathematical systems constructed from the functionals. Finally, it is shown that if the response functional of a simple material is sufficiently smooth in a certain sense, then BOLTZMANN'S equations of linear visco-elasticity result as an approximation in motions whose histories are nearly constant. Thus the general theory of simple materials is seen to furnish a properly invariant generalization of classical visco-elasticity to arbitrary states of deformation and flow; likewise it includes not only as special cases but also in suitable senses of approximation the classical theories of finite elasticity and linear viscosity.

In statics, the stress in any simple material reduces to a function of the finite strain. Materials having this property also in time-dependent deformations are said to be *elastic*, and most of Chapter D is devoted to them. Here we present the theory of *finite elastic strain*, not only its principles but also its general theorems and the known special exact solutions or approximate methods, in generality and completeness not before attempted. When, as proposed by GREEN, the work done in elastic deformation is stored as internal energy, so that the stresses are derivable from a stored-energy function as a potential, the material is called *hyperelastic*. Nearly all previous studies concerned this case exclusively. While we develop its distinguishing properties and general theorems, our emphasis lies on the more embracing concept, due to CAUCHY. Generalizations of hyperelasticity to allow for thermal conduction, polarization, and couple stresses are then sketched. The last sections of the chapter concern the partly more general and partly exclusive concept of *hypo-elasticity*, according to which the time-rate-of-change of stress is an explicit function of the stretchings, shearings, and spin at a material element, along with the present stress. The behavior of a hypo-elastic material depends essentially upon the initial stress.

Chapter E concerns *fluidity*. Most of its contents is given over to an exhaustive survey of what is known about *simple fluids*. These are distinguished from other simple materials by having the maximum possible isotropy group; all are in fact isotropic. While they are capable of exhibiting complicated effects of stress-relaxation and long-range memory, these are proved to have no influence on certain special kinds of flow, which turn out to include all those customarily used in viscometric tests. For these special flows, the response functional is shown to manifest itself only through three *viscometric functions*. One of these may be interpreted as a non-linear shear viscosity; the other two, as differences of normal stresses. The exact solutions of the dynamical equations are developed for these flows, as well as for some others of similar kind. The chapter closes with consideration of materials embodying various other concepts of fluidity.

By including general effects of rates and of relaxation, we cover a broader range of physical phenomena than did *The Mechanical Foundations*, although we narrow the topic by omitting most special or approximate theories. The main difference, however, is one of depth. In the present treatise the method of inquiry and formulation, far less formal than the approaches known in 1952, goes straight to the physics of each situation. We have sought, and we believe we have often succeeded in finding, *simple and clear* mathematical expression for the physical principles or hypotheses.

**6. Terminology and general scheme of notation.** We employ at will the notations of RICCI'S tensor calculus[1], of linear vector and matrix algebra, and of GIBBS'

[1] Short introductions to tensor analysis have been given by LICHNÉROWICZ [1950, *10*] and REICHHARDT [1957, *17*]. A large body of definitions, identities, and theorems especially useful in continuum physics is presented in the Appendix to CFT.