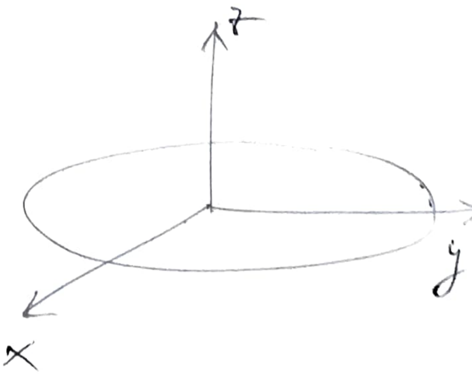
Kvadriky v $\mathbb{R}^3$ (viz G1)

KM1

I.) Nedegenerované:

• **Elipsoid**: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Pozn: sféra pro $r=a=b=c$.

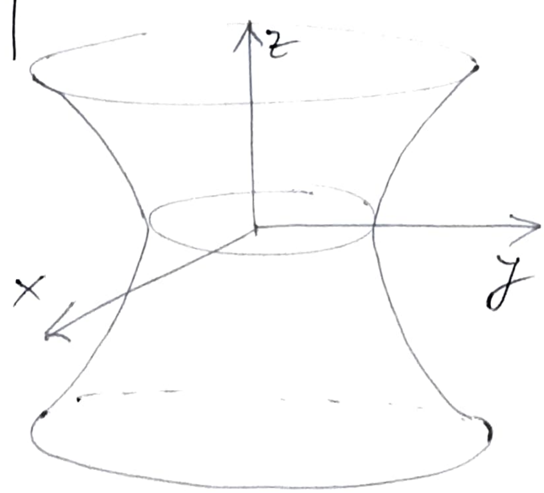


• **Hyperboloid**: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=\pm 1$

jednodílný / dvoudílný

$x^2+y^2=z^2\pm 1$

$\boxed{+}$



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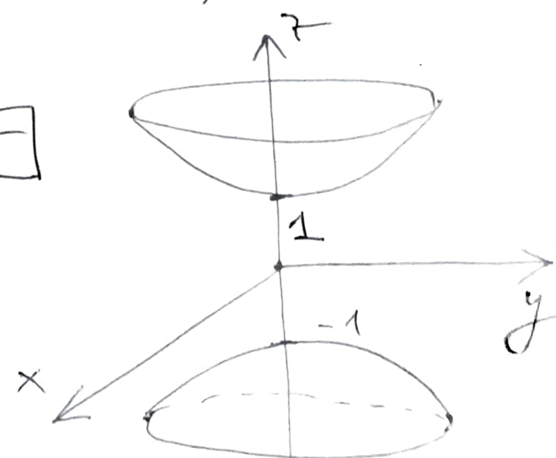

• **Paraboloid**: $z=\frac{x^2}{a^2}\pm\frac{y^2}{b^2}$ eliptický / hyperbolický

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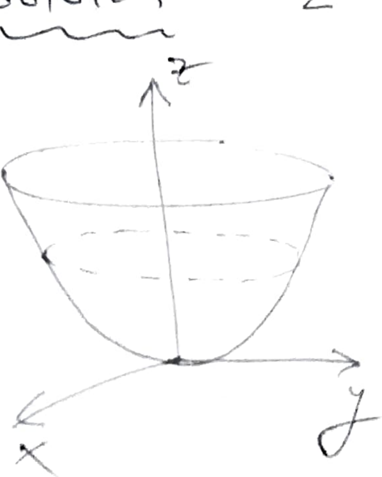

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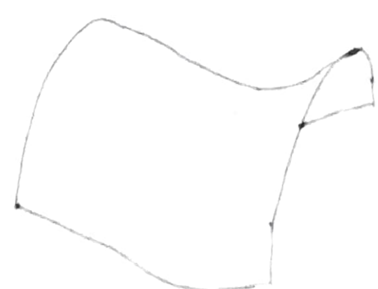
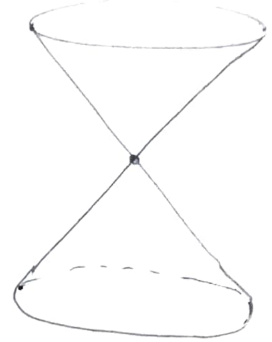


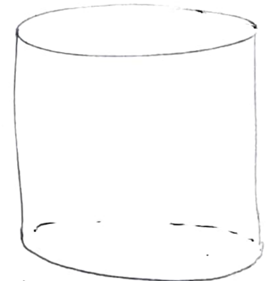
II. Degenerované:

- **Kužel eliptický:** $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ $(z\neq 0)$.



- **Válcové plochy:** eliptické/hyperbolické/parabolické

$$\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1\,,\quad x^2+2ay=0$$



Ukažte, že jde o 2-plochy v $\mathbb{R}^3$. Najděte atlas těchto ploch.

[WIKIPEDIA, Quadric]

$\boxed{\text{TP1}}$

Tečný prostor

Popište tečný prostor $T_x S$ k ploše S v bodě $x \in S$, pokud S je zadaná (lokálně)

(1.) mapou, tzn. $\varphi : U \subset \mathbb{R}^k \to S$, $x = \varphi(u)$:

Potom $T_x S = D\varphi(u)(\mathbb{R}^k) = Lo\left\{ \frac{\partial \varphi}{\partial u_1}(u), \dots, \frac{\partial \varphi}{\partial u_k}(u) \right\}$;

(2.) jako graf funkce, tzn. $S = \operatorname{graf}(f)$,

$f : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$ tř. $\mathcal{C}^\infty$: Potom je

$\varphi(u) := (u, f(u))$, $u \in U$ mapa S a

$$D\varphi(u) = \begin{array}{cl} k & \\ \left(\begin{array}{c} I_k \\ \hline Df(u) \end{array}\right) & \begin{array}{l} k \\ n-k \end{array} \\ k & \end{array}$$

jednotková matice

(3.) implicitně, tzn. $S = \{ x \in \mathbb{R}^n \mid F(x) = 0 \}$,

kde $F : G \subset \mathbb{R}^n \to \mathbb{R}^{n-k}$: Potom platí

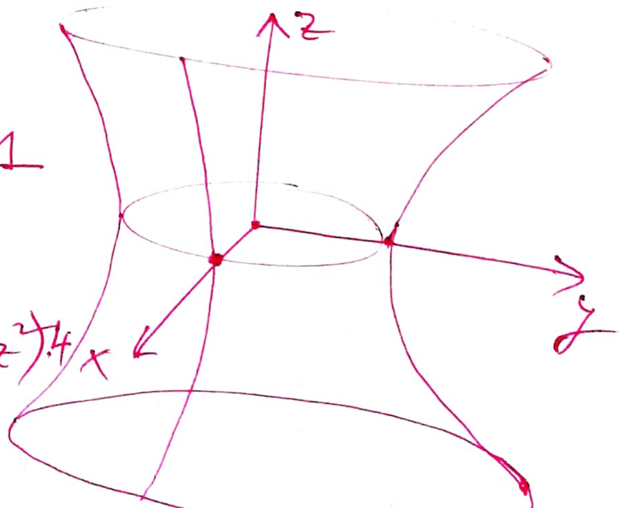
$$T_x S^\perp = Lo\{ \underbrace{\nabla F_1(x), \dots, \nabla F_{n-k}(x)}_{\text{lin. nezávislé řádky v } DF(x)} \}.$$

TP2

[Skutočne, nech $v \in T_x S$. Potom ex. krivka $c: I \to S$ a $t_0 \in I$ taká, že $c(t_0) = x$ a $c'(t_0) = v$. Potom $F(c(t)) = 0$, $t \in I$ a $\left(F(c(t))\right)'\big|_{t=t_0} = DF(x)v = 0$, neboli $v \in \mathbb{R}^n$ je kolmý na vš. $DF_i(x)$. ▨]

Př. Napíšte rovnicu tečnej roviny jednodielneho hyperboloidu $S \subset \mathbb{R}^3$ zadaného implicitne $x^2+y^2-z^2=1$ v bode (i) $(1,0,0)$, (ii) $(1,1,1)$, (iii) $A=(a,b,c)$.

— x —



$x^2+y^2 = 1+z^2$

(1.) $F(x,y,z) := x^2+y^2-z^2-1$

$DF = 2(x,y,-z) \neq 0$ na S

$\|DF\|^2 = 4(x^2+y^2+z^2) = (1+2z^2)4 \neq 0$ na S

Potom $(T_A S)^\perp = Lo\{(a,b,-c)\}$ a tečná rovina v $A \in S$ je $a(x-a)+b(y-b)-c(z-c)=0$.

Pre $A=(1,0,0)$: $x=1$

$A=(1,1,1)$: $(x-1)+(y-1)-(z-1)=0$

$x+y-z-1=0$

TP3

Pozn: $N(X) := \nabla F(X) / \|\nabla F(X)\| = \frac{1}{\sqrt{1+2z^2}}(x,y,-z)$ je jednotkový normálový vektor v $X \in S$, $X = (x,y,z)$.

(2.) Pro $z > 0$ je $z = f(x,y) := \sqrt{x^2+y^2-1}$,
$(x,y) \in U := \mathbb{R}^2 \setminus \overline{B(0,1)}$.

Pro $\varphi(x,y) = (x,y,f(x,y))$ je

$$\frac{\partial \varphi}{\partial x} = \left(1,0,\frac{x}{\sqrt{x^2+y^2-1}}\right) = \left(1,0,\frac{x}{z}\right)$$ tečné vektory

$$\frac{\partial \varphi}{\partial y} = \left(0,1,\frac{y}{\sqrt{\dots}}\right) = \left(0,1,\frac{y}{z}\right)$$

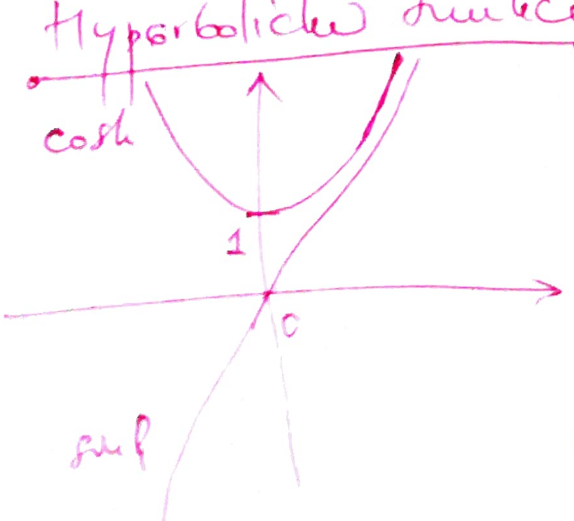
Potom $T_A S = Lo\{(c,0,a),(0,c,b)\}$,
$T_A S^{\perp} = Lo\{(a,b,-c)\}$, protože

$$(c,0,a) \times (0,c,b) = (-ac,-bc,c^2), \quad c > 0$$

Hyperbolické funkce: $e^u = \cosh u + \sinh u$, kde

$$\cosh u := \frac{e^u + e^{-u}}{2}, \quad u \in \mathbb{R}$$

$$\sinh u := \frac{e^u - e^{-u}}{2}$$



(i) $\cosh' = \sinh$, $\sinh' = \cosh$

(ii) $\cosh^2(u) - \sinh^2(u) = 1$

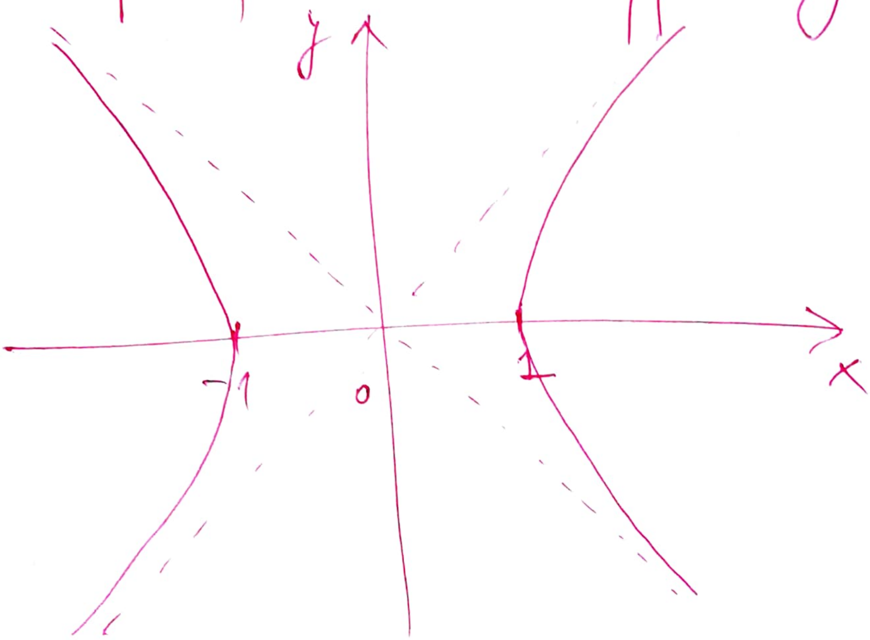
(iii) $\quad + \quad = \cosh(2u)$

atd.

TP4

Pozn: $x = \cosh v,\ y = \sinh v,\ v \in \mathbb{R}$ je mapa pravé větve hyperboly $x^2 - y^2 = 1$ v $\mathbb{R}^2$

$y = \pm\sqrt{x^2-1}$



(3.) $x^2 + y^2 = 1 + z^2 = r^2$, kde $r = r(z) := \sqrt{1+z^2} \geq 1$

$x = r \cdot \cos v \qquad r^2 - z^2 = 1, \quad r = \cosh u$

$y = r \cdot \sin v \qquad\qquad\qquad z = \sinh u$

$$\varphi: \begin{cases} x = \cosh(u)\cdot\cos(v) \\ y = \cosh(u)\cdot\sin(v), \quad u,v \in \mathbb{R} \\ z = \sinh(u) \end{cases}$$

(i) φ je tř. C^∞, $\varphi: \mathbb{R}^2 \xrightarrow{na} S \subset \mathbb{R}^3$

$$\frac{\partial\varphi}{\partial u} = (\sinh u \cos v,\ \sinh u \sin v,\ \cosh u)$$

$$\frac{\partial\varphi}{\partial v} = (-\cosh u \sin v,\ \cosh u \cos v,\ 0)$$

$$\frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v} = (-\cosh^2 u \cos v,\ -\cosh^2 u \sin v,\ \cosh u \sinh u)$$

$$\left\|\frac{\partial\varphi}{\partial u}\times\frac{\partial\varphi}{\partial v}\right\| = \cosh u \cdot \left(\cosh(2u)\right)^{1/2} > 0. \qquad \boxed{+P\surd}$$

Pozn: $N(X) = -\left(\frac{\partial\varphi}{\partial u}\times\frac{\partial\varphi}{\partial v}\right) / \left\|\frac{\partial\varphi}{\partial u}\times\frac{\partial\varphi}{\partial v}\right\|$ v X, je-li $N(X)$ jako v 1.

(ii) $\psi := \varphi|_U$ je mapa, je-li $U := \mathbb{R}\times(-\pi+v_0, v_0+\pi)$ pro libovolné $v_0 \in \mathbb{R}$.

Např. pro $v_0 = 0$: $u = \sinh^{-1} z$

$v = \arg(x, y)$, kde

$\arg : \mathbb{R}^2 \xrightarrow{na} (-\pi, \pi]$ je tzv. hlavní hodnota argumentu z UKA. Víme, že arg je spojitou na $\mathbb{R}^2 \setminus (-\infty, 0] \times \{0\}$.