Harmonic functions

HAR 1

We study $f: \mathbb{C} \to \mathbb{C}$.

Since $\mathbb{C} \simeq \mathbb{R}^2$ we have $z = x + iy$ with $x = \operatorname{Re} z$, $y = \operatorname{Im} z$

$f = u + iv$ with $u = \operatorname{Re} f$, $v = \operatorname{Im} f$.

Observation If $f, g \in \mathcal{H}(G)$, $G \subset \mathbb{C}$ is a domain, and $\operatorname{Re} f = \operatorname{Re} g$ on G, then there is $c \in \mathbb{R}$ s.t. $\operatorname{Im} f = \operatorname{Im} g + c$ on G.

Similarly when $\operatorname{Im} f = \operatorname{Im} g$.

[Pf follows easily from CR conditions (HW)

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$
]

Question What are real parts of holomorphic functions? What properties do they have?

Recall: the total differential

$$df(z_0)h = \partial f(z_0)h + \bar{\partial} f(z_0)\bar{h}$$

where $\partial f = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$

$$\bar{\partial} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

$\boxed{\text{Cauchy–Riemann}}$ $f'(z_0)$ exists iff $df(z_0)$ exists and $\bar\partial f(z_0) = 0$

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$\boxed{\text{LEMMA}}$ If $f \in C^2(G)$ and $G \subset \mathbb{C}$ is open then

$$\partial\bar\partial f = \bar\partial\partial f = \frac{1}{4}\Delta f,$$

where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator.

Pf: $\partial\bar\partial f = \frac{1}{4}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = \frac{1}{4}\Delta f.$ ▨

$\boxed{\text{DEF.}}$ If $G \subset \mathbb{C}$ is open, we say that $u \in C^2(G)$ is harmonic if $\Delta u = 0$ on G.

$\boxed{\text{Ex 1}}$ If $f \in \mathcal{H}(G)$, then $\operatorname{Re} f$ and $\operatorname{Im} f$ are harmonic.

⌈Indeed, $0 = \partial\bar\partial f = \frac{1}{4}\big(\underbrace{\Delta(\operatorname{Re} f)}_{\in\mathbb{R}} + i\underbrace{\Delta(\operatorname{Im} f)}_{\in\mathbb{R}}\big) \Rightarrow$

$$\Delta(\operatorname{Re} f) = 0 = \Delta(\operatorname{Im} f).$$ ⌋

$\boxed{\text{Ex 2}}$ If $f \in \mathcal{H}(G)$, $f \neq 0$ on G and G is a simply connected domain in $\mathbb{C}$, then

$$\log|f| = \operatorname{Re} F$$

for some $F \in \mathcal{H}(G)$; in particular, $\log|f|$ is harmonic.

⌈We know that there is a continuous holomorphic branch F of $\log f$, i.e., $F \in \mathcal{H}(G)$ and $f = e^F$. Then $|f| = e^{\operatorname{Re} F}$. ⌋

$\boxed{\text{Cor}}$ If $G\subset\mathbb{C}$ is open, $f\in\mathcal{H}(G)$ and $f\neq 0$ on G, then $\log|f|$ is harmonic on G.

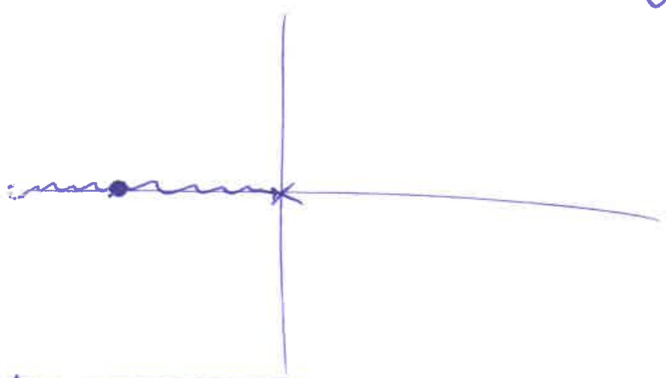
⌈ By $\boxed{\text{Ex. 2}}$, $\log|f|$ is harmonic on any open ball U in G. ⌋

$\boxed{\text{Ex. 3}}$ $f(z)=\log|z|$, $z\in\mathbb{C}\setminus\{0\}$ is harmonic but it is not the real part of any holomorphic function $F\in\mathcal{H}(\mathbb{C}\setminus\{0\})$.

⌈ ?? Assume that $F\in\mathcal{H}(\mathbb{C}\setminus\{0\})$ and $\operatorname{Re}F=f$.

On $\mathbb{C}\setminus(-\infty,0]$, $\operatorname{Re}F=\operatorname{Re}(\log)$ implies
$\operatorname{Im}F=\operatorname{Im}(\log)+c$ for some $c\in\mathbb{R}$,

so we have $F=\log+ic$, which is impossible. ↯ ⌋



$\boxed{\text{Theorem}}$ If G is a simply connected domain and $u: G\to\mathbb{R}$ is harmonic on G, then there is $f\in\mathcal{H}(G)$ s.t. $\operatorname{Re}f=u$.

Re (i) Every harmonic function is locally the real part of a holomorphic function.

(ii) If $f \in \mathcal{H}(G)$, then $f' = \partial f = \cancel{\ldots}$ $= \partial(f + \bar{f}) = 2\partial(\mathrm{Re}\, f)$ because $0 = \overline{(\bar{\partial} f)} = \partial \bar{f}$.

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__Pf:__ We have that $\partial u \in \mathcal{H}(G)$ because

$$\bar{\partial}(\partial u) = \frac{1}{4}\Delta u = 0.$$

Then there is $f_0 \in \mathcal{H}(G)$ s.t. $f_0' = 2(\partial u)$. By (ii) we have $2\partial(\mathrm{Re}\, f_0) = f_0' = 2(\partial u)$

$$\partial(\underbrace{\mathrm{Re}\, f_0 - u}_{\in \mathbb{R}}) = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial x}(\mathrm{Re}\, f_0 - u) = 0, \qquad \frac{\partial}{\partial y}(\mathrm{Re}\, f_0 - u) = 0$$

on the domain G.

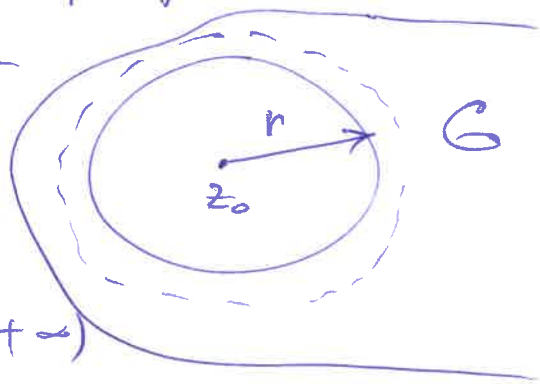
Hence $u = \mathrm{Re}\, f_0 + c$ for some $c \in \mathbb{R}$.

Put $f := f_0 + c$. ▨

$\boxed{\text{Cor}}$ Let $G \subset \mathbb{C}$ be open and u be a harmonic function on G. Then $u \in \mathcal{C}^\infty(G)$ and u satisfies the mean value property, i.e.,

(MV) $$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,dt$$

whenever $\overline{U(z_0, r)} \subset G$.



__Pf:__ Let $\overline{U(z_0,r)} \subset G$. Take $R \in (r, +\infty)$ s.t. $U(z_0, R) \subset G$. Then $u = \mathrm{Re}\, F$ for some $F \in \mathcal{H}(U(z_0,R))$ so $u \in \mathcal{C}^\infty(U(z_0,R))$ and, by the Cauchy integral formula,

$$F(z_0) = \frac{1}{2\pi i} \int_{\varphi} \frac{F(z)}{z - z_0}\, dz \quad \text{with} \quad \varphi(t) = z_0 + r e^{it}, \quad t \in [0, 2\pi)$$

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$$= \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} F(z_0 + r e^{it}) \frac{\cancel{i} \cdot \cancel{r e^{it}}\, dt}{\cancel{r e^{it}}},$$

which implies (MV). ▨

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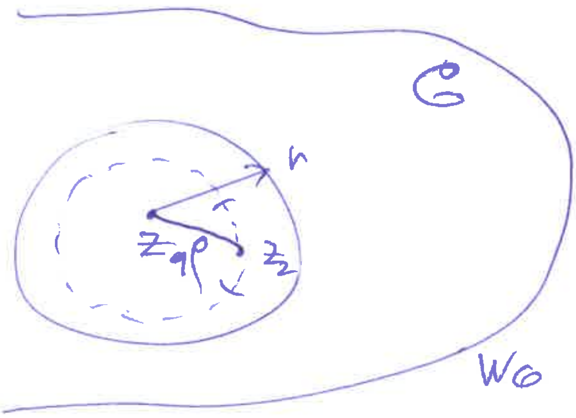
$\boxed{\text{Theorem}}$ (the ~~maximum~~ extremum principle)

Let $G\subset\mathbb{C}$ be a domain and $u: G\to\mathbb{R}$ be continuous with the property (MV). If u is <u>not constant</u>, then u does not attain an <u>extremum</u> in G.

<u>Re</u>: An extremum means a maximum or a minimum.

<u>Pf</u>: Assume that $z_0\in G$ and $u(z_0)\geq u$ on G.



Put $M:=\{z\in G \mid u(z)=u(z_0)\}$. Obviously, $M\neq\emptyset$ and M is closed in G. If we show that M is open, we get $M=G$.

Let $z_1\in M$ and $U(z_1,r)\subset G$. We show that $U(z_1,r)\subset G$.

Indeed, on the contrary, assume that there is $z_2\in U(z_1,r)\setminus M$. By (MV),

$$u(z_0)=u(z_1)=\frac{1}{2\pi}\int_0^{2\pi}u(z_1+\rho e^{it})\,dt<u(z_0)$$

where $\rho:=|z_2-z_1|$. The last inequality is strict because $u\leq u(z_0)$ on G and $u<u(z_0)$ on a nbh. of z_2. This is impossible. ∎

Cor. Let $G \subset \mathbb{C}$ be a bounded open set, HARG

$u \in \mathcal{C}(\overline{G})$ and u be harmonic on G. Then

$$\min_{\partial G} u \le u \le \max_{\partial G} u$$

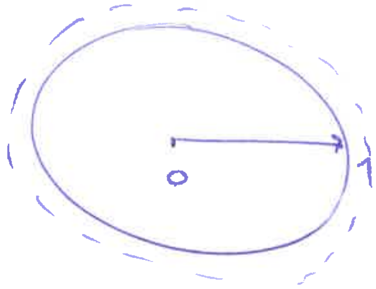
Re: The assumption of harmonicity can be replaced with (MV).

Pf: Let $z_0 \in G$ and $u(z_0) = \max_{\overline{G}} u$

Then u is constant on the component G_0 of G containing z_0, so u attains the maximum at the boundary. ▨

The Poisson integral

Let $u : \overline{\mathbb{D}} \to \mathbb{R}$ be harmonic, i.e., u is harmonic on $U(0,r)$ for some $r \in (1, +\infty)$.



Then there is $f \in \mathcal{H}(\overline{\mathbb{D}})$ s.t.

$$u = \operatorname{Re} f \text{ on } \overline{\mathbb{D}} \text{ and}$$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| \le 1.$$

For $|z| \le 1$, $z = re^{i\theta}$, we have

$$u(z) = \operatorname{Re} f(z) = \operatorname{Re} a_0 + \sum_{n=1}^{\infty} \frac{1}{2} \left(a_n r^n e^{in\theta} + \overline{a_n} r^n e^{-in\theta} \right)$$

because $\operatorname{Re} z = \frac{1}{2}(z + \bar{z})$. Hence

$$(1) \quad u(z) = \sum_{n=-\infty}^{+\infty} b_n r^{|n|} e^{in\theta}, \qquad \text{Fourier series}$$

where
$$b_n := \operatorname{Re} a_0, \quad n = 0;$$
$$:= \frac{1}{2} a_n, \quad n > 0;$$
$$:= \frac{1}{2} \overline{a_n}, \quad n < 0.$$

In addition, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{it}) e^{-imt} \, dt \overset{(1)}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{+\infty} b_n e^{i(n-m)t} \, dt =$$

$$= \sum_{n=-\infty}^{+\infty} b_n \cdot \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)t} \, dt}_{= \begin{cases} 1 & n = m \\ 0 & n \ne m \end{cases}} = b_m \qquad (2)$$

Put (2) into (1): For $z = re^{i\theta}$, $r \in [0,1)$, | MAR 8

we get

$$u(z) = \sum_{n=-\infty}^{+\infty} r^{|n|} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{it})\, e^{in(\theta - t)}\, dt \underset{r<1}{=}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{+\infty} r^{|n|} e^{in(\theta-t)} \right) u(e^{it})\, dt. \qquad (3)$$

$\boxed{\text{Notation}}$ (i) the Poisson kernel $P_r(\theta) := \sum_{n=-\infty}^{+\infty} r^{|n|} e^{in\theta}$

for $0 \le r < 1$, $\theta \in \mathbb{R}$.

(ii) the Poisson integral: $[Pu](re^{i\theta}) :=$

$$(PI) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)\, u(e^{it})\, dt, \quad 0 \le r < 1, \ \theta \in \mathbb{R}.$$

$\boxed{\text{Exercise}}$ $P_r(\theta) = \mathrm{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \frac{1-r^2}{1+r^2-2r\cos\theta}$

Γ $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \Rightarrow 2\mathrm{Re}\left(\frac{1}{1-z}\right) \underset{z=re^{i\theta}}{=} 2 + \sum_{n=1}^{\infty} (r^n e^{in\theta} + r^n e^{-in\theta})$

$$= 1 + P_r(\theta)$$

$P_r(\theta) = \mathrm{Re}\left(\frac{2}{1-z} - 1\right) = \mathrm{Re}\left(\frac{1+z}{1-z}\right) = \ldots$ easy $\blacksquare$ $\rfloor$

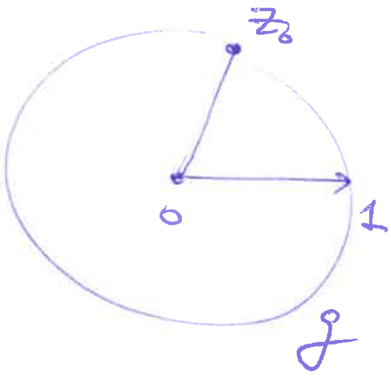
$\boxed{\text{Thm}}$ If u is harmonic on $\mathbb{D}$, then

$$u = Pu \text{ on } \mathbb{D}.$$

Re: (i) $\boxed{\text{the Poisson formula}}$ for harmonic fun's is an analogue of the Cauchy integral formula for holomorphic fun's, (ii) $P1 = 1$ on $\mathbb{D}$

$\boxed{\text{Thm}}$ (boundary behaviour of PI)

Let $g \in L^1(\mathbb{T})$. Here $\mathbb{T} = \partial \mathbb{D}$. Then



1. Pg is harmonic on $\mathbb{D}$;
2. If g is continuous at $z_0 \in \mathbb{T}$ then $\displaystyle\lim_{\substack{z \to z_0 \\ z \in \mathbb{D}}} [Pg](z) = g(z_0)$.
3. (FATOU) For almost all $\theta \in \mathbb{R}$,
$$\lim_{r \to 1-} [Pg](re^{i\theta}) = g(e^{i\theta}) \in \mathbb{R}.$$

Pf: WLOG, assume that g is real.

1. For $|z|<1$, $z = re^{i\theta}$, we have

$$Pg(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}\left(\frac{1+re^{i(\theta-t)}}{1-re^{i(\theta-t)}}\right) g(e^{it})\,dt =$$

$$= \mathrm{Re}\Bigg(\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1+ze^{-it}}{1-ze^{-it}}\, g(e^{it})\,dt}_{\overset{!!\ \text{def.}}{=}\ f(z)}\Bigg)$$

So we have $f \in \mathcal{H}(\mathbb{D})$ and $Pg = \mathrm{Re} f$ is harmonic on $\mathbb{D}$.

(2.) WLOG, $g(z_0)=0$ $\quad$ (otherwise, consider $g-g(z_0)$ and use the fact that $P1=1$). $\quad$ $\boxed{\text{HAR 1.}}$

$z_0 = e^{i\theta_0}$

Let $\varepsilon>0$. Take $\delta\in(0,\pi)$ s.t. $|g(e^{it})|<\varepsilon \quad \forall\, t\in(\theta_0-\delta,\theta_0+\delta)$

Let $z\in\mathbb{D}$, $z=re^{i\theta}$ and $|\theta-\theta_0|<\frac{\delta}{2}$. Then

$$Pg(z) = \frac{1}{2\pi}\int_{\theta_0-\delta}^{\theta_0+\delta} P_r(\theta-t)\,g(e^{it})\,dt \;+$$

$$+\;\frac{1}{2\pi}\int_A P_r(\theta-t)\,g(e^{it})\,dt \;=\; I_1+I_2$$

with $A:=(\theta_0-\pi,\theta_0-\delta)\cup(\theta_0+\delta,\theta_0+\pi)$.

Of course, $P_r>0$ and $\displaystyle\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)\,dt = 1$.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)\,dt = [P(1)](re^{i\theta})$$

Hence $|I_1|<\varepsilon$. Next we have

(*) $$|I_2|\le\frac{1}{2\pi}\int_A P_r(\theta-t)\,|g(e^{it})|\,dt \le P_r\left(\frac{\delta}{2}\right)\cdot\|g\|_{L^1}\xrightarrow{r\to1^-}0$$

because

$$0<P_r(\theta-t)=\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}\le P_r\left(\frac{\delta}{2}\right)$$

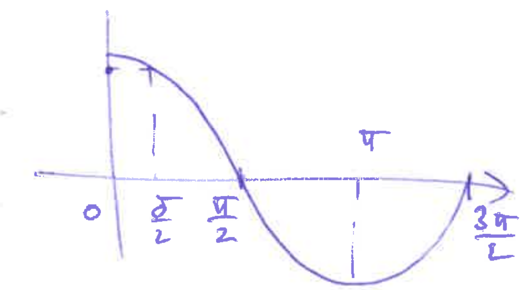
$$\Uparrow$$

$$\frac{\delta}{2}\le|\theta-t|\le\frac{3\pi}{2}$$

$$\Uparrow$$

$$\theta_0+\delta<t<\theta_0+\pi \quad \& \quad -\frac{\delta}{2}<\theta_0-\theta<\frac{\delta}{2}$$

$$\delta<t-\theta_0<\pi$$



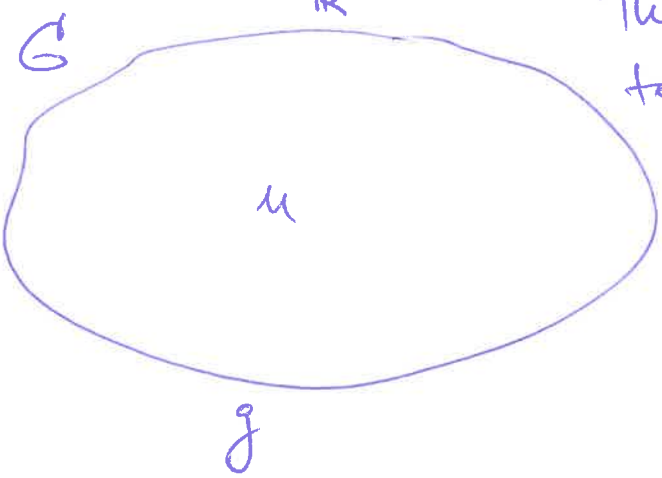
By (*), take $r_0 \in (0,1)$ s.t. $|I_2| < \varepsilon$ $\forall r \in (r_0, 1)$. Then for all $z = re^{i\theta}$ with $|\theta - \theta_0| < \frac{\delta}{2}$ and $r \in (r_0, 1)$, we have

$$|P_g(z)| \le |I_1| + |I_2| < 2\varepsilon. \quad \boxtimes$$

Re: A proof of (3) is difficult and we do not give it.

The Dirichlet problem

Let $G \subset \mathbb{C}$ ($\mathbb{R}^n$) be open and bounded. Let $g \in C(\partial G)$. The Dirichlet problem (DP) is to find $u \in C(\overline{G})$ s.t. u is harmonic on G and $u = g$ on ∂G.

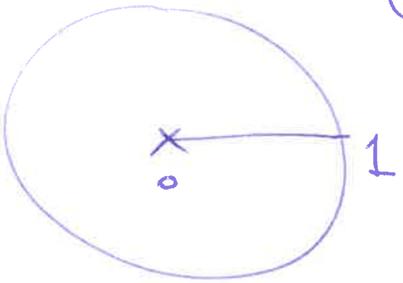


Re (i) DP corresponds to many problems in physics, e.g., a charge distribution g on ∂G gives an electric potential u on G.

(ii) By the maximum principle, (DP) has at most one solution. Indeed, let u_1, u_2 be solutions of (DP). Then, for $u := u_1 - u_2$, we have

$$0 = \min_{\partial G} u \le u \le \max_{\partial G} u = 0 \text{ on } G.$$

(ii) (DP) has not always (classical) solution, e.g. for $G = \mathbb{D} \setminus \{0\}$ and $g := 0$ on $\mathbb{T}$
(ZAREMBA) $:= 1$ at 0.



(iii) (DP) has a unique solution on "nice" domains, e.g., on domains with Lipschitz boundaries.

Theorem (DP on $\mathbb{D}$)

DP has a unique solution on $\mathbb{D}$.
Indeed, let $g \in \mathcal{C}(\mathbb{T})$. Then there is a unique $u \in \mathcal{C}(\bar{\mathbb{D}})$ s.t. u is harmonic on $\mathbb{D}$ and $u = g$ on $\mathbb{T}$. In addition,

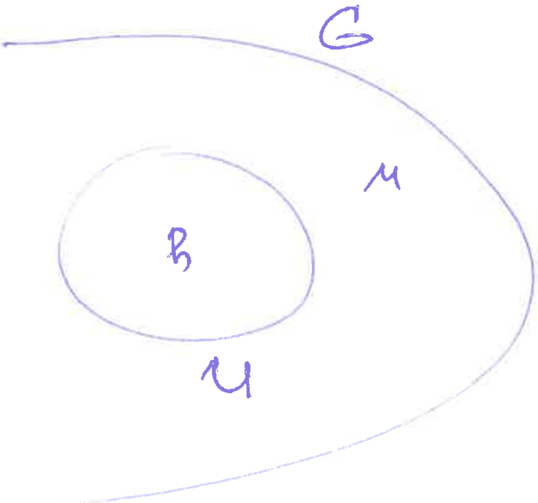
$$u = Pg \text{ on } \mathbb{D}.$$

Pf: It follows from Thm on boundary behaviour of PT. ▨

Re: Obviously, DP has a unique solution on any open disc in $\mathbb{C}$.

Thm Let $G \subset \mathbb{C}$ be open. Then $u : G \to \mathbb{R}$ is harmonic iff u is continuous on G and u satisfies the mean value property (MV).

__Pf:__ $\boxed{\Rightarrow}$ We know this. $\boxed{\Leftarrow}$ Let $u \in \mathcal{C}(G)$ and u have (MV) on G. Let $U := U(z_0, r)$ with $\overline{U} \subset G$. Let h be a unique solution of (DP) on U with the boundary data $u|_{\partial U}$.



Then $v := u - h$ is continuous on $\overline{U}$, satisfies (MV) on U and $v = 0$ on ∂U. By the maximum principle, we get $v = 0$ on $\overline{U}$
$u = h$ on $\overline{U}$.

In particular, u is harmonic on $\overline{U}$. ∎