Hypercomplex Analysis
Selected Topics

(Habilitation Thesis)

ROMAN LÁVIČKA

Field: Mathematics — Mathematical Analysis

Faculty of Mathematics and Physics
Charles University in Prague

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Contents

Chapter 1

Introduction

Without any doubt, complex analysis belongs to the most important areas of mathematics. From its beginnings, there have been several attempts to generalize this deep theory to higher dimensions. One possibility is to study functions of several complex variables and this field of research has already become a classical part of mathematics. The other possibility is to deal with functions of one hypercomplex variable. The first attempt in this direction was of course made in dimension 4 where W. R. Hamilton discovered an analogue of complex multiplication and thus the non-commutative field of real quaternions $\mathbb{H}$. In 1930s quaternionic functions of one quaternionic variable were investigated by G. C. Moisil, N. Théodoresco and, mainly, by R. Fueter and his school. Unfortunately, this theory was almost forgotten for many years and it has become popular again since 1970s, see e.g. the influential paper [104] of A. Sudbery or the books [70] and [27]. Quaternionic analysis studies properties of solutions of the Fueter equation

$$\bar{D}f = \partial_{x_0} f + i\,\partial_{x_1} f + j\,\partial_{x_2} f + k\,\partial_{x_3} f = 0.$$

Here f is an $\mathbb{H}$-valued function defined in $\mathbb{H} \simeq \mathbb{R}^4$ and a quaternion $q \in \mathbb{H}$ is of the form $q = x_0 + x_1 i + x_2 j + x_3 k$ where i, j, k are the imaginary units and $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4$. At the first sight, this equation is an obvious generalization of the Cauchy-Riemann equation from complex analysis. The Fueter operator $\bar{D}$ factorizes the Laplacian Δ and hence all solutions of the Fueter equation are (componentwise) harmonic. Moreover, the Fueter equation is conformally invariant and, as is well-known, the only conformal mappings in this case are quaternionic Möbius transformations. Let us note that in the group of quaternionic Möbius transformations reversible maps are classified in [L1], see also [90]. Recall that the reversible elements of a group are those elements that are conjugate to their own inverse. Reversibility is also studied in other groups, see [99, 89, 66, 65].

At the same time R. Fueter started to work on his theory similar ideas were the centre of attention in physics. Indeed, in 1928 P. Dirac factorized the Klein-Gordon equation (see [47]) and since then the resulting equation called now after him has belonged to the foundations of physics. Furthermore, the Fueter equation may be understood as the Euclidean version of the (massless) Dirac equation (see [102, 103] for details). The latter equation may be naturally defined not only in any dimension but also on certain Riemannian manifolds and the corresponding function theory thus generalizes both complex and quaternionic analysis.

In 1970s R. Delanghe started a systematic study of solutions of the Dirac equation in the Euclidean space $\mathbb{R}^m$ which take values in the Clifford algebra $\mathcal{C}\ell_m$ over $\mathbb{R}^m$ such that $e_je_k + e_ke_j = -2\delta_{jk}$ (see [38]). Here $(e_1, \dots, e_m)$ is an orthonormal basis of $\mathbb{R}^m$ and the Dirac equation is defined by

$$\partial f = e_1 \partial_{x_1} f + \cdots + e_m \partial_{x_m} f = 0.$$

We call solutions of the Dirac equation in $\mathbb{R}^m$ monogenic functions. On the one hand, as we have mentioned, monogenic functions are a higher dimensional analogue of holomorphic functions of one complex variable. It turns out that most basic results from classical function theory are still valid in this framework, including Cauchy's theorem, Cauchy's integral formula, Taylor and Laurent series, residue theory etc. On the other hand, as the Dirac operator ∂ factorizes the Laplacian Δ in the sense that $\Delta = -\partial^2$ theory of monogenic functions which is nowadays called Clifford analysis also refines harmonic analysis. Clifford analysis is still very active area of research full of interesting results and with a wide range of applications, see e.g. the books [14, 43, 64, 70, 67] on this subject.

Now we give an outline of this thesis. In Chapter 2, we recall briefly basic facts and results from Clifford analysis needed later on. As monogenic functions are real analytic it is obviously important to understand the structure of monogenic polynomials called often spherical monogenics. Actually, in Chapter 3, we construct orthogonal bases for spherical monogenics taking values in a certain subspace V of $\mathcal{C}\ell_m$. When it is not necessary to specify the subspace V we refer to V-valued monogenic functions simply as special monogenic functions. We shall see that it is the most interesting to take a subspace V of $\mathcal{C}\ell_m$ invariant with respect to the so-called L-action or the H-action of the Pin group $Pin(m)$ (or the Spin group $Spin(m)$).

It is well-known (see e.g. [43]) that, under the L-action, the spaces of homogeneous spinor valued spherical monogenics form irreducible modules and play a role of building blocks in this case (see Section 2.1.1). When, in the even dimensional case $m = 2n$, the symmetry is given by the L-action restricted to the unitary group $U(n)$ (realized as a certain subgroup of $Spin(m)$), the same role is played by homogeneous Hermitian monogenic polynomials studied in Hermitian Clifford analysis (see Section 2.1.3). Hermitian Clifford analysis has recently become a well-established field of research, see the books [96, 37] and the papers [97, 48, 49, 50, 12, 13, 24, 19, 18, 25, 101, 16, 15, 20, L10]. Let us mention that holomorphic functions of several complex variables are a special case of Hermitian monogenic functions.

Under the H-action, important examples of modules are given by the spaces of homogeneous solutions of generalized Moisil-Théodoresco systems. To be more explicit, let $S \subset \{0, 1, \dots, m\}$ be given and put

$$\mathcal{C}\ell_m^S = \bigoplus_{s \in S} \mathcal{C}\ell_m^s,$$

where $\mathcal{C}\ell_m^s$ is the space of s-vectors in $\mathcal{C}\ell_m$. Then the so-called generalized Moisil-Théodoresco system introduced by R. Delanghe is defined as the Dirac equation $\partial f = 0$ for $\mathcal{C}\ell_m^S$-valued functions f. In recent years, there has been a growing interest in the study and better understanding of these systems (see [6, 42, 41, 44, 45, L8, 46, L9, 94, 95]). In this case, the spaces of homogeneous s-vector valued spherical monogenics form irreducible modules and play a role of building blocks, see [44, 45, L8, 46]. Let us note that, for s-vector valued functions f, the Dirac equation $\partial f = 0$ is equivalent to the Hodge-de Rham system of equations. For 1-vector valued functions, we get the Riesz system, which has been carefully studied (see [35, 40, 28, 29, 80, 108, 88, 69]). See Section 2.1.2 for details.

In Chapter 3, we construct complete orthogonal Appell systems for the most important cases of special monogenic functions. Let us explain what it means in the case of holomorphic functions. As is well-known, we can expand a given holomorphic function f on the unit disc $\mathbb{B}_2$ into its Taylor series

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, z^k.$$

The coefficients of this Taylor series may be expressed directly by the complex derivatives of the function f at the origin due to the fact that $(z^k)' = kz^{k-1}$. In general, we say that basis elements possess the Appell property or form the Appell system if their derivatives are equal to a multiple of another basis element. Moreover, the powers z^k form an orthogonal basis for holomorphic functions in $L^2(\mathbb{B}_2)$, the space of square-integrable functions on $\mathbb{B}_2$. In what follows, we suggest a proper analogue of the powers z^k for several cases of special monogenic functions. Actually, we show that the so-called Gelfand-Tsetlin bases form complete orthogonal Appell systems in these cases. The notion of the Gelfand-Tsetlin basis comes from representation theory. It is known that even any irreducible finite dimensional module over a classical simple Lie group has its Gelfand-Tsetlin basis (see Section 3.1.1 for details).

Already R. Fueter came with the idea to use certain homogeneous monogenic polynomials as a generalization of the complex powers z^k and studied the corresponding Taylor series expansions, namely, series expansions into the so-called Fueter polynomials (see [53] and also [14, 83]). But the Fueter polynomials are not orthogonal with respect to the L^2-inner product, which is not convenient for numerical calculations. First Appell systems of paravector-valued monogenic polynomials were constructed by H. Malonek et. al. (see [82], [51], [52]). These systems were orthogonal but not complete with respect to the L_2-inner product. Moreover, Appell systems are discussed also in [4, 91, 92, 71, 98]. Let us note that the Appell property is connected with the so-called hypercomplex derivative (see [83, 86, 68]). Actually, for monogenic functions, the hypercomplex derivative coincides with the partial derivative with respect to one (the last) variable. In [34], I. Cação and H. Malonek succeeded in constructing an orthogonal Appell basis for the solutions of the Riesz system in dimension 3. For the Riesz system, R. Delanghe considered similar questions, see [40] and also [108], [88]. Later on S. Bock and K. Gürlebeck described orthogonal Appell bases for quaternion valued monogenic functions in $\mathbb{R}^3$ and $\mathbb{R}^4$ (see [10], [9], [7], [8]). In [L5], it is observed that the complete orthogonal Appell system in $\mathbb{R}^3$ constructed in [10] can be considered as a Gelfand-Tsetlin basis. In [L6], the Gelfand-Tsetlin bases for spherical monogenics in dimension 3 are obtained in yet another way.

The first construction of orthogonal bases for Clifford algebra valued monogenic functions even in any dimension was given by F. Sommen, see [100, 43]. Moreover, in dimension 3, explicit constructions using the standard bases of spherical harmonics were done also by I. Cação [28], S. Bock and K. Gürlebeck and H. Malonek (see [30], [32], [31]). In [43, pp. 254-264], orthogonal bases for monogenic functions in $\mathbb{R}^{p+q}$ are constructed when these orthogonal bases are known in $\mathbb{R}^p$ and $\mathbb{R}^q$. The construction is based on solving a Vekua-type system of partial differential equations. In [105, L7], these bases are interpreted as $Spin(p) \times Spin(q)$-invariant orthogonal bases and are obtained using extremal projections. It turns out that the Gelfand-Tsetlin bases correspond to the case when $p = 1$.

In Chapter 3, we show that Gelfand-Tsetlin bases form complete orthogonal Appell systems for spherical harmonics and for Clifford algebra valued, spinor valued, s-vector valued and Hermitian monogenic polynomials. In each of these cases, we describe the Gelfand-Tsetlin construction of orthogonal bases. According to Section 3.1.1, it is clear that the construction is based on the branching of the corresponding homogeneous polynomials. For example, the branching for spherical harmonics in $\mathbb{R}^m$ is nothing else than their decomposition into spherical harmonics in $\mathbb{R}^{m-1}$ multiplied by certain embedding factors and, analogously, for other cases. The branching and thus the Gelfand-Tsetlin basis may be obtained by the Cauchy-Kovalevskaya method (see [43, Theorem 2.2.3, p. 315], [L5], [L9] and [L10, 20, 21, 22, 23] for Clifford algebra valued, spinor valued, s-vector valued and Hermitian monogenic polynomials, respectively). Except for the hermitian case, we determine the embedding factors in the branching quite explicitly and study the corresponding Taylor series expansions. In the hermitian case, we are able so far to describe only an inductive algorithm for a construction of Gelfand-Tsetlin bases and to obtain explicit formulas for basis elements just in complex dimension 2, see [L10]. Let us note that, in Section 3.4, we give an alternative proof of the branching for Hodge-de Rham systems. Moreover, it is worth mentioning that, according to [L5], elements of the Gelfand-Tsetlin bases for spinor valued spherical monogenics in dimension 3 possess the Appell property even with respect to all variables, see Section 3.3.1. An analogous result holds for Hodge-de Rham systems in dimension 3 (see Section 3.4.1) and for Hermitian monogenics in complex dimension 2 (see [L10]). This might be a great advantage for numerical calculations.

In Chapter 4, we review results about finely monogenic functions obtained in a series of the papers [74, L2, L4, 75, L3, 76]. Finely monogenic functions are a generalization of B. Fuglede's finely holomorphic functions to higher dimensions in the context of Clifford analysis. Another possibility is to extend finely holomorphic functions to several complex variables, see [60] and cf. [59]. In Section 4.1, we recall briefly the theory of finely holomorphic functions which has been developed by B. Fuglede, A. Debiard, B. Gaveau, T. J. Lyons and A. G. O'Farrell since 1970s. These functions are an extension of holomorphic functions to plane domains open in a topology finer than the Euclidean one, namely, the fine topology from potential theory. This idea goes back to É. Borel who tried to extend holomorphic functions to more general domains (no longer open) in such a way that the unique continuation property was preserved, see [11]. But his domains were rather special and his theory never became too popular. On the other hand, it gave inspiration to the creation of the important theory of quasi-analytic classes (on the real line) by Denjoy, Carleman and Mandelbrojt.

In Section 4.2, we define finely monogenic functions. Moreover, we generalize several equivalent characterizations of finely holomorphic functions to higher dimensions and we are interested in relations between the obtained conditions. In Section 4.3, we recall some results on fine differentiability. In particular, we know that functions in $\mathbb{R}^m$ which have zero fine differential on a fine domain must be constant and that finely continuously differentiable functions are finely locally extendable to usual continuously differentiable functions. It is quite surprising that these results have been obtained quite recently, see [L3] for the case of dimension $m = 2$ and [62] for the general case. Let us mention that, in [L4], the results in any dimension were also obtained but under a mild additional assumption. Finally, in Theorem 21, we show that, for finely continuously differentiable functions, all the conditions of Section 4.2 are equivalent to each other. It is known that finely holomorphic functions are infinitely fine differentiable and have the unique continuation property. It would be interesting to clear up to what extent these properties remain true for finely monogenic functions in any dimension.

The papers [L1] – [L10] are reprinted in the rest of this thesis.

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Chapter 2

Preliminaries of hypercomplex analysis

In this chapter, we recall some basic concepts and facts from hypercomplex analysis needed later on.

2.1 Clifford analysis

For an account of Clifford analysis, we refer to the books [14, 43, 64, 70, 67].

The Clifford algebras $\mathbb{R}_{0,m}$ and $\mathbb{C}_m$. Let $(e_1, \ldots, e_m)$ be an orthonormal basis of the Euclidean space $\mathbb{R}^m$. Let $\mathbb{R}_{0,m}$ stand for the real Clifford algebra constructed over $\mathbb{R}^m$ such that

$$e_j e_k + e_k e_j = -2\delta_{jk} \quad , \quad j, k = 1, \ldots, m. \tag{2.1}$$

As a basis for $\mathbb{R}_{0,m}$, one takes, for any set $A = \{j_1, \ldots, j_h\} \subset \{1, \ldots, m\}$, the element $e_A = e_{j_1} \ldots e_{j_h}$, with $1 \leq j_1 < j_2 < \cdots < j_h \leq m$, together with $e_\emptyset = 1$, the identity element. The Euclidean space $\mathbb{R}^m$ is embedded in $\mathbb{R}_{0,m}$ by identifying $(x_1, \ldots, x_m)$ with the Clifford vector $x = \sum_{j=1}^m e_j\, x_j$, for which it holds that $x^2 = -|x|^2$. The corresponding complex Clifford algebra $\mathbb{C}_m$ is defined as $\mathbb{C}_m = \mathbb{R}_{0,m} \otimes_{\mathbb{R}} \mathbb{C}$. Any Clifford number a in $\mathbb{R}_{0,m}$ (resp. $\mathbb{C}_m$) may thus be written as

$$a = \sum_A e_A a_A, \ a_A \in \mathbb{R} \ \text{(resp. } a_A \in \mathbb{C}).$$

In what follows, we denote by $\mathcal{C}\ell_m$ either the Clifford algebra $\mathbb{R}_{0,m}$ or $\mathbb{C}_m$. Moreover, for $s = 0, \ldots, m$, a Clifford number $a \in \mathcal{C}\ell_m$ is called an s-vector if $a = \sum_{|A|=s} e_A a_A$. Here $|A|$ is the number of elements of a set A. For each $a \in \mathcal{C}\ell_m$, we have that $a = \sum_{s=0}^m [a]_s$ for some uniquely determined s-vectors $[a]_s$. We call $[a]_s$ the s-vector part of a. For each $a \in \mathcal{C}\ell_m$, define

$$|a| = \left(\sum_A |a_A|^2 \right)^{1/2} \quad \text{and} \quad \bar{a} = \sum_A \bar{e}_A \bar{a}_A$$

where, for $A = \{j_1, \dots, j_h\}$, we put $\bar{e}_A = (-1)^h e_{j_h} \dots e_{j_1}$ and $\bar{a}_A$ is the complex conjugate to a_A. In particular, for $a, b \in \mathcal{C}\ell_m$, we have that $[\bar{a}b]_0 = \sum_A \bar{a}_A b_A$ and $|a|^2 = [\bar{a}a]_0$.

It is easy to see that $\mathbb{R}_{0,1} \simeq \mathbb{C}$ and $\mathbb{R}_{0,2} \simeq \mathbb{H}$ where $\mathbb{H}$ is the algebra of real quaternions with the imaginary units $i = e_1$, $j = e_2$ and $k = e_1 e_2$.

Monogenic functions. A function f defined and continuously differentiable in an open region Ω of $\mathbb{R}^m$ and taking values in the Clifford algebra $\mathcal{C}\ell_m$ is called left monogenic in Ω if it satisfies the Dirac equation $\partial f = 0$ in Ω. Here the Dirac operator ∂ is defined as

$$\partial = e_1 \partial_{x_1} + \dots + e_m \partial_{x_m} \tag{2.2}$$

and $\partial f = e_1 \partial_{x_1} f + \dots + e_m \partial_{x_m} f$. Due to non-commutativity of the Clifford multiplication, we study also right monogenic functions, that is, solutions of the equation $f\partial = (\partial_{x_1} f)\, e_1 + \dots + (\partial_{x_m} f)\, e_m = 0$. In what follows, monogenic always means left monogenic unless otherwise stated. Obviously, the Dirac operator ∂ factorizes the Laplacian $\Delta = \partial^2_{x_1} + \dots + \partial^2_{x_m}$ in the sense that $\Delta = -\partial^2$. As a consequence, each monogenic function f is (componentwise) harmonic, namely, it satisfies the Laplace equation $\Delta f = 0$.

It turns out to be interesting to study special monogenic functions, that is, monogenic functions taking values in a given subspace V of $\mathcal{C}\ell_m$. We shall see below that it is the most interesting to take a subspace V of $\mathcal{C}\ell_m$ invariant in a certain sense. For a general subspace V of $\mathcal{C}\ell_m$, denote by $\mathcal{M}_k(\mathbb{R}^m, V)$ the space of k-homogeneous monogenic polynomials $P : \mathbb{R}^m \to V$. These spaces are at least locally basic building blocks for V-valued monogenic functions. To illustrate this fact, let $\mathbb{B}_m$ be the unit ball in $\mathbb{R}^m$ and let $L^2(\mathbb{B}_m, \mathcal{C}\ell_m)$ be the space of square-integrable functions $f : \mathbb{B}_m \to \mathcal{C}\ell_m$, endowed with a $\mathcal{C}\ell_m$-valued inner product, resp. a (scalar) norm, by

$$(f, g)_{\mathcal{C}\ell_m} = \int_{\mathbb{B}_m} \bar{f} g \, d\lambda^m, \text{ resp. } \quad \|f\| = \left(\int_{\mathbb{B}_m} [\bar{f} g]_0 \, d\lambda^m \right)^{\frac{1}{2}}. \tag{2.3}$$

Here λ^m is the Lebesgue measure in $\mathbb{R}^m$. Moreover, let $L^2(\mathbb{B}_m, V) \cap \operatorname{Ker} \partial$ be the space of L^2-integrable monogenic functions $f : \mathbb{B}_m \to V$. The space $L^2(\mathbb{B}_m, \mathcal{C}\ell_m) \cap \operatorname{Ker} \partial$ is understood as a right $\mathcal{C}\ell_m$-linear Hilbert space. For a general subspace V of $\mathcal{C}\ell_m$, the space $L^2(\mathbb{B}_m, V) \cap \operatorname{Ker} \partial$ forms at least a (real or complex) Hilbert space with respect to the scalar inner product defined by

$$(f, g) = [(f, g)_{\mathcal{C}\ell_m}]_0. \tag{2.4}$$

As is well-known, the orthogonal direct sum

$$\bigoplus_{k=0}^{\infty} \mathcal{M}_k(\mathbb{R}^m, V)$$

is dense in the space $L^2(\mathbb{B}_m, V) \cap \operatorname{Ker} \partial$. Hence to construct an orthogonal basis for the space $L^2(\mathbb{B}_m, V) \cap \operatorname{Ker} \partial$ it is sufficient to find orthogonal bases in all finite dimensional subspaces $\mathcal{M}_k(\mathbb{R}^m, V)$. In particular, in Section

3.2, we construct an orthogonal basis for the right $\mathcal{C}\ell_m$-linear Hilbert space $L^2(\mathbb{B}_m, \mathcal{C}\ell_m) \cap \operatorname{Ker} \partial$.

On the space $\mathcal{M}_k(\mathbb{R}^m, V)$, we also use another inner products, namely, the L_2-inner product on the unit sphere S^m in $\mathbb{R}^m$ and the Fischer inner product defined, respectively, by

$$(P,Q)_{S^m} = \int_{S^m} [\bar{P}Q]_0 \, d\sigma^m \text{ and } (P,Q)_F = [(\overline{P}(\partial_{x_1}, \dots, \partial_{x_m})Q)(x)|_{x=0}]_0. \tag{2.5}$$

Here $d\sigma^m$ is the elementary surface element on S^m. It is well-known that, on the space $\mathcal{M}_k(\mathbb{R}^m, V)$, the inner products $(\cdot,\cdot)$, $(\cdot,\cdot)_{S^m}$ and $(\cdot,\cdot)_F$ are all the same up to a multiple.

Subgroups of $\mathcal{C}\ell_m$ and their representations. As is well-known, inside the Clifford algebra $\mathcal{C}\ell_m$ we can realize the Pin group $Pin(m)$ as the set of finite products of unit vectors of $\mathbb{R}^m$ endowed with the Cliffford multiplication. Moreover, the Spin group $Spin(m)$ is the subgroup of $Pin(m)$ consisting of finite products of even number of unit vectors of $\mathbb{R}^m$. The group $Pin(m)$ (resp. $Spin(m)$) is a double cover of the orthogonal group $O(m)$ (resp. $SO(m)$). For $\mathcal{C}\ell_m$-valued functions $f(x)$, there are two natural actions of the group $Pin(m)$, namely, the so-called L-action, given by

$$[L(r)(f)](x) = r\, f(r^{-1}x\, r), \; r \in Pin(m) \; \text{ and } \; x \in \mathbb{R}^m, \tag{2.6}$$

and the H-action, given by

$$[H(r)(f)](x) = r\; f(r^{-1}x\; r)\; r^{-1}, \; r \in Pin(m) \; \text{ and } \; x \in \mathbb{R}^m. \tag{2.7}$$

Now we recall basic notions from representation theory needed later on. Let G be a compact Lie group and let $\mathcal{V}$ be a real or complex finite-dimensional vector space. Then a representation of G is a pair $(\mathcal{V}, \tau)$ such that τ is a homomorphism from G into the group $\operatorname{Aut}(\mathcal{V})$ of invertible linear transformations on $\mathcal{V}$. In addition, we assume that the action τ of the group G on $\mathcal{V}$ is continuous. For $g \in G$ and $v \in \mathcal{V}$, we often write shortly gv instead of $\tau(g)v$ and we consider the representation $(\mathcal{V}, \tau)$ as a (left) G-module.

Let $\mathcal{V}$ be a G-module. A subspace $\mathcal{U}$ of $\mathcal{V}$ is called a submodule of $\mathcal{V}$ if it is G-invariant in the sense that $gu \in \mathcal{U}$ for each $g \in G$ and $u \in \mathcal{U}$. The module $\mathcal{V}$ is said to be irreducible if it contains no submodules than $\{0\}$ and $\mathcal{V}$. Moreover, let $\mathcal{V}'$ be a second G-module. Then a linear mapping $T : \mathcal{V} \to \mathcal{V}'$ is called equivariant if $T(gv) = g(Tv)$ for each $g \in G$ and $v \in \mathcal{V}$. The G-modules $\mathcal{V}$ and $\mathcal{V}'$ are said to be equivalent if there is an equivariant isomorphism of $\mathcal{V}$ onto $\mathcal{V}'$. The following simple but very useful result is known as Schur's lemma.

Lemma 1. *Let $\mathcal{V}$ and $\mathcal{V}'$ be irreducible finite-dimensional G-modules.*

(i) Then every equivariant mapping $T : \mathcal{V} \to \mathcal{V}'$ is either 0 or an isomorphism.

(ii) If $\mathcal{V}$ is a complex module and $T : \mathcal{V} \to \mathcal{V}$ is an equivariant mapping, then there is a complex number λ such that $Tv = \lambda v$, $v \in \mathcal{V}$.

Furthermore, it is well-known that, on a given G-module $\mathcal{V}$, there exists a G-invariant inner product $(\cdot,\cdot)_G$ in the sense that $(gu, gv)_G = (u, v)$ for each $g \in G$ and $u, v \in \mathcal{V}$. In addition, the module $\mathcal{V}$ can be always written as the direct sum of irreducible submodules which is orthogonal with respect to the inner product $(\cdot,\cdot)_G$.

It is known that all possible irreducible $Spin(m)$-representations can be realized by means of monogenic and harmonic polynomials of several vector variables, see [106]. In what follows, we deal with some explicit examples of spin modules of monogenic functions of one vector variable. Namely, we study G-modules $\mathcal{M}_k(\mathbb{R}^m, V)$ when the group G is $Spin(m)$, $Pin(m)$ or the unitary group $U(n)$ (realized as a subgroup of $Spin(m)$), the action of G is either the L-action (2.6) or the H-action (2.7) and V is a G-invariant subspace of $\mathcal{C}\ell_m$. Note that the inner products (2.4) and (2.5) are invariant on these G-modules.

2.1.1 Euclidean Clifford analysis

Euclidean Clifford analysis may be understood as the study of monogenic functions in the Euclidean space $\mathbb{R}^m$ under the L-action of the group $Spin(m)$. Let V be an arbitrary L-invariant subspace of $\mathcal{C}\ell_m$. Then the space V has an orthogonal decomposition into irreducible pieces $V = \mathbb{S}_1 \oplus \cdots \oplus \mathbb{S}_p$ with each $\mathbb{S}_j$ being equivalent to a basic spinor representation for $Spin(m)$. As a direct consequence, the module $\mathcal{M}_k(\mathbb{R}^m, V)$ has an orthogonal decomposition

$$\mathcal{M}_k(\mathbb{R}^m, V) = \mathcal{M}_k(\mathbb{R}^m, \mathbb{S}_1) \oplus \cdots \oplus \mathcal{M}_k(\mathbb{R}^m, \mathbb{S}_p).$$

Hence, in this case, it is sufficient to study only spinor valued monogenic functions. Let $\mathbb{S}$ be a basic spinor representation of the group $Spin(m)$. Then, under the L-action of $Spin(m)$, the spaces $\mathcal{M}_k(\mathbb{R}^m, \mathbb{S})$ form irreducible modules and are mutually inequivalent. As we know, to construct an orthogonal basis for the complex Hilbert space $L^2(\mathbb{B}_m, \mathbb{S}) \cap \operatorname{Ker} \partial$ it is sufficient to find orthogonal bases in the spaces $\mathcal{M}_k(\mathbb{R}^m, \mathbb{S})$, which is done in Section 3.3.

2.1.2 Generalized Moisil-Théodoresco systems

For $\mathcal{C}\ell_m$-valued monogenic functions, we now consider the H-action of $Pin(m)$. The Clifford algebra $\mathcal{C}\ell_m$ can be viewed naturally as the graded associative algebra

$$\mathcal{C}\ell_m = \bigoplus_{s=0}^{m} \mathcal{C}\ell_m^s.$$

Here $\mathcal{C}\ell_m^s$ stands for the space of s-vectors in $\mathcal{C}\ell_m$. Actually, under the H-action, the spaces $\mathcal{C}\ell_m^s$ are mutually inequivalent irreducible submodules of $\mathcal{C}\ell_m$. For a 1-vector u and an s-vector v, the Clifford product uv splits into the sum of an $(s-1)$-vector $u \bullet v$ and an $(s+1)$-vector $u \wedge v$. Indeed, we have that $uv = u \bullet v + u \wedge v$ with

$$u \bullet v = \frac{1}{2}(uv - (-1)^s vu) \quad \text{and} \quad u \wedge v = \frac{1}{2}(uv + (-1)^s vu).$$

By linearity, we extend the so-called inner product $u \bullet v$ and the outer product $u \wedge v$ for a 1-vector u and an arbitrary Clifford number $v \in \mathcal{C}\ell_m$. In particular, we can split the left multiplication by a 1-vector x into the outer multiplication $(x\,\wedge)$ and the inner multiplication $(x\,\bullet)$, that is,

$$x = (x\,\wedge) + (x\,\bullet).$$

Analogously, the Dirac operator ∂ can be split also into two parts $\partial = \partial^+ + \partial^-$ where

$$\partial^+ P = \sum_{j=1}^m e_j \wedge (\partial_{x_j} P) \quad \text{and} \quad \partial^- P = \sum_{j=1}^m e_j \bullet (\partial_{x_j} P).$$

Obviously, for s-vector valued polynomials P, the Dirac equation $\partial P = 0$ is equivalent to the system of equations

$$\partial^+ P = 0, \quad \partial^- P = 0$$

we call the Hodge-de Rham system. This terminology is legitimate because, after the translation into the language of differential forms explained in [14], the operators ∂^+ and ∂^- correspond to the standard de Rham differential d and its codifferential d^* for differential forms.

In contrast with the spinor case, it is interesting to study V-valued monogenic functions not only for irreducible subspaces V of the Clifford algebra $\mathcal{C}\ell_m$. Let V be an arbitrary H-invariant subspace of $\mathcal{C}\ell_m$, that is, for some subset S of $\{0, 1, \ldots, m\}$, we have that $V = \mathcal{C}\ell_m^S$ where

$$\mathcal{C}\ell_m^S = \bigoplus_{s \in S} \mathcal{C}\ell_m^s.$$

Then the so-called generalized Moisil-Théodoresco system introduced by R. Delanghe is defined as the Dirac equation $\partial f = 0$ for V-valued functions f. In particular, denote $\mathcal{H}_k^s(\mathbb{R}^m) = \mathcal{M}_k(\mathbb{R}^m, \mathcal{C}\ell_m^s)$. Then the spaces $\mathcal{H}_k^s(\mathbb{R}^m)$ are just formed by homogeneous solutions of the Hodge-de Rham system. It is well-known that, under the H-action, the spaces $\mathcal{H}_k^s(\mathbb{R}^m)$ form irreducible modules. Moreover, all non-trivial modules $\mathcal{H}_k^s(\mathbb{R}^m)$ are mutually inequivalent, see [44]. The following result shows that the spaces $\mathcal{H}_k^s(\mathbb{R}^m)$ play a role of building blocks in this case (see [L8, 45]).

Theorem 1. *Let $S \subset \{0, 1, \ldots, m\}$ and let $S' = \{s : s \pm 1 \in S\}$. Under the H-action of $Pin(m)$, the space $\mathcal{M}_k(\mathbb{R}^m, \mathcal{C}\ell_m^S)$ decomposes into inequivalent irreducible pieces as*

$$\mathcal{M}_k(\mathbb{R}^m, \mathcal{C}\ell_m^S) = \left(\bigoplus_{s \in S} \mathcal{H}_k^s(\mathbb{R}^m) \right) \oplus \left(\bigoplus_{s \in S'} ((x\,\wedge) + \beta_{k-1}^{s,m} (x\,\bullet))\, \mathcal{H}_{k-1}^s(\mathbb{R}^m) \right)$$

with $\beta_k^{s,m} = -(k+m-s)/(k+s)$.

In particular, this decomposition is orthogonal with respect to any invariant inner product, including the L^2-inner product and the Fischer inner product, see (2.4) *and* (2.5).

By Theorem 1, to construct an orthogonal basis for the (real or complex) Hilbert space $L^2(\mathbb{B}_m, \mathcal{C}\ell_m^S) \cap \operatorname{Ker} \partial$ it obviously suffices to find orthogonal bases in the spaces $\mathcal{H}_k^s(\mathbb{R}^m)$, see Theorem 14 of Section 3.4.

2.1.3 Hermitian Clifford analysis

For an acccount of Hermitian Clifford analysis, we refer to [96, 37, 12, 13, 24, 19]. The transition to Hermitian Clifford analysis consists in adding a complex structure J to the above Euclidean setting, that is, an $SO(m)$-element J for which $J^2 = -\mathbf{1}$. Note that a complex structure can exist only in the even dimensional case $m = 2n$. In what follows, the complex structure J is chosen to act upon the generators $e_1, \ldots, e_{2n}$ of $\mathbb{C}_{2n}$ as $J[e_j] = -e_{n+j}$ and $J[e_{n+j}] = e_j$ for $j = 1, \ldots, n$. Moreover, a vector variable in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ is now alternatively expressed by the real variables $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ or the complex variables $(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)$ with $z_j = x_j + iy_j$ and $\bar{z}_j = x_j - iy_j$. Then the isotropic Witt basis elements $(\mathfrak{f}_j, \mathfrak{f}_j^\dagger)_{j=1}^n$ for $\mathbb{C}_{2n}$ are defined by

$$\mathfrak{f}_j = \frac{1}{2}(e_j - i\, e_{n+j}) \quad \text{and} \quad \mathfrak{f}_j^\dagger = -\frac{1}{2}(e_j + i\, e_{n+j}) \quad \text{for } j = 1, \ldots, n.$$

The Hermitian Clifford variables $\underline{z}$ and $\underline{z}^\dagger$ are given by

$$\underline{z} = \sum_{j=1}^n \mathfrak{f}_j\, z_j \quad \text{and} \quad \underline{z}^\dagger = \sum_{j=1}^n \mathfrak{f}_j^\dagger\, \bar{z}_j.$$

Finally, we define the Hermitian Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^\dagger$ by

$$\partial_{\underline{z}}^\dagger = \sum_{j=1}^n \mathfrak{f}_j\, \partial_{\bar{z}_j} \quad \text{and} \quad \partial_{\underline{z}} = \sum_{j=1}^n \mathfrak{f}_j^\dagger\, \partial_{z_j}$$

with $\partial_{\bar{z}_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ and $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$. In particular, we have that the Dirac operator ∂ in $\mathbb{R}^{2n}$ splits into the Hermitian Dirac operators as

$$\partial = 2(\partial_{\underline{z}}^\dagger - \partial_{\underline{z}}).$$

As explained above, for studying monogenic functions in $\mathbb{R}^{2n}$ we can restrict ourselves to spinor valued ones. The spinor space $\mathbb{S}$ is realized within the Clifford algebra $\mathbb{C}_{2n}$ as

$$\mathbb{S} = \mathbb{C}_{2n} I \cong \mathbb{C}_n I$$

where I is a suitable primitive idempotent, say $I = I_1 \ldots I_n$ with $I_j = \mathfrak{f}_j \mathfrak{f}_j^\dagger$, $j = 1, \ldots, n$. As $\mathfrak{f}_j I = 0$, $j = 1, \ldots, n$, we also have that $\mathbb{S} \cong \bigwedge_n^\dagger I$ where $\bigwedge_n^\dagger$ stands for the complex Grassmann algebra generated by $\{\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger\}$. Hence spinor space $\mathbb{S}$ decomposes further into homogeneous parts as

$$\mathbb{S} = \bigoplus_{r=0}^n \mathbb{S}^r \tag{2.8}$$

with $\mathbb{S}^r = (\bigwedge_n^\dagger)^{(r)} I$. Here $(\bigwedge_n^\dagger)^{(r)}$ is the space of r-vectors of $\bigwedge_n^\dagger$.

In comparison with the Euclidean setting the symmetry in the Hermitian framework is given not by the L-action of the whole group $Spin(2n)$ but only its subgroup $Spin_J(2n)$. The subgroup $Spin_J(2n)$ is a double cover of the group $SO_J(2n)$, the subgroup of rotations in $\mathbb{R}^{2n}$ commuting with the complex structure J. Let us note that the group $SO_J(2n)$ can be seen as a realization of the unitary group $U(n)$. Moreover, under the action of the group $U(n)$, (2.8) is the decomposition of $\mathbb{S}$ into inequivalent irreducible pieces $\mathbb{S}^r$.

It is easy to see that, for $\mathbb{S}^r$-valued functions g, the Dirac equation $\partial g = 0$ is equivalent to the system of equations

$$\partial_{\underline{z}}\, g = 0 \quad \text{and} \quad \partial_{\underline{z}}^{\dagger}\, g = 0. \tag{2.9}$$

A continuously differentiable function g in an open region Ω of $\mathbb{R}^{2n}$ with values in the complex Clifford algebra $\mathbb{C}_{2n}$ is called (left) Hermitian monogenic in Ω if and only if it satisfies in Ω the system (2.9). As $\partial = 2(\partial_{\underline{z}}^{\dagger} - \partial_{\underline{z}})$ Hermitian monogenicity can be regarded as a refinement of monogenicity.

Let V be an arbitrary $U(n)$-invariant subspace of the spinor space $\mathbb{S}$, that is, for some subset R of $\{0, 1, \dots, n\}$, we have that $V = \mathbb{S}^R$ where

$$\mathbb{S}^R = \bigoplus_{r \in R} \mathbb{S}^r. \tag{2.10}$$

Moreover, denote by $\mathcal{M}^r_{a,b}(\mathbb{C}^n)$ the space of $\mathbb{S}^r$-valued (Hermitian) monogenic polynomials in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ which are (a, b)-homogeneous, that is, a-homogeneous in the variables $(z_1, \dots, z_n)$ and at the same time b-homogeneous in the variables $(\bar{z}_1, \dots, \bar{z}_n)$. It is well-known that, under the action of the group $U(n)$, the spaces $\mathcal{M}^r_{a,b}(\mathbb{C}^n)$ are mutually inequivalent irreducible modules. By the following theorem, the spaces $\mathcal{M}^r_{a,b}(\mathbb{C}^n)$ are actually basic building blocks in the hermitian case.

Theorem 2. *Let $R \subset \{0, 1, \dots, n\}$, let $R' = \{r : r \pm 1 \in R\}$ and let $\mathbb{S}^R$ be as in* (2.10)*. Then, under the action of the group $U(n)$, the space $\mathcal{M}_k(\mathbb{R}^{2n}, \mathbb{S}^R)$ has a multiplicity free irreducible decomposition*

$$\mathcal{M}_k(\mathbb{R}^{2n}, \mathbb{S}^R) = \left(\bigoplus_{r \in R} \bigoplus_{a=0}^{k} \mathcal{M}^r_{a,k-a} \right) \oplus \left(\bigoplus_{r \in R'} \bigoplus_{a=0}^{k-1} \left(\underline{z} + \gamma^{r,n}_{a,k-1-a} \underline{z}^{\dagger}\right) \mathcal{M}^r_{a,k-1-a} \right)$$

with $\mathcal{M}^r_{a,b} = \mathcal{M}^r_{a,b}(\mathbb{C}^n)$ and $\gamma^{r,n}_{a,b} = (a + n - r)/(b + r)$. In particular, this decomposition is orthogonal with respect to any invariant inner product, including the L^2-inner product and the Fischer inner product, see (2.4) *and* (2.5)*.*

Proof. See [26, 36] for a proof in the case when $R = \{0, 1, \dots, n\}$. Then a general case is obvious. $\square$

By Theorem 2, to construct an orthogonal basis for the complex Hilbert space $L^2(\mathbb{B}_{2n}, \mathbb{S}^R) \cap \operatorname{Ker} \partial$ it is obviously sufficient to find orthogonal bases in the spaces $\mathcal{M}^r_{a,b}(\mathbb{C}^n)$, which is done in [L10] (see Section 3.5).

2.2 Quaternionic analysis

Quaternionic analysis developed by R. Fueter in 1930's can be considered as a special case of Clifford analysis. For an account of quaternionic analysis, we refer to [104, 70, 27].

Denote by $\mathbb{H}$ the field of real quaternions. The field $\mathbb{H}$ can be viewed as the Euclidean space $\mathbb{R}^4$ endowed with a non-commutative multiplication. A quaternion q can be written in the form $q = x_0 + x_1 i + x_2 j + x_3 k$ where x_0, x_1, x_2, x_3 are real numbers and i, j, k are the imaginary units such that

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

Quaternionic analysis studies $\mathbb{H}$-valued functions f defined in $\mathbb{R}^4$ and satisfying the so-called Fueter equation

$$\bar{D}f = \partial_{x_0} f + i\,\partial_{x_1} f + j\,\partial_{x_2} f + k\,\partial_{x_3} f = 0,$$

which is an obvious generalization of the Cauchy-Riemann equations from complex analysis. These functions called by R. Fueter regular have analogous properties like monogenic functions studied in Clifford analysis. For example, the Fueter equation is conformally invariant (see [27, Theorem 4.5.1]). Recall that, for $m \geq 3$, the only conformal mappings in $\mathbb{R}^m$ are conformal Möbius transformations, that is, compositions of translations, dilatations and the inversion $x \to x/|x|^2$. This was first shown in $\mathbb{R}^3$ by Liouville in 1850. As is known from a series of papers written by L. V. Ahlfors in the 1980s (see e.g. [2]), Möbius transformations in $\mathbb{R}^m$ may be represented by using Clifford algebras.

Moreover, in $\mathbb{R}^4$, we can use quaternions to represent Möbius transformations similarly as in $\mathbb{R}^2$ complex numbers. Indeed, the conformal Möbius transformations in $\mathbb{R}^4$ can be identified with bijections of $\mathbb{H}_\infty$ (the one-point compactification of $\mathbb{H}$) of the form

$$q \to (aq + b)(cq + d)^{-1}$$

where $a, b, c, d \in \mathbb{H}$. Let us note that, in [L1], reversible maps are classified in the group of Möbius transformations in $\mathbb{R}^4$. Recall that the reversible elements of a group are those elements that are conjugate to their own inverse.

Chapter 3

Complete orthogonal Appell systems

In this chapter, we show that Gelfand-Tsetlin bases form complete orthogonal Appell systems for spherical harmonics (see Section 3.1), for Clifford algebra valued spherical monogenics (see Section 3.2), for spinor valued spherical monogenics (see Section 3.3), for s-vector valued spherical monogenics (see Section 3.4) and, finally, for Hermitian monogenics (see Section 3.5).

3.1 Spherical harmonics

Following [79], we construct a complete orthogonal Appell system for spherical harmonics. Let us recall a standard construction of orthogonal bases in this case. Denote by $\mathcal{H}_k(\mathbb{R}^m)$ the space of complex valued harmonic polynomials in $\mathbb{R}^m$ which are k-homogeneous. Let $(e_1, \ldots, e_m)$ be an orthonormal basis of the Euclidean space $\mathbb{R}^m$. Then the construction of an orthogonal basis for the space $\mathcal{H}_k(\mathbb{R}^m)$ is based on the following decomposition (see [64, p. 171])

$$\mathcal{H}_k(\mathbb{R}^m) = \bigoplus_{j=0}^{k} F_{m,j}^{(k-j)} \mathcal{H}_j(\mathbb{R}^{m-1}). \tag{3.1}$$

This decomposition is orthogonal with respect to the L^2-inner product, say, on the unit ball $\mathbb{B}_m$ in $\mathbb{R}^m$ and the embedding factors $F_{m,j}^{(k-j)}$ are defined as the polynomials

$$F_{m,j}^{(k-j)}(x) = \frac{(j+1)_{k-j}}{(m-2+2j)_{k-j}} \, |x|^{k-j} C_{k-j}^{m/2+j-1}(x_m/|x|), \ x \in \mathbb{R}^m. \tag{3.2}$$

Here $x = (x_1, \ldots, x_m)$, $|x| = \sqrt{x_1^2 + \cdots + x_m^2}$ and C_k^ν is the Gegenbauer polynomial given by

$$C_k^\nu(z) = \sum_{i=0}^{[k/2]} \frac{(-1)^i (\nu)_{k-i}}{i!(k-2i)!} (2z)^{k-2i} \quad \text{with} \quad (\nu)_k = \nu(\nu+1)\cdots(\nu+k-1). \tag{3.3}$$

The decomposition (3.1) shows that spherical harmonics in $\mathbb{R}^m$ can be expressed in terms of spherical harmonics in $\mathbb{R}^{m-1}$, that is, for each $P \in$

$\mathcal{H}_k(\mathbb{R}^m)$, we have that

$$P(x) = P_k(\underline{x}) + F^{(1)}_{m,k-1}(x)P_{k-1}(\underline{x}) + \cdots + F^{(k)}_{m,0}(x)P_0(\underline{x}),\ x = (\underline{x}, x_m) \in \mathbb{R}^m$$

for some polynomials $P_j \in \mathcal{H}_j(\mathbb{R}^{m-1})$. Of course, here $F^{(0)}_{m,k} = 1$ and $\underline{x} = (x_1, \ldots, x_{m-1})$.

Applying the decomposition (3.1), we easily construct an orthogonal basis of the space $\mathcal{H}_k(\mathbb{R}^m)$ by induction on the dimension m. Indeed, as the polynomials $(x_1 \mp ix_2)^k$ form an orthogonal basis of the space $\mathcal{H}_k(\mathbb{R}^2)$ an orthogonal basis of the space $\mathcal{H}_k(\mathbb{R}^m)$ is formed by the polynomials

$$h_{k,\mu}(x) = (x_1 \mp ix_2)^{k_2} \prod_{r=3}^{m} F^{(k_r - k_{r-1})}_{r,k_{r-1}} \tag{3.4}$$

where μ is an arbitrary sequence of integers $(k_{m-1}, \ldots, k_3, \pm k_2)$ such that $k = k_m \geq k_{m-1} \geq \cdots \geq k_3 \geq k_2 \geq 0$. Furthermore, we have taken the normalization of the embedding factors $F^{(k-j)}_{m,j}$ so that the basis elements $h_{k,\mu}$ possess the following Appell property.

Theorem 3. *Let $m \geq 3$ and let $h_{k,\mu}$ be the basis elements of the spaces $\mathcal{H}_k(\mathbb{R}^m)$ defined in* (3.4) *with $\mu = (k_{m-1}, \ldots, k_3, \pm k_2)$. Then we have that*

(i) $\partial_{x_m} h_{k,\mu} = 0$ *for* $k = k_{m-1}$*;*

(ii) $\partial_{x_m} h_{k,\mu} = k\ h_{k-1,\mu}$ *for* $k > k_{m-1}$*;*

(iii) $\partial_{\pm}^{k_2}\, \partial_{x_3}^{k_3-k_2} \cdots \partial_{x_m}^{k-k_{m-1}} h_{k,\mu} = k!$ *where* $\partial_{\pm} = (1/2)(\partial_{x_1} \pm i\partial_{x_2})$.

Proof. The statement (i) follows from the fact that $F^{(0)}_{m,j} = 1$. Using standard formulas for Gegenbauer polynomials (see [3]), it is easy to verify that, for $k > j$, $\partial_{x_m} F^{(k-j)}_{m,j} = k\ F^{(k-1-j)}_{m,j}$, which implies (ii). Finally, we get (iii) by applying (ii) several times and by the fact that $\partial_{\pm}\ (x_1 \mp ix_2)^k = k\ (x_1 \mp ix_2)^{k-1}$. $\square$

To summarize, we have constructed a complete orthogonal Appell system for the complex Hilbert space $L^2(\mathbb{B}_m, \mathbb{C}) \cap \operatorname{Ker} \Delta$ of L^2-integrable harmonic functions $g : \mathbb{B}_m \to \mathbb{C}$. Here $\mathbb{B}_m$ is the unit ball in $\mathbb{R}^m$. Indeed, we have the following result.

Theorem 4. *Let $m \geq 3$ and, for each $k \in \mathbb{N}_0$, denote by N^m_k the set of sequences $(k_{m-1}, \ldots, k_3, \pm k_2)$ of integers such that $k \geq k_{m-1} \geq \cdots \geq k_3 \geq k_2 \geq 0$.*

(a) Then an orthogonal basis of the space $L^2(\mathbb{B}_m, \mathbb{C}) \cap \operatorname{Ker} \Delta$ is formed by the polynomials $h_{k,\mu}$ for $k \in \mathbb{N}_0$ and $\mu \in N^m_k$. Here the basis elements $h_{k,\mu}$ are defined in (3.4)*.*

(b) Each function $g \in L^2(\mathbb{B}_m, \mathbb{C}) \cap \operatorname{Ker} \Delta$ has a unique orthogonal series expansion

$$g = \sum_{k=0}^{\infty} \sum_{\mu \in N^m_k} \mathbf{t}_{k,\mu}(g)\ h_{k,\mu} \tag{3.5}$$

for some complex coefficients $\mathbf{t}_{k,\mu}(g)$.

In addition, for $\mu = (k_{m-1}, \ldots, k_3, \pm k_2) \in N_k^m$, *we have that*

$$\mathbf{t}_{k,\mu}(g) = \frac{1}{k!}\, \partial_{\pm}^{k_2} \partial_{x_3}^{k_3-k_2} \cdots \partial_{x_m}^{k-k_{m-1}} g(x)|_{x=0} \tag{3.6}$$

with $\partial_{\pm} = (1/2)(\partial_{x_1} \pm i\partial_{x_2})$.

Proof. It is well-known that the orthogonal direct sum

$$\bigoplus_{k=0}^{\infty} \mathcal{H}_k(\mathbb{R}^m)$$

is dense in the space $L^2(\mathbb{B}_m, \mathbb{C}) \cap \operatorname{Ker} \Delta$, which gives (a). The formula (3.6) then follows directly from the Appell property of the basis elements, namely, from (iii) of Theorem 3. $\square$

For a function $g \in L^2(\mathbb{B}_m, \mathbb{C}) \cap \operatorname{Ker} \Delta$, we call the orthogonal series expansion (3.5) its generalized Taylor series.

From the point of view of representation theory, the space $\mathcal{H}_k(\mathbb{R}^m)$ forms an irreducible module under the action of the group $Spin(m)$, defined by

$$[h(s)(P)](x) = P(s^{-1}xs), \ s \in Spin(m) \ \text{ and } \ x \in \mathbb{R}^m,$$

when $m \geq 3$. Under the action of $Spin(2)$, the module $\mathcal{H}_k(\mathbb{R}^2)$ decomposes as $\mathcal{H}_k(\mathbb{R}^2) = \langle (x_1 + ix_2)^k \rangle \cup \langle (x_1 - ix_2)^k \rangle$. Here $\langle M \rangle$ stands for the linear span of a set M. See [64, Chapter 3] for details. We show that the constructed basis (3.4) is actually a Gelfand-Tsetlin basis of the $Spin(m)$-module $\mathcal{H}_k(\mathbb{R}^m)$.

3.1.1 Gelfand-Tsetlin bases for spin modules

Now we recall an abstract definition of a Gelfand-Tsetlin basis for any given irreducible finite dimensional $Spin(m)$-module V (see [63, 87]). We assume that the space V is endowed with an invariant inner product.

The first step of the construction of a Gelfand-Tsetlin basis consists in reducing the symmetry to the group $Spin(m-1)$, realized as the subgroup of $Spin(m)$ describing rotations fixing the last vector e_m. It turns out that, under the action of the group $Spin(m-1)$, the module V is reducible and decomposes into a multiplicity free direct sum of irreducible $Spin(m-1)$-submodules

$$V = \bigoplus_{\mu_{m-1}} V(\mu_{m-1}). \tag{3.7}$$

This irreducible decomposition is multiplicity free and so it is orthogonal. Let us remark that, in representation theory, the decomposition (3.7) is called the branching of the module V.

Of course, we can further reduce the symmetry to the group $Spin(m-2)$, the subgroup of $Spin(m)$ describing rotations fixing the last two vectors e_{m-1}, e_m. Then we can again decompose each piece $V(\mu_{m-1})$ of the decomposition (3.7) into irreducible $Spin(m-2)$-submodules $V(\mu_{m-1}, \mu_{m-2})$ and so on. Hence we end up with the decomposition of the given $Spin(m)$-module V into irreducible $Spin(2)$-modules $V(\mu)$. Moreover, any such module $V(\mu)$ is uniquely determined by the sequence of labels

$$\mu = (\mu_{m-1}, \ldots, \mu_2). \tag{3.8}$$

To summarize, the given module V is the direct sum of irreducible $Spin(2)$-modules

$$V = \bigoplus_{\mu} V(\mu). \tag{3.9}$$

Moreover, the decomposition (3.9) is obviously orthogonal. Now it is easy to obtain an orthogonal basis of the space V. Indeed, as each irreducible $Spin(2)$-module $V(\mu)$ is one-dimensional we easily construct a basis of the space V by taking a non-zero vector $e(\mu)$ from each piece $V(\mu)$. The obtained basis $E = \{e(\mu)\}_\mu$ is called a Gelfand-Tsetlin basis of the module V. By construction, the basis E is orthogonal with respect to any invariant inner product given on the module V. Moreover, each vector $e(\mu) \in E$ is uniquely determined by its index μ up to a scalar multiple. In other words, for the given orthonormal basis $(e_1, \ldots, e_m)$ of $\mathbb{R}^m$, the Gelfand-Tsetlin basis E is uniquely determined up to a normalization.

It is easily seen that, for the $Spin(m)$-module $\mathcal{H}_k(\mathbb{R}^m)$, the decomposition (3.1) is nothing else than its branching and, consequently, the basis (3.4) is obviously its Gelfand-Tsetlin basis, uniquely determined by the property (iii) of Theorem 3. Moreover, the Appell property described in Theorem 3 is not a coincidence but, by Schur's lemma (see Lemma 1), the consequence of the fact that ∂_{x_m} is an invariant operator under the action of the subgroup $Spin(m-1)$.

3.2 Clifford algebra valued spherical monogenics

Following [79], we construct a complete orthogonal Appell system for Clifford algebra valued spherical monogenics. Denote by $\mathcal{C}\ell_m$ either the Clifford algebra $\mathbb{R}_{0,m}$ or $\mathbb{C}_m$. For notation, see Section 2.1. First we construct an orthogonal basis for the space $\mathcal{M}_k(\mathbb{R}^m, \mathcal{C}\ell_m)$ of k-homogeneous monogenic polynomials $P : \mathbb{R}^m \to \mathcal{C}\ell_m$, endowed with a $\mathcal{C}\ell_m$-valued inner product (2.3). We want to proceed as in the harmonic case so we need to express spherical monogenics in $\mathbb{R}^m$ in terms of spherical monogenics in $\mathbb{R}^{m-1}$, which is done in the following theorem.

Theorem 5. *The space $\mathcal{M}_k(\mathbb{R}^m, \mathcal{C}\ell_m)$ has the orthogonal decomposition*

$$\mathcal{M}_k(\mathbb{R}^m, \mathcal{C}\ell_m) = \bigoplus_{j=0}^{k} X_{m,j}^{(k-j)} \mathcal{M}_j(\mathbb{R}^{m-1}, \mathcal{C}\ell_m). \tag{3.10}$$

Here the embedding factors $X_{m,j}^{(k-j)}$ are defined as the polynomials

$$X_{m,j}^{(k-j)}(x) = F_{m,j}^{(k-j)} + \frac{j+1}{m+2j-1}\, F_{m,j+1}^{(k-j-1)}\, \underline{x} e_m,\ x = (\underline{x}, x_m) \in \mathbb{R}^m \tag{3.11}$$

with $\underline{x} = x_1e_1 + \cdots + x_{m-1}e_{m-1}$, $F^{(k-j)}_{m,j}$ *defined in* (3.2) *and* $F^{(-1)}_{m,k+1} = 0$.

Proof. See [43, Theorem 2.2.3, p. 315] for a proof. Denote by $\mathcal{P}_k(\mathbb{R}^{m-1}, \mathcal{C}\ell_m)$ the space of k-homogeneous polynomials $P : \mathbb{R}^{m-1} \to \mathcal{C}\ell_m$. Then, in the proof, the decomposition (3.10) is obtained by applying the Cauchy-Kovalevskaya extension operator $CK = e^{x_m e_m \underline{\partial}}$ to the Fischer decomposition of the space $\mathcal{P}_k(\mathbb{R}^{m-1}, \mathcal{C}\ell_m)$, that is,

$$\mathcal{P}_k(\mathbb{R}^{m-1}, \mathcal{C}\ell_m) = \bigoplus_{j=0}^{k} (\underline{x}e_m)^{k-j} \mathcal{M}_j(\mathbb{R}^{m-1}, \mathcal{C}\ell_m). \tag{3.12}$$

Here $\underline{\partial} = e_1\partial_{x_1} + \cdots + e_{m-1}\partial_{x_{m-1}}$. Indeed, it holds that

$$CK(\mathcal{P}_k(\mathbb{R}^{m-1}, \mathcal{C}\ell_m)) = \mathcal{M}_k(\mathbb{R}^m, \mathcal{C}\ell_m)$$

and, for each $P \in \mathcal{M}_j(\mathbb{R}^{m-1}, \mathcal{C}\ell_m)$, we have that

$$CK((\underline{x}e_m)^{k-j} P(\underline{x})) = \mu^{(k-j)}_{m,j} X^{(k-j)}_{m,j}(x) P(\underline{x})$$

where the non-zero constants $\mu^{(k-j)}_{m,j}$ are defined as $\mu^{(2l)}_{m,j} = (-1)^l (C^{m/2+j-1}_{2l}(0))^{-1}$ and $\mu^{(2l+1)}_{m,j} = (-1)^l \frac{m+2j+2l-1}{m+2j-2} (C^{m/2+j}_{2l}(0))^{-1}$ (see [L5, Lemma 1]). We want to have a decomposition analogous to (3.10) also for spinor valued polynomials (see (3.17) below) and therefore we have used the Fischer decomposition (3.12) given in terms of powers of $\underline{x}e_m$ and not $\underline{x}$ as usual. Moreover, we have chosen a different normalization of the embedding factors $X^{(k-j)}_{m,j}$ than in [43, L5], namely, we have omitted the constants $\mu^{(k-j)}_{m,j}$. $\square$

Using the decomposition (3.10), we easily construct an orthogonal basis of the space $\mathcal{M}_k(\mathbb{R}^m, \mathcal{C}\ell_m)$ by induction on the dimension m as explained in [43, pp. 262-264]. Indeed, as the polynomial $(x_1 - e_{12}x_2)^{k_2}$ forms a basis of $\mathcal{M}_{k_2}(\mathbb{R}^2, \mathcal{C}\ell_2)$ an orthogonal basis of the space $\mathcal{M}_k(\mathbb{R}^m, \mathcal{C}\ell_m)$ is formed by the polynomials

$$f_{k,\mu} = X^{(k-k_{m-1})}_{m,k_{m-1}} X^{(k_{m-1}-k_{m-2})}_{m-1,k_{m-2}} \cdots X^{(k_3-k_2)}_{3,k_2} (x_1 - e_{12}x_2)^{k_2} \tag{3.13}$$

where μ is an arbitrary sequence of integers $(k_{m-1}, \ldots, k_2)$ such that $k = k_m \geq k_{m-1} \geq \cdots \geq k_3 \geq k_2 \geq 0$. Here $e_{12} = e_1e_2$. Due to non-commutativity of the Clifford multiplication the order of factors in the product (3.13) is important. It is easy to see that the basis elements $f_{k,\mu}$ possess again the Appell property.

Theorem 6. *Let* $m \geq 3$ *and let* $f_{k,\mu}$ *be the basis elements of the spaces* $\mathcal{M}_k(\mathbb{R}^m, \mathcal{C}\ell_m)$ *defined in* (3.13) *with* $\mu = (k_{m-1}, \ldots, k_2)$. *Then we have that*

(i) $\partial_{x_m} f_{k,\mu} = 0$ *for* $k = k_{m-1}$;

(ii) $\partial_{x_m} f_{k,\mu} = k\ f_{k-1,\mu}$ *for* $k > k_{m-1}$;

(iii) $\partial^{k_2}_{12}\, \partial^{k_3-k_2}_{x_3} \cdots \partial^{k-k_{m-1}}_{x_m} f_{k,\mu} = k!$ *where* $\partial_{12} = (1/2)(\partial_{x_1} + e_{12}\partial_{x_2})$.

Proof. It is obvious from the fact that, for $k > j$, $\partial_{x_m} X^{(k-j)}_{m,j} = k\ X^{(k-j-1)}_{m,j}$ and $X^{(0)}_{m,j} = 1$. $\square$

Actually, we have constructed a complete orthogonal Appell system for the right $\mathcal{C}\ell_m$-linear Hilbert space $L^2(\mathbb{B}_m, \mathcal{C}\ell_m) \cap \operatorname{Ker} \partial$ of L^2-integrable monogenic functions $g : \mathbb{B}_m \to \mathcal{C}\ell_m$. Indeed, we have obtained the following result.

Theorem 7. *Let $m \geq 3$ and, for each $k \in \mathbb{N}_0$, denote by J_k^m the set of sequences $(k_{m-1}, k_{m-2}, \ldots, k_2)$ of integers such that $k \geq k_{m-1} \geq \cdots \geq k_3 \geq k_2 \geq 0$.*

(a) Then an orthogonal basis of the space $L^2(\mathbb{B}_m, \mathcal{C}\ell_m) \cap \operatorname{Ker} \partial$ is formed by the polynomials $f_{k,\mu}$ for $k \in \mathbb{N}_0$ and $\mu \in J_k^m$. Here the basis elements $f_{k,\mu}$ are defined in (3.13)*.*

(b) Each function $g \in L^2(\mathbb{B}_m, \mathcal{C}\ell_m) \cap \operatorname{Ker} \partial$ has a unique orthogonal series expansion

$$g = \sum_{k=0}^{\infty} \sum_{\mu \in J_k^m} f_{k,\mu}\, \mathbf{t}_{k,\mu}(g) \tag{3.14}$$

for some coefficients $\mathbf{t}_{k,\mu}(g)$ of $\mathcal{C}\ell_m$.

In addition, for $\mu = (k_{m-1}, \ldots, k_2) \in J_k^m$, we have that

$$\mathbf{t}_{k,\mu}(g) = \frac{1}{k!}\, \partial_{12}^{k_2} \partial_{x_3}^{k_3-k_2} \cdots \partial_{x_m}^{k-k_{m-1}} g(x)|_{x=0}$$

with $\partial_{12} = (1/2)(\partial_{x_1} + e_{12}\partial_{x_2})$.

For a function $g \in L^2(\mathbb{B}_m, \mathcal{C}\ell_m) \cap \operatorname{Ker} \partial$, we call the orthogonal series expansion (3.14) its generalized Taylor series.

In the next section, we show that the studied bases can be interpreted as Gelfand-Tsetlin bases at least for spinor valued spherical monogenics.

3.3 Spinor valued spherical monogenics

Following [79], we adapt the results obtained in the previous section for spinor valued spherical monogenics. As is well known, the Lie algebra $\mathfrak{spin}(m)$ of the group $Spin(m)$ can be realized as the space of bivectors, that is,

$$\mathfrak{spin}(m) = \langle e_{12}, e_{13}, \ldots, e_{m-1,m} \rangle$$

with $e_{ij} = e_i e_j$. Let $\mathbb{S}$ be a basic spinor representation of the group $Spin(m)$ and let $\mathcal{M}_k(\mathbb{R}^m, \mathbb{S})$ be the space of k-homogeneous monogenic polynomials $P : \mathbb{R}^m \to \mathbb{S}$. Then it is well-known that the space $\mathcal{M}_k(\mathbb{R}^m, \mathbb{S})$ forms an irreducible module under the L-action of the group $Spin(m)$. Now we recall an explicit realization of the space $\mathbb{S}$. For $j = 1, \ldots, n$, put

$$w_j = \frac{1}{2}(e_{2j-1} + ie_{2j}), \quad \overline{w}_j = \frac{1}{2}(-e_{2j-1} + ie_{2j}) \quad \text{and} \quad I_j = \overline{w}_j w_j.$$

Then $I_1, \ldots, I_n$ are mutually commuting idempotent elements in $\mathbb{C}_{2n}$. Moreover, $I = I_1 I_2 \cdots I_n$ is a primitive idempotent in $\mathbb{C}_{2n}$ and $\mathbb{S}_{2n} = \mathbb{C}_{2n} I$ is a minimal left ideal in $\mathbb{C}_{2n}$. Putting $W = \langle w_1, \ldots, w_n \rangle$, we have that

$$\mathbb{S}_{2n} = \Lambda(W)I, \quad \mathbb{S}_{2n}^+ = \Lambda^+(W)I \quad \text{and} \quad \mathbb{S}_{2n}^- = \Lambda^-(W)I \tag{3.15}$$

where $\Lambda(W)$ is the exterior algebra over W with the even part $\Lambda^+(W)$ and the odd part $\Lambda^-(W)$. Putting $\theta_{2n} = (-i)^n e_1 e_2 \cdots e_{2n}$, we have that

$$\mathbb{S}^{\pm}_{2n} = \{u \in \mathbb{S}_{2n} : \theta_{2n} u = \pm u\}. \tag{3.16}$$

Let us recall that $\mathbb{S}^{\pm}_{2n}$ are just two inequivalent basic spinor representations of the group $Spin(2n)$. On the other hand, there exists only a unique basic spinor representation $\mathbb{S}$ of the group $Spin(2n-1)$ and, as $Spin(2n-1)$-modules, the modules $\mathbb{S}^{\pm}_{2n}$ are both equivalent to $\mathbb{S}$. See [43, pp. 114-118] for details.

In what follows, we explicitly construct a Gelfand-Tsetlin basis for the space $\mathcal{M}_k(\mathbb{R}^m, \mathbb{S})$. First we recall the branching for spherical monogenics described in [L5]. When you adapt the decomposition (3.10) for spinor valued polynomials you get obviously

$$\mathcal{M}_k(\mathbb{R}^m, \mathbb{S}) = \bigoplus_{j=0}^{k} X^{(k-j)}_{m,j} \mathcal{M}_j(\mathbb{R}^{m-1}, \mathbb{S}). \tag{3.17}$$

Indeed, it is easy to see that by multiplying $\mathbb{S}$-valued polynomials in $\mathbb{R}^{m-1}$ with the embedding factors $X^{(k-j)}_{m,j}$ from the left you get $\mathbb{S}$-valued polynomials in $\mathbb{R}^m$. In the even dimensional case $m = 2n$, the decomposition (3.17) describes the branching of the module $\mathcal{M}_k(\mathbb{R}^m, \mathbb{S})$, that is, its decomposition into $Spin(m-1)$-irreducible submodules. In the odd dimensional case $m = 2n-1$, under the action of $Spin(2n-2)$, the module $\mathbb{S}$ splits into two inequivalent submodules $\mathbb{S}^{\pm} \simeq \mathbb{S}^{\pm}_{2n-2}$ and so each module $\mathcal{M}_j(\mathbb{R}^{2n-2}, \mathbb{S})$ in (3.17) decomposes further as

$$\mathcal{M}_j(\mathbb{R}^{2n-2}, \mathbb{S}) = \mathcal{M}_j(\mathbb{R}^{2n-2}, \mathbb{S}^+) \oplus \mathcal{M}_j(\mathbb{R}^{2n-2}, \mathbb{S}^-).$$

See [L5, Theorems 1 and 2] for details.

Using the decomposition (3.17), it is easy to construct Gelfand-Tsetlin bases for the module $\mathcal{M}_k(\mathbb{R}^m, \mathbb{S})$ by induction on the dimension m. Let $m = 2n$ or $m = 2n-1$. To do this we need to describe a Gelfand-Tsetlin basis of the space $\mathbb{S}$ itself. The space $\mathbb{S}$ is a basic spinor representation for $Spin(m)$. As $Spin(2n-2)$-module, the space $\mathbb{S}$ has the irreducible decomposition $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$. By reducing the symmetry to $Spin(2n-4)$, the pieces $\mathbb{S}^{\pm}$ themselves further decompose and so on. Indeed, for $j = 0, \dots, n-1$, denote by $\mathcal{S}_j$ the set of sequences of the length j consisting of the signs $\pm$. For each $\nu \in \mathcal{S}_j$, define (by induction on j) the subset $\mathbb{S}^{\nu}$ of the set $\mathbb{S}$ such that $\mathbb{S}^{\emptyset} = \mathbb{S}$ and, for $\nu = (\underline{\nu}, \pm)$, we have that $\mathbb{S}^{\nu} = (\mathbb{S}^{\underline{\nu}})^{\pm}$. Put $\mathcal{S}^m = \mathcal{S}_{n-1}$. Then we get the following decomposition of the space $\mathbb{S}$ into irreducible $Spin(2)$-submodules

$$\mathbb{S} = \bigoplus_{\nu \in \mathcal{S}^m} \mathbb{S}^{\nu} \quad \text{with} \quad \mathbb{S}^{\nu} = \langle v^{\nu} \rangle \tag{3.18}$$

where, in each 1-dimensional piece $\mathbb{S}^{\nu}$, we have chosen an arbitrary non-zero element v^{ν}. The last ingredient is to describe Gelfand-Tsetlin bases for spherical monogenics in dimension 2. Obviously, for a given $\nu \in \mathcal{S}^m$ and $k \in \mathbb{N}_0$, the polynomial $(x_1 - e_{12}x_2)^k v^{\nu}$ forms a Gelfand-Tsetlin basis of $\mathcal{M}_k(\mathbb{R}^2, \mathbb{S}^{\nu})$. To summarize, we have proved the following result.

Theorem 8. *Let $m \geq 3$ and let $\mathbb{S}$ be a basic spinor representation of $Spin(m)$.*

(i) Then a Gelfand-Tsetlin basis of the $Spin(m)$-module $\mathcal{M}_k(\mathbb{R}^m, \mathbb{S})$ is formed by the polynomials

$$f^\nu_{k,\mu} = f_{k,\mu}\, v^\nu \tag{3.19}$$

where $\nu \in \mathcal{S}^m$ and $\mu \in J^m_k$. Here $f_{k,\mu}$ are as in Theorem 7 and $\mathcal{S}^m$ and v^ν as in (3.18).

In addition, the basis (3.19) *is orthogonal with respect to any invariant inner product on the module $\mathcal{M}_k(\mathbb{R}^m, \mathbb{S})$, including the L^2-inner product and the Fischer inner product.*

(ii) The Gelfand-Tsetlin basis (3.19) *is uniquely determined by the property that, for each $\nu = (\underline{\nu}, \pm) \in \mathcal{S}^m$ and $\mu = (k_{m-1}, k_{m-2}, \dots, k_2) \in J^m_k$,*

$$\partial^{k_2}_{\pm}\, \partial^{k_3-k_2}_{x_3} \cdots \partial^{k-k_{m-1}}_{x_m} f^\nu_{k,\mu} = k!\; v^\nu. \tag{3.20}$$

Here $\partial_\pm = (1/2)(\partial_{x_1} \pm i\partial_{x_2})$.

Proof. The statement (i) is obvious. The Appell property (3.20) follows directly from the statement (iii) of Theorem 6 and the fact that, for $\nu = (\underline{\nu}, \pm)$, we have that $e_{12}v^\nu = \pm i v^\nu$ and hence $(x_1 - e_{12}x_2)^k v^\nu = (x_1 \mp i x_2)^k v^\nu$. □

Remark 1. In (3.15) above, we realize the space $\mathbb{S} = \mathbb{S}^\pm_{2n}$ inside the Clifford algebra $\mathbb{C}_{2n}$ as

$$\mathbb{S} = \Lambda^s(w_1, \dots, w_n)I \quad \text{with } s = \pm.$$

It is not difficult to find generators of 1-dimensional pieces $\mathbb{S}^\nu$ of $\mathbb{S}$. Indeed, we have that

$$\mathbb{S}^\pm = \Lambda^\pm(w_1, \dots, w_{n-1})I^\pm$$

where, for $s = +$, we put $I^+ = I$ and $I^- = w_n I$ and, for $s = -$, obviously $I^+ = w_n I$ and $I^- = I$. Hence, by induction on j, we deduce easily that, for $s, t \in \{\pm\}$ and $\nu = (\underline{\nu}, s, t) \in \mathcal{S}_j$, we have that

$$\mathbb{S}^\nu = \Lambda^t(w_1, \dots, w_{n-j})I^\nu$$

where we put $I^{(\underline{\nu},+,+)} = I^{(\underline{\nu},+)}$, $I^{(\underline{\nu},+,-)} = w_{n-j+1}I^{(\underline{\nu},+)}$, $I^{(\underline{\nu},-,+)} = w_{n-j+1}I^{(\underline{\nu},-)}$ and $I^{(\underline{\nu},-,-)} = I^{(\underline{\nu},-)}$. In particular, we have that $\mathbb{S}^\nu \simeq \mathbb{S}^t_{2(n-j)}$. Finally, for each $\nu \in \mathcal{S}^m = \mathcal{S}_{n-1}$, the 1-dimensional piece $\mathbb{S}^\nu$ is generated by the element

$$v^\nu = \begin{cases} I^\nu, & \nu = (\underline{\nu}, +); \\ w_1 I^\nu, & \nu = (\underline{\nu}, -). \end{cases} \tag{3.21}$$

It is easy to see that
for $\mathbb{S} = \mathbb{S}^+_4$, we have that $v^+ = I$ and $v^- = w_1w_2I$;
for $\mathbb{S} = \mathbb{S}^-_4$, we have that $v^+ = w_2I$ and $v^- = w_1I$;
for $\mathbb{S} = \mathbb{S}^+_6$, $v^{++} = I$, $v^{+-} = w_1w_2I$, $v^{-+} = w_2w_3I$, $v^{--} = w_1w_3I$;
for $\mathbb{S} = \mathbb{S}^-_6$, $v^{++} = w_3I$, $v^{+-} = w_1w_2w_3I$, $v^{-+} = w_2I$ and $v^{--} = w_1I$.

In fact, we have constructed a complete orthogonal Appell system for the complex Hilbert space $L^2(\mathbb{B}_m,\mathbb{S})\cap\operatorname{Ker}\partial$ of L^2-integrable monogenic functions $g:\mathbb{B}_m\to\mathbb{S}$. Indeed, using Theorem 8, we easily obtain the following result.

Theorem 9. *Let $m\geq 3$ and let $\mathbb{S}$ be a basic spinor representation for $Spin(m)$.*

(a) Then an orthogonal basis of the space $L^2(\mathbb{B}_m,\mathbb{S})\cap\operatorname{Ker}\partial$ is formed by the polynomials $f^\nu_{k,\mu}$ for $k\in\mathbb{N}_0$, $\mu\in J^m_k$ and $\nu\in\mathcal{S}^m$. Here the basis elements $f^\nu_{k,\mu}$ are defined in Theorem 8.

(b) Each function $g\in L^2(\mathbb{B}_m,\mathbb{S})\cap\operatorname{Ker}\partial$ has a unique orthogonal series expansion

$$g=\sum_{k=0}^{\infty}\sum_{\nu\in\mathcal{S}^m}\sum_{\mu\in J^m_k}\mathbf{t}^\nu_{k,\mu}(g)\ f^\nu_{k,\mu}\tag{3.22}$$

for some complex coefficients $\mathbf{t}^\nu_{k,\mu}(g)$.
In addition, let $g=\sum_{\nu\in\mathcal{S}^m}g^\nu v^\nu$ for some complex functions g^ν on $\mathbb{B}_m$. Then, for $\mu=(k_{m-1},\dots,k_2)\in J^m_k$ and $\nu=(\underline{\nu},\pm)\in\mathcal{S}^m$, we have that

$$\mathbf{t}^\nu_{k,\mu}(g)=\frac{1}{k!}\ \partial^{k_2}_{\pm}\ \partial^{k_3-k_2}_{x_3}\cdots\partial^{k-k_{m-1}}_{x_m}g^\nu(x)|_{x=0}$$

with $\partial_\pm=(1/2)(\partial_{x_1}\pm i\partial_{x_2})$.

For a function $g\in L^2(\mathbb{B}_m,\mathbb{S})\cap\operatorname{Ker}\partial$, we call the orthogonal series expansion (3.22) its generalized Taylor series.

Remark 2. Of course, there is a close connection between the generalized Taylor series expansions from Theorem 7 and Theorem 9. Indeed, we can always realize the spinor space $\mathbb{S}$ inside the Clifford algebra $\mathbb{C}_m$ and then, for each $g\in L^2(\mathbb{B}_m,\mathbb{S})\cap\operatorname{Ker}\partial$, we have that

$$\mathbf{t}_{k,\mu}(g)=\sum_{\nu\in\mathcal{S}^m}\mathbf{t}^\nu_{k,\mu}(g)\ v^\nu.$$

3.3.1 The generalized Appell property in dimension 3

According to [L5], we recall briefly a construction of Gelfand-Tsetlin bases for $Spin(3)$-modules $\mathcal{M}_k(\mathbb{R}^3,\mathbb{S})$ using the Cauchy-Kovalevskaya method and the fact that the basis elements in this case possess the Appell property even with respect to all variables. As a $Spin(2)$-module, the space $\mathbb{S}$ is reducible and decomposes into two parts $\mathbb{S}=\mathbb{S}^+\oplus\mathbb{S}^-$. Let $v^\pm$ be generators of $\mathbb{S}^\pm$, that is, $\mathbb{S}^\pm=\langle v^\pm\rangle$. Put $z=x_1+ix_2$ and $\overline{z}=x_1-ix_2$. As is well-known, the $Spin(2)$-modules $\mathcal{P}_k(\mathbb{R}^2,\mathbb{S}^\pm)$ decompose into inequivalent irreducible $Spin(2)$-submodules as

$$\mathcal{P}_k(\mathbb{R}^2,\mathbb{S}^\pm)=\bigoplus_{j=0}^{k}\langle z^j\overline{z}^{k-j}v^\pm\rangle.\tag{3.23}$$

In particular, we have that $\mathcal{M}_k(\mathbb{R}^2,\mathbb{S}^+)=\langle\overline{z}^k v^+\rangle$ and $\mathcal{M}_k(\mathbb{R}^2,\mathbb{S}^-)=\langle z^k v^-\rangle$. Applying the Cauchy-Kovalevskaya extension operator to the Fischer decomposition (3.23) we get the following result (see [L5]).

Proposition 1. *Denote* $\underline{\partial} = e_1\partial_{x_1} + e_2\partial_{x_2}$. *Then, for each* $k \in \mathbb{N}_0$, *the polynomials*

$$f_{2j}^k = e^{x_3e_3\underline{\partial}}\,(\frac{z^j\overline{z}^{k-j}}{j!(k-j)!}\,v^+) \text{ and } f_{2j+1}^k = e^{x_3e_3\underline{\partial}}\,(\frac{z^j\overline{z}^{k-j}}{j!(k-j)!}\,v^-),\ j = 0,\dots,k$$

form a Gelfand-Tsetlin basis of the irreducible $Spin(3)$*-module* $\mathcal{M}_k(\mathbb{R}^3, \mathbb{S})$.

For the sake of explicitness, we limit ourselves to the case when $\mathbb{S} = \mathbb{S}_4^+$ or $\mathbb{S} = \mathbb{S}_4^-$. In the former case, we put $v^+ = I$ and $v^- = w_1w_2I$. In the latter case, we put $v^+ = w_2I$ and $v^- = w_1I$. See Remark 1 of Section 3.3. In [L5], the basis elements f_j^k are expressed explicitly using hypergeometric series ${}_2F_1$. Moreover, in [L6], explicit formulae for the basis elements in spherical coordinates are given in terms of the Legendre polynomials. In Theorem 8 above, we expressed elements of Gelfand-Tsetlin bases for spherical monogenics in terms of Gegenbauer polynomials not only in dimension 3 but even in any dimension. In Proposition 2 below, we collect basic properties of Gelfand-Tsetlin bases in dimension 3.

Proposition 2. *Let* $\{f_0^k, \dots, f_{2k+1}^k\}$ *be the Gelfand-Tsetlin bases of* $\mathcal{M}_k(\mathbb{R}^3, \mathbb{S}_4^{\pm})$ *defined in Proposition 1. Then the following statements hold.*

(a) For each $k \in \mathbb{N}$ *and* $j = 0, \dots, 2k+1$, *we have that*

$$\partial_z f_j^k = f_{j-2}^{k-1}, \quad \partial_{\overline{z}} f_j^k = f_j^{k-1} \quad \text{and} \quad \partial_{x_3} f_j^k = \mp(-1)^j 2\, f_{j-1}^{k-1}.$$

Here $f_j^{k-1} = 0$ *unless* $j = 0, \dots, 2k+1$.

(b) For $k \in \mathbb{N}_0$ *and* $j = 0, \dots, 2k+1$, *there are non-zero constants* d_j^k *such that the polynomials* $\hat{f}_j^k = d_j^k f_j^k$ *satisfy*

(b1) $\hat{f}_0^k = \overline{z}^k v^+$ *and* $\hat{f}_{2k+1}^k = z^k v^-$

(b2) For $j = 1, \dots, 2k$, *we have that* $\partial_{x_3}\hat{f}_j^k = k\, \hat{f}_{j-1}^{k-1}$.

(c) Moreover, $\{\hat{f}_0^k, \dots, \hat{f}_{2k+1}^k\}$ *are the Gelfand-Tsetlin bases of the modules* $\mathcal{M}_k(\mathbb{R}^3, \mathbb{S})$, *uniquely determined by the properties* (b1) *and* (b2).

For a proof of Proposition 2, see [L5]. Let us remark only that the statement (a) of Proposition 2 follows easily from the formula

$$\partial_{x_3}(e^{x_3e_3\underline{\partial}} f) = e^{x_3e_3\underline{\partial}}(e_3\underline{\partial} f)$$

and from the fact that the derivatives ∂_z and $\partial_{\overline{z}}$ both commute with the operator $e^{x_3e_3\underline{\partial}}$.

In Figure 3.1, structural properties of the Gelfand-Tsetlin bases in this case are shown. In the k-th column of Figure 3.1, the decomposition of the $Spin(3)$-module

$$\mathcal{M}_k = \mathcal{M}_k(\mathbb{R}^3, \mathbb{S})$$

into irreducible $Spin(2)$-submodules can be found. By Proposition 2, we know that the application of the derivative ∂_{x_3} to basis elements causes the shift in a given row to the left, the derivative $\partial_{\overline{z}}$ moves them diagonally

$$
\begin{tikzcd}
\mathcal{M}_0 & \mathcal{M}_1 & \mathcal{M}_2 & \cdots \\
 & & \langle \hat{f}_0^2 \rangle \arrow[dl, "\partial_{\overline{z}}"'] \arrow[d, leftrightarrow] & \\
 & \langle \hat{f}_0^1 \rangle \arrow[dl] \arrow[d, leftrightarrow] & \langle \hat{f}_1^2 \rangle \arrow[l] \arrow[dl] \arrow[d, leftrightarrow] & \cdots \\
\langle \hat{f}_0^0 \rangle \arrow[d, leftrightarrow] & \langle \hat{f}_1^1 \rangle \arrow[l, "\partial_{x_3}"'] \arrow[dl] \arrow[d, leftrightarrow] & \langle \hat{f}_2^2 \rangle \arrow[l] \arrow[dl] \arrow[ul] \arrow[d, leftrightarrow] & \\
\langle \hat{f}_1^0 \rangle & \langle \hat{f}_2^1 \rangle \arrow[l, "\partial_{x_3}"'] \arrow[ul] \arrow[d, leftrightarrow] & \langle \hat{f}_3^2 \rangle \arrow[l] \arrow[dl] \arrow[ul] \arrow[d, leftrightarrow] & \cdots \\
 & \langle \hat{f}_3^1 \rangle \arrow[ul] & \langle \hat{f}_4^2 \rangle \arrow[l] \arrow[ul] \arrow[d, leftrightarrow] & \\
 & & \langle \hat{f}_5^2 \rangle \arrow[ul, "\partial_z"] & \cdots
\end{tikzcd}
\qquad
\begin{array}{c}
\uparrow \tilde{X}^+ \\[2em]
\downarrow \tilde{X}^-
\end{array}
$$

Figure 3.1: The decomposition of the modules $\mathcal{M}_k = \mathcal{M}_k(\mathbb{R}^3, \mathbb{S})$.

downward and ∂_z diagonally upward. In other words, the Gelfand-Tsetlin bases in this case possess the Appell property not only with respect to the last real variable x_3 but also with respect to the complex variables z and $\overline{z}$. Furthermore, in [L6], it is shown that the spaces $\mathcal{M}_k(\mathbb{R}^3, \mathbb{S})$ can be also considered as irreducible modules over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ generated by the operators $\tilde{H} = -i(x_2\partial_{x_1} - x_1\partial_{x_2} + e_{12}/2)$,

$$\tilde{X}^+ = -2x_3\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial x_3} + \frac{1}{2}(e_{31} + ie_{23}) \quad \text{and} \quad \tilde{X}^- = 2x_3\frac{\partial}{\partial \overline{z}} - z\frac{\partial}{\partial x_3} - \frac{1}{2}(e_{31} - ie_{23}).$$

In particular, the operators $\tilde{X}^+$ and $\tilde{X}^-$ move basis elements in a given column upward and downward, respectively. Moreover, each basis element $\hat{f}_j^k$ is an eigenvector of the operator $\tilde{H}$ with the eigenvalue $k - j + 1/2$ and all eigenvectors $\hat{f}_j^k$ with the same eigenvalue are collected in the same row.

3.4 Hodge-de Rham systems

In this section, we construct Gelfand-Tsetlin bases for the spaces $\mathcal{H}_k^s(\mathbb{R}^m)$ of k-homogeneous monogenic polynomials $P : \mathbb{R}^m \to \mathcal{C}\ell_m^s$. Here $\mathcal{C}\ell_m^s$ stands for the space of s-vectors in $\mathcal{C}\ell_m$. For notation, see Section 2.1.2.

Remark 3. Obviously, we have that $\mathcal{H}_k^s(\mathbb{R}^m) = \{0\}$ for $s \in \{0, m\}$ and $k \geq 1$. In the case when $\mathcal{C}\ell_m = \mathbb{R}_{0,m}$ (resp. $\mathbb{C}_m$), we have that $\mathcal{H}_0^0(\mathbb{R}^m) = \mathbb{R}$ (resp. $\mathbb{C}$) and $\mathcal{H}_0^m(\mathbb{R}^m) = \mathbb{R}\, e_M^*$ (resp. $\mathbb{C}\, e_M^*$) with $e_M^* = e_m e_{m-1} \cdots e_1$.

As we know, the space $\mathcal{H}_k^s(\mathbb{R}^m)$ forms an irreducible module under the H-action of the Pin group $Pin(m)$. Moreover, all non-trivial modules $\mathcal{H}_k^s(\mathbb{R}^m)$

are mutually inequivalent, see [44]. The key step for constructing the Gelfand-Tsetlin bases is to understand the branching of the module $\mathcal{H}^s_k(\mathbb{R}^m)$.

Theorem 10. *Let $m \geq 3$, $s = 0, \dots, m$ and $k \in \mathbb{N}_0$. Denote by $N^{s,m}_k$ the set of pairs $(t,j) \in \{0, \dots, m-1\} \times \{0, \dots, k\}$ such that $t \in \{s-1, s\}$ and, if $t \in \{0, m-1\}$ then $j = 0$. Then, under the H-action of $Pin(m-1)$, we have the following multiplicity free irreducible decomposition*

$$\mathcal{H}^s_k(\mathbb{R}^m) = \bigoplus_{(t,j)\in N^{s,m}_k} X^{s,t,m}_{k,j}\ \mathcal{H}^t_j(\mathbb{R}^{m-1}). \tag{3.24}$$

Here the embedding factors $X^{s,t,m}_{k,j}$ are defined as the polynomials

$$X^{s,t,m}_{k,j}(x) = X^{(k-j)}_{m,j}(x) e^{s-t}_m + \alpha\, X^{(k-1-j)}_{m,j+1}(x)(\beta^{t-s}(\underline{x}\wedge) + \beta^{t-s+1}\ (\underline{x}\bullet))\ e^{s-t+1}_m$$

where $x = (\underline{x}, x_m) \in \mathbb{R}^m$, $X^{(k-j)}_{m,j}$ are given in (3.11)*, $X^{(-1)}_{m,k+1} = 0$, $\beta = -(j+m-1-t)/(j+t)$ and*

$$\alpha = \begin{cases} -(j+1)/(m+2j-1) & \textit{unless} \quad t = 0, m-1; \\ 0 & \textit{if} \quad t = 0, m-1. \end{cases}$$

In particular, this decomposition is orthogonal.

Proof. In [L9], a proof is given by the Cauchy-Kovalevskaya method. Let $\mathcal{I}^s_k$ be the set of initial polynomials for the space $\mathcal{H}^s_k(\mathbb{R}^m)$, that is, $\mathcal{I}^s_k$ is a subset of $\mathcal{P}_k(\mathbb{R}^{m-1}, \mathcal{C}\ell_m)$ such that $CK(\mathcal{I}^s_k) = \mathcal{H}^s_k(\mathbb{R}^m)$. Then, in [L9], it is shown that

$$\mathcal{I}^s_k = \mathrm{Ker}^s_k\, \underline{\partial}^+ \oplus (\mathrm{Ker}^{s-1}_k\, \underline{\partial}^-) e_m$$

where $\mathrm{Ker}^s_k\, \underline{\partial}^\pm = \{u \in \mathcal{P}_k(\mathbb{R}^{m-1}, \mathcal{C}\ell^s_m) \mid \underline{\partial}^\pm u = 0\}$. The branching (3.24) for $\mathcal{H}^s_k(\mathbb{R}^m)$ is obtained by applying the operator $CK = e^{x_m e_m \underline{\partial}}$ to the irreducible decomposition of the module $\mathcal{I}^s_k$ under the H-action of $Pin(m-1)$. See [L9] for details.

Now we give an alternative proof. It is a well-known fact from representation theory that, under the H-action of $Pin(m-1)$, the module $\mathcal{H}^s_k(\mathbb{R}^m)$ possesses the decomposition

$$\mathcal{H}^s_k(\mathbb{R}^m) = \bigoplus_{(t,j)\in N^{s,m}_k} \tilde{\mathcal{H}}^t_j$$

into irreducible submodules $\tilde{\mathcal{H}}^t_j$ equivalent to $\mathcal{H}^t_j(\mathbb{R}^{m-1})$. To get (3.24) we need to describe explicitly the pieces $\tilde{\mathcal{H}}^t_j$ in the decomposition. To do this, we recall that, under the H-action of $Pin(m)$, the space $\mathcal{M}_k(\mathbb{R}^m, \mathcal{C}\ell_m)$ decomposes into inequivalent irreducible pieces as

$$\mathcal{M}_k(\mathbb{R}^m, \mathcal{C}\ell_m) = \left(\bigoplus_{s=0}^{m} \mathcal{H}^s_k(\mathbb{R}^m)\right) \oplus \left(\bigoplus_{s=1}^{m-1} ((x\wedge) + \beta^{s,m}_{k-1}(x\bullet))\, \mathcal{H}^s_{k-1}(\mathbb{R}^m)\right) \tag{3.25}$$

with $\beta_k^{s,m} = -(k+m-s)/(k+s)$ (see Theorem 1). On the other hand, by (3.10), we have that

$$\mathcal{M}_k(\mathbb{R}^m, \mathcal{C}\ell_m) = \bigoplus_{j=0}^{k} X_{m,j}^{(k-j)} \mathcal{M}_j(\mathbb{R}^{m-1}, \mathcal{C}\ell_m). \tag{3.26}$$

As $\mathcal{C}\ell_m = \mathcal{C}\ell_{m-1} \oplus e_m\ \mathcal{C}\ell_{m-1}$ each space $\mathcal{M}_j(\mathbb{R}^{m-1}, \mathcal{C}\ell_m)$ in (3.26) decomposes further as $\mathcal{M}_j(\mathbb{R}^{m-1}, \mathcal{C}\ell_m) = \mathcal{M}_j(\mathbb{R}^{m-1}, \mathcal{C}\ell_{m-1}) \oplus e_m \mathcal{M}_j(\mathbb{R}^{m-1}, \mathcal{C}\ell_{m-1})$. Finally, the spaces $\mathcal{M}_j(\mathbb{R}^{m-1}, \mathcal{C}\ell_{m-1})$ possess the irreducible decompositions (3.25) under the H-action of $Pin(m-1)$. As a result, we have decomposed the space $\mathcal{M}_k(\mathbb{R}^m, \mathcal{C}\ell_m)$ into $Pin(m-1)$-irreducible pieces

$$X_{m,j}^{(k-j)} e_m^r\ \mathcal{H}_j^t(\mathbb{R}^{m-1}) \text{ and } X_{m,j}^{(k-j)}((\underline{x}\ \wedge) + \beta_{j-1}^{t,m-1}(\underline{x}\ \bullet)) e_m^r\ \mathcal{H}_{j-1}^t(\mathbb{R}^{m-1}) \tag{3.27}$$

where $t = 0, \dots, m-1$, $j = 0, \dots, k$ and $t = 1, \dots, m-2$, $j = 1, \dots, k$, respectively, and $r = 0, 1$. Now it is easy to find the submodule $\tilde{\mathcal{H}}_j^t$ equivalent to $\mathcal{H}_j^t(\mathbb{R}^{m-1})$ inside $\mathcal{H}_k^s(\mathbb{R}^m)$. Indeed, by (3.27), it is sufficient to choose a constant $\tilde{\alpha}$ such that, for

$$X_{k,j}^{s,t,m}(x) = X_{m,j}^{(k-j)}(x) e_m^{s-t} + \tilde{\alpha}\ X_{m,j+1}^{(k-1-j)}(x)((\underline{x}\ \wedge) + \beta_j^{t,m-1}\ (\underline{x}\ \bullet))\ e_m^{s-t+1},$$

we have that $X_{k,j}^{s,t,m}\ \mathcal{H}_j^t(\mathbb{R}^{m-1}) \subset \mathcal{H}_k^s(\mathbb{R}^m)$. As we know that

$$X_{k,j}^{s,t,m}\ \mathcal{H}_j^t(\mathbb{R}^{m-1}) \subset \mathcal{M}_k(\mathbb{R}^m, \mathcal{C}\ell_m)$$

it is sufficient to take the constant $\tilde{\alpha}$ such that the piece $X_{k,j}^{s,t,m}\ \mathcal{H}_j^t(\mathbb{R}^{m-1})$ contains only s-vector valued polynomials. But, recalling the definition (3.11) of the factors $X_{m,j}^{(k-j)}$, this is not difficult to do. □

Using Theorem 10, we easily construct Gelfand-Tsetlin bases of the modules $\mathcal{H}_k^s(\mathbb{R}^m)$ by induction on the dimension m.

Example 1. First we construct Gelfand-Tsetlin bases for Hodge-de Rham systems in dimension 2. Indeed, the following statements are obvious.

(i) For $s \in \{0, 2\}$, the Gelfand-Tsetlin basis for $\mathcal{H}_0^s(\mathbb{R}^2)$ is formed by the unique element $f_0^s = e^{s,0}$ with $e^{0,0} = 1$ and $e^{2,2} = e_{21}$.

(ii) Let $\mathcal{C}\ell_2 = \mathbb{R}_{0,2}$. Then, for $k \in \mathbb{N}_0$, the Gelfand-Tsetlin basis for $\mathcal{H}_k^1(\mathbb{R}^2)$ consists of two polynomials $f_k^{\pm 1}(x) = (x_1 - e_{12}x_2)^k e^{1,\pm 1}$ with $e^{1,1} = e_1$ and $e^{1,-1} = e_2$.

(iii) Let $\mathcal{C}\ell_2 = \mathbb{C}_2$. Then, for $k \in \mathbb{N}_0$, the Gelfand-Tsetlin basis for $\mathcal{H}_k^1(\mathbb{R}^2)$ consists of two polynomials $f_k^{\pm 1}(x) = (x_1 \mp ix_2)^k e^{1,\pm 1}$ with $e^{1,\pm 1} = e_1 \mp ie_2$.

(iv) Otherwise, the spaces $\mathcal{H}_k^s(\mathbb{R}^2)$ are trivial.

Theorem 11. *Let $m \geq 3$. Denote by $I_k^{s,m}$ the set of pairs (ν, μ) such that the sequences*

$$\nu = (s_{m-1}, \dots, s_3, t_2), \quad \mu = (k_{m-1}, k_{m-1}, \dots, k_2)$$

of integers, $k_m = k$, $s_m = s$ and $s_2 = |t_2|$ satisfy, for each $r = 3, \dots, m$, that $(s_{r-1}, k_{r-1}) \in N_{k_r}^{s_r,r}$ and $(t_2, k_2) \in \{(0,0), (2,0)\} \cup \{(\pm 1, k) |\ k \in \mathbb{N}_0\}$.

Then a Gelfand-Tsetlin basis of the $Pin(m)$-module $\mathcal{H}^s_k(\mathbb{R}^m)$ is formed by the polynomials

$$f^{s,\nu}_{k,\mu} = X^{s,s_{m-1},m}_{k,k_{m-1}} X^{s_{m-1},s_{m-2},m-1}_{k_{m-1},k_{m-2}} \cdots X^{s_3,s_2,3}_{k_3,k_2}\ f^{t_2}_{k_2},\ (\nu,\mu) \in I^{s,m}_k.$$

Here the embedding factors $X^{s,t,m}_{k,j}$ are given in Theorem 10 and $f^{t_2}_{k_2}$ in Example 1.

The basis elements $f^{s,\nu}_{k,\mu}$ have the following Appell property.

Theorem 12. *Let $m \geq 3$ and let $f^{s,\nu}_{k,\mu}$ be the basis elements of the spaces $\mathcal{H}^s_k(\mathbb{R}^m)$ given in Theorem 11 with $\nu = (s_{m-1},\dots,s_3,t_2)$, $\mu = (k_{m-1},k_{m-1},\dots,k_2)$ and $s_2 = |t_2|$. Then we have that*

(i) $\partial_{x_m} f^{s,\nu}_{k,\mu} = 0$ *for* $k = k_{m-1}$*;*

(ii) $\partial_{x_m} f^{s,\nu}_{k,\mu} = k\ f^{s,\nu}_{k-1,\mu}$ *for* $k > k_{m-1}$*;*

(iii) $\partial^{k_2}_{t_2}\ \partial^{k_3-k_2}_{x_3} \cdots \partial^{k-k_{m-1}}_{x_m}\ f^{s,\nu}_{k,\mu} = k!\ e^{s,\nu}$
with $e^{s,\nu} = e^{s-s_{m-1}}_m \cdots e^{s_3-s_2}_3 e^{s_2,t_2}$ *and*

$$\partial_{t_2} = \begin{cases} (1/2)(\partial_{x_1} + e_{12}\partial_{x_2}) & \text{if} \quad \mathcal{C}\ell_m = \mathbb{R}_{0,m} \text{ and } t_2 = \pm 1; \\ (1/2)(\partial_{x_1} \pm i\partial_{x_2}) & \text{if} \quad \mathcal{C}\ell_m = \mathbb{C}_m \text{ and } t_2 = \pm 1. \end{cases} \tag{3.28}$$

Note that $k_2 = 0$ unless $t_2 = \pm 1$.

Proof. It is obvious from the fact that, for $k > j$, $\partial_{x_m} X^{s,t,m}_{k,j} = k\ X^{s,t,m}_{k-1,j}$ and $X^{s,t,m}_{k,k} = e^{s-t}_m$. □

Remark 4. For $s = 0,\dots,m$, let us denote by $J^{s,m}$ the set of sequences $\nu = (s_{m-1},\dots,s_3,t_2)$ of integers such that, putting $s_m = s$ and $s_2 = |t_2|$,

$$0 \leq s_r \leq r \quad \text{and} \quad s_{r+1} - 1 \leq s_r \leq s_{r+1}$$

for each $r = 2,\dots,m-1$. Obviously, the set $\{e^{s,\nu}|\ \nu \in J^{s,m}\}$ is a basis of the space $\mathcal{C}\ell^s_m$ of s-vectors. Here $e^{s,\nu}$ are given in Theorem 12. Then each $a \in \mathcal{C}\ell^s_m$ can be uniquely written as

$$a = \sum_{\nu \in J^{s,m}} a^\nu e^{s,\nu}$$

for some (real or complex) numbers a^ν.

To summarize, we have constructed a complete orthogonal Appell system for the Hilbert space $L^2(\mathbb{B}_m, \mathcal{C}\ell^s_m) \cap \operatorname{Ker}\partial$ of L^2-integrable monogenic functions $g : \mathbb{B}_m \to \mathcal{C}\ell^s_m$. Indeed, using Theorems 11 and 12, we easily obtain the following result.

Theorem 13. *Let $m \geq 3$, let $\mathbb{B}_m$ be the unit ball in $\mathbb{R}^m$ and let $s = 0,\dots,m$.*

(a) Then an orthogonal basis of the space $L^2(\mathbb{B}_m, \mathcal{C}\ell^s_m) \cap \operatorname{Ker}\partial$ is formed by the polynomials $f^{s,\nu}_{k,\mu}$ for $k \in \mathbb{N}_0$ and $(\nu,\mu) \in I^{s,m}_k$. Here the basis elements $f^{s,\nu}_{k,\mu}$ are defined in Theorem 11.

(b) *Each function $g \in L^2(\mathbb{B}_m, \mathcal{C}\ell_m^s) \cap \operatorname{Ker} \partial$ has a unique orthogonal series expansion*

$$g = \sum_{k=0}^{\infty} \sum_{(\nu,\mu) \in I_k^{s,m}} \mathbf{t}_{k,\mu}^{s,\nu}(g)\ f_{k,\mu}^{s,\nu} \tag{3.29}$$

for some complex coefficients $\mathbf{t}_{k,\mu}^{s,\nu}(g)$. In addition, by Remark 4, we have that

$$g = \sum_{\nu \in J^{s,m}} g^\nu e^{s,\nu}$$

for some complex functions g^ν on $\mathbb{B}_m$. Then, for $(\nu, \mu) \in I_k^{s,m}$, it holds that

$$\mathbf{t}_{k,\mu}^{s,\nu}(g) = \frac{1}{k!}\, \partial_{t_2}^{k_2}\, \partial_{x_3}^{k_3-k_2} \cdots \partial_{x_m}^{k-k_{m-1}}\ g^\nu(x)|_{x=0} \tag{3.30}$$

where ∂_{t_2} is defined in (3.28).

For a function $g \in L^2(\mathbb{B}_m, \mathcal{C}\ell_m^s) \cap \operatorname{Ker} \partial$, we call the orthogonal series expansion (3.29) its generalized Taylor series.

Now we construct orthogonal bases for solutions of an arbitrary generalized Moisil-Théodoresco systems. For a subset S of $\{0, 1, \ldots, m\}$, put

$$\mathcal{C}\ell_m^S = \bigoplus_{s \in S} \mathcal{C}\ell_m^s.$$

It is easy to see that, using Theorems 1 and 11, we obtain the following result.

Theorem 14. *Let S be a subset of $\{0, 1, \ldots, m\}$ and let $S' = \{s : s \pm 1 \in S\}$. Then an orthogonal basis of the Hilbert space $L^2(\mathbb{B}_m, \mathcal{C}\ell_m^S) \cap \operatorname{Ker} \partial$ is formed by the polynomials*

$$f_{k,\mu}^{s,\nu} \ \ \textit{for } s \in S,\ k \in \mathbb{N}_0 \textit{ and } (\nu, \mu) \in I_k^{s,m}$$

together with the polynomials

$$((x \wedge) + \beta_{k-1}^{s,m}(x\, \bullet))\, f_{k-1,\mu}^{s,\nu} \ \ \textit{for } s \in S',\ k \in \mathbb{N} \textit{ and } (\nu, \mu) \in I_{k-1}^{s,m}.$$

Here $\beta_k^{s,m} = -(k + m - s)/(k + s)$.

3.4.1 The Riesz system in dimension 3

In this section, we recall a construction of Gelfand-Tsetlin bases for Hodge-de Rham systems in dimension 3 using the Cauchy-Kovalevskaya method and the fact that the basis elements in this case possess the Appell property even with respect to all variables, see [L9, 78]. In what follows, we assume that $\mathcal{C}\ell_3 = \mathbb{C}_3$. Obviously, we have that, for $s \in \{0, 3\}$ and $k \geq 1$, $\mathcal{H}_k^s(\mathbb{R}^3) = \{0\}$, $\mathcal{H}_0^0(\mathbb{R}^3) = \mathbb{C}$ and $\mathcal{H}_0^3(\mathbb{R}^3) = \mathbb{C}\, e_{321}$ with $e_{321} = e_3 e_2 e_1$. As $\mathcal{H}_k^2(\mathbb{R}^3) = \mathcal{H}_k^1(\mathbb{R}^3)\, e_{321}$ we may limit ourselves to the spaces $\mathcal{H}_k^1(\mathbb{R}^3)$ of k-homogeneous solutions of the Riesz system in dimension 3.

According to [L9], the operator $CK = e^{x_3 e_3 \underline{\partial}}$ is an isomorphism of the space $\mathcal{I}_k^1$ onto the space $\mathcal{H}_k^1(\mathbb{R}^3)$. Here $\mathcal{I}_k^1 = \operatorname{Ker}_k^1 \underline{\partial}^+ \oplus (\operatorname{Ker}_k^0 \underline{\partial}^-) e_3$ is the space of initial polynomials for $\mathcal{H}^1_k(\mathbb{R}^3)$. Next we need to describe a decomposition of the space $\mathcal{I}^1_k$ into irreducible $Spin(2)$-submodules. Put $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$ and, for $k \in \mathbb{N}_0$, denote

$$p^k_j = \frac{z^j\overline{z}^{k-j}}{j!(k-j)!}e_3,\ j = 0,\dots,k \quad \text{and} \quad q^k_j = \frac{1}{2}e_3\underline{\partial}(p^{k+1}_j),\ j = 0,\dots,k+1. \tag{3.31}$$

Then the space $\mathcal{I}^1_k$ decomposes into 1-dimensional irreducible $Spin(2)$-submodules as follows:

$$\mathcal{I}^1_k = \bigoplus_{j=0}^{k+1} \langle q^k_j\rangle \ \oplus\ \bigoplus_{j=0}^{k} \langle p^k_j\rangle. \tag{3.32}$$

To get the Gelfand-Tsetlin basis for $\mathcal{H}^1_k(\mathbb{R}^3)$ it is now sufficient to apply the Cauchy-Kovalevskaya extension operator to the polynomials p^k_j and q^k_j. Indeed, we have the following result (see [L9, 78]).

Proposition 3. *Let $k \in \mathbb{N}_0$ and let the polynomials p^k_j and q^k_j be as in* (3.31). *Then the polynomials*

$$g^k_{2j+2} = e^{x_3e_3\underline{\partial}}(p^k_j),\ j = 0,\dots,k \quad \textit{and} \quad g^k_{2j+1} = e^{x_3e_3\underline{\partial}}(q^k_j),\ j = 0,\dots,k+1$$

form a Gelfand-Tsetlin basis of the irreducible $Spin(3)$-module $\mathcal{H}^1_k(\mathbb{R}^3)$.

It is not difficult to obtain explicit formulas for the basis elements in terms of hypergeometric series. Recall that the hypergeometric series ${}_2F_1(a,b,c;y)$ is given by

$${}_2F_1(a,b,c;y) = \sum_{s=0}^{\infty}\frac{(a)_s(b)_s}{(c)_s\ s!}y^s \quad \text{with} \quad (a)_s = a(a+1)\cdots(a+s-1).$$

Corollary 1. *(See [78].) Let $\{g^k_j|\ j = 1,\dots,2k+3\}$ be the Gelfand-Tsetlin basis of the module $\mathcal{H}^1_k(\mathbb{R}^3)$ defined in Proposition 3. Let $v_\pm = (e_1 \pm ie_2)/2$. Then we have that $g^k_1 = (\overline{z}^k/k!)v_-$, $g^k_{2k+3} = (z^k/k!)v_+$ and, in general,*

$$\begin{aligned}
g^k_{2j+2} \;&=\; \frac{1}{j!(k-j)!}\,({}_2F_1(-j,-k+j,\frac{1}{2},-\frac{x_3^2}{|z|^2})\ z^j\overline{z}^{k-j}e_3 + \\
&\quad +2\ {}_2F_1(-j+1,-k+j,\frac{3}{2},-\frac{x_3^2}{|z|^2})\ jx_3z^{j-1}\overline{z}^{k-j}v_+ + \\
&\quad +2\ {}_2F_1(-j,-k+j+1,\frac{3}{2},-\frac{x_3^2}{|z|^2})\ (k-j)x_3z^j\overline{z}^{k-j-1}v_-); \\
g^k_{2j+1} \;&=\; \frac{1}{j!(k+1-j)!}\,({}_2F_1(-j+1,-k-1+j,\frac{1}{2},-\frac{x_3^2}{|z|^2})\ jz^{j-1}\overline{z}^{k+1-j}v_+ + \\
&\quad -2\ {}_2F_1(-j+1,-k+j,\frac{3}{2},-\frac{x_3^2}{|z|^2})\ j(k+1-j)x_3z^{j-1}\overline{z}^{k-j}e_3 + \\
&\quad +{}_2F_1(-j,-k+j,\frac{1}{2},-\frac{x_3^2}{|z|^2})\ (k+1-j)z^j\overline{z}^{k-j}v_-).
\end{aligned}$$

Here $|z|^2 = z\overline{z}$.

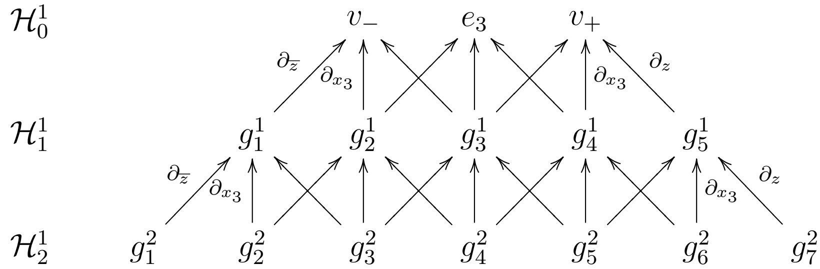

Figure 3.2: Generalized Appell property

Remark 5. It is well-known that orthogonal bases for the spaces $\mathcal{H}_k^1(\mathbb{R}^3)$ can be also obtained by applying the Dirac operator ∂ to standard bases of spherical harmonics in $\mathbb{R}^3$ (see [35, 28, 29, 88, 69]). Indeed, let us recall that a standard basis of k-homogeneous spherical harmonics in $\mathbb{R}^3$ is formed by the polynomials

$$h_{\pm j}^k = |x|^{k-j} C_{k-j}^{1/2+j}(x_3/|x|)\ (x_1 \pm i x_2)^j, \quad j = 0, \dots, k,$$

see Section 3.1. Then the Gelfand-Tsetlin basis for $\mathcal{H}_k^1(\mathbb{R}^3)$ given in Proposition 3 coincides, up to a normalization, with

$$\{\partial h_{\pm j}^{k+1} \mid j = 0, \dots, k+1\}$$

(see [L9] for details).

By [78], the basis elements from Proposition 3 possess the Appell property even with respect to all variables z, $\overline{z}$ and x_3.

Proposition 4. *For each $k \in \mathbb{N}_0$, let $\{g_j^k \mid j = 1, \dots, 2k+3\}$ be the Gelfand-Tsetlin basis of the module $\mathcal{H}_k^1(\mathbb{R}^3)$ defined in Proposition 3. Then, for each $k \in \mathbb{N}$ and $j = 1, \dots, 2k+3$, we have that*

$$\partial_z\, g_j^k = g_{j-2}^{k-1}, \quad \partial_{\overline{z}}\, g_j^k = g_j^{k-1}, \quad \partial_{x_3}\, g_j^k = (-1)^j 2\, g_{j-1}^{k-1}.$$

Here $g_j^{k-1} = 0$ unless $j = 1, \dots, 2k+1$.

Figure 3.2 shows structural properties of Gelfand-Tsetlin bases in this case. Indeed, the k-th row of Figure 3.2 contains the basis elements g_j^k of the space $\mathcal{H}_k^1 = \mathcal{H}_k^1(\mathbb{R}^3)$. Then, according to Proposition 4, the application of the derivative ∂_{x_3} to basis elements causes upward shift in a given column, the derivative $\partial_{\overline{z}}$ moves them diagonally upward to the right and ∂_z diagonally upward to the left.

3.5 Hermitian monogenics

In this section, we describe an algorithm for a construction of Gelfand-Tsetlin bases of homogeneous Hermitian monogenic polynomials given in [L10]. The construction is based on the Cauchy-Kovalevskaya method. The first step is to generalize the Cauchy-Kovalevskaya extension to this setting, which is done in [20].

The Cauchy-Kovalevskaya extension. We denote by $\mathcal{P}^r_{a,b}(\mathbb{C}^n)$ the space of (a,b)-homogeneous polynomials P in $\mathbb{C}^n$ taking values in the part $\mathbb{S}^r$ of the spinor space $\mathbb{S}$. For notation, see Section 2.1.3. Recall that the space $\mathcal{M}^r_{a,b}(\mathbb{C}^n)$ consists of (Hermitian) monogenic polynomials of $\mathcal{P}^r_{a,b}(\mathbb{C}^n)$ and that, under the action of the group $U(n)$, the spaces $\mathcal{M}^r_{a,b}(\mathbb{C}^n)$ are mutually inequivalent irreducible modules. Moreover, the definition of Gelfand-Tsetlin bases for irreducible $U(n)$-modules is quite analogous as for spin modules (see [L10]).

For the case $r=0$, resp. $r=n$, the notion of Hermitian monogenicity coincides with the notion of antiholomorphy, resp. holomorphy, in n complex variables. Hence we can assume that $1\leq r\leq n-1$ from now on. The idea of the Cauchy-Kovalevskaya extension is simply to characterize solutions of a given system of PDE's by their restriction, sometimes together with the restrictions of some of their derivatives, to a submanifold of codimension one. In [20], this is done for the hermitian case, namely, homogeneous Hermitian monogenic polynomials in $\mathbb{C}^n$ are characterized by their restrictions, together with the restrictions of some of their derivatives, to the hyperplane of complex codimension 1. Indeed, we single out the variables $(z_n,\overline{z}_n)$ and split $\underline{z}$, $\underline{z}^\dagger$, $\partial_{\underline{z}}$ and $\partial^\dagger_{\underline{z}}$ as $\underline{z}=\widetilde{\underline{z}}+\mathfrak{f}_n z_n$, $\underline{z}^\dagger=\widetilde{\underline{z}}^\dagger+\mathfrak{f}^\dagger_n\overline{z}_n$, $\partial_{\underline{z}}=\widetilde{\partial}_{\underline{z}}+\mathfrak{f}^\dagger_n\partial_{z_n}$ and $\partial^\dagger_{\underline{z}}=\widetilde{\partial}^\dagger_{\underline{z}}+\mathfrak{f}_n\partial_{\overline{z}_n}$. We consider restrictions to the hyperplane $\{z\in\mathbb{C}^n\,|\,z_n=\overline{z}_n=0\}$, identified with $\mathbb{C}^{n-1}$. We may then split the value space $\mathbb{S}^r=(\bigwedge^\dagger_n)^{(r)}\,I$ as

$$\mathbb{S}^r=(\textstyle\bigwedge^\dagger_{n-1})^{(r)}\,I\oplus\mathfrak{f}^\dagger_n\,(\bigwedge^\dagger_{n-1})^{(r-1)}\,I$$

Hence any polynomial p with values in $(\bigwedge^\dagger_n)^{(r)}I$ can be split as

$$p=p^0+\mathfrak{f}^\dagger_n\,p^1$$

where p^0 has values in $(\bigwedge^\dagger_{n-1})^{(r)}I$ and p^1 has values in $(\bigwedge^\dagger_{n-1})^{(r-1)}I$. Now consider $M_{a,b}\in\mathcal{M}^r_{a,b}(\mathbb{C}^n)$ and denote the restrictions of some of its derivatives to $\mathbb{C}^{n-1}$ as

$$\frac{\partial^i M_{a,b}}{\partial {z_n}^i}|_{\mathbb{C}^{n-1}} \;=\; p_{a-i,b} \;=\; p^0_{a-i,b}+\mathfrak{f}^\dagger_n\,p^1_{a-i,b},\quad i=0,\dots,a, \tag{3.33}$$

$$\frac{\partial^j M_{a,b}}{\partial {\overline{z}_n}^j}|_{\mathbb{C}^{n-1}} \;=\; p_{a,b-j} \;=\; p^0_{a,b-j}+\mathfrak{f}^\dagger_n\,p^1_{a,b-j},\quad j=0,\dots,b. \tag{3.34}$$

In [20], the Hermitian Cauchy-Kovalevskaya extension operator is introduced and the following theorem is proved.

Theorem 15. *(i) Any $M_{a,b}\in\mathcal{M}^r_{a,b}(\mathbb{C}^n)$ is uniquely determined by the initial polynomials $p^0_{a,b-j}$, $j=0,\dots,b$, and $p^1_{a-i,b}$, $i=0,\dots,a$, defined in (3.33)-(3.34). Moreover, the initial data satisfy the compatibility conditions*

$$\widetilde{\partial}_{\underline{z}}p^0_{a,b-j}=0 \textit{ for } r<n-1 \quad \textit{and} \quad \widetilde{\partial}^\dagger_{\underline{z}}\,p^1_{a-i,b}=0 \textit{ for } r>1.$$

(ii) On the other hand, denote $\mathcal{A}^r_{a,b-j}=\mathrm{Ker}^r_{a,b-j}(\widetilde{\partial}_{\underline{z}})$ and $\mathcal{B}^r_{a-i,b}=\mathrm{Ker}^{r-1}_{a-i,b}(\widetilde{\partial}^\dagger_{\underline{z}})$ where, for example, we put

$$\mathrm{Ker}^r_{a,b-j}(\widetilde{\partial}_{\underline{z}})=\mathrm{Ker}(\widetilde{\partial}_{\underline{z}})\cap\mathcal{P}^r_{a,b-j}(\mathbb{C}^{n-1}).$$

Then the Hermitian Cauchy-Kovalevskaya extension operator CK is an isomorphism from the space of initial data

$$\bigoplus_{j=0}^{b} \mathcal{A}^r_{a,b-j} \oplus \bigoplus_{i=0}^{a} \mathcal{B}^r_{a-i,b}$$

onto the space $\mathcal{M}^r_{a,b}(\mathbb{C}^n)$ commuting with the action of $U(n-1)$.

In particular, the operator CK yields a splitting of $\mathcal{M}^r_{a,b}(\mathbb{C}^n)$ into a direct sum of $U(n-1)$-invariant subspaces.

The Fischer decomposition for two kernels. The last ingredient for the construction of the Gelfand-Tsetlin basis is the decomposition of both spaces of initial data $\mathcal{A}^r_{a,b-j}$ and $\mathcal{B}^r_{a-i,b}$ into irreducible components under the action of $U(n-1)$. The tool needed here is the Fischer decomposition for the kernels of the Hermitian Dirac operators (see [21]).

Theorem 16. *Let $1 \leq r \leq n-2$ and let $\widetilde{\mathcal{M}}^r_{a,b}$ stand for $\mathcal{M}^r_{a,b}(\mathbb{C}^{n-1})$. Then the following statements hold.*

(i) Under the action of $U(n-1)$, the space $\mathrm{Ker}^r_{a,b}(\widetilde{\partial}_{\underline{z}})$ has the multiplicity free irreducible decomposition

$$\begin{aligned}
\mathrm{Ker}^r_{a,b}(\widetilde{\partial}_{\underline{z}}) \;&=\; \widetilde{\mathcal{M}}^r_{a,b} \oplus \bigoplus_{j=0}^{\min(a,b-1)} |\widetilde{\underline{z}}|^{2j} \widetilde{\underline{z}}^\dagger \widetilde{\mathcal{M}}^{r-1}_{a-j,b-j-1} \\
&\oplus \bigoplus_{j=0}^{\min(a-1,b-1)} |\widetilde{\underline{z}}|^{2j} (\widetilde{\underline{z}}^\dagger \widetilde{\underline{z}} + \frac{(a-j-1+r)}{(a+r)} \widetilde{\underline{z}}\, \widetilde{\underline{z}}^\dagger) \widetilde{\mathcal{M}}^r_{a-j-1,b-j-1}
\end{aligned}$$

(ii) Under the action of $U(n-1)$, the space $\mathrm{Ker}^{r-1}_{a,b}(\widetilde{\partial}^\dagger_{\underline{z}})$ has the multiplicity free irreducible decomposition

$$\begin{aligned}
\mathrm{Ker}^{r-1}_{a,b}(\widetilde{\partial}^\dagger_{\underline{z}}) \;&=\; \widetilde{\mathcal{M}}^{r-1}_{a,b} \oplus \bigoplus_{j=0}^{\min(a-1,b)} |\widetilde{\underline{z}}|^{2j} \widetilde{\underline{z}} \widetilde{\mathcal{M}}^r_{a-j-1,b-j} \\
&\oplus \bigoplus_{j=0}^{\min(a-1,b-1)} |\widetilde{\underline{z}}|^{2j} (\widetilde{\underline{z}} \widetilde{\underline{z}}^\dagger + \frac{(b-j-1+n-r)}{(b+n-r)} \widetilde{\underline{z}}\, \widetilde{\underline{z}}^\dagger) \widetilde{\mathcal{M}}^{r-1}_{a-j-1,b-j-1}
\end{aligned}$$

The construction. Now we are ready to construct Gelfand-Tsetlin bases of homogeneous Hermitian monogenics in $\mathbb{C}^n$ by induction on the dimension n.

(i) The case $n = 1$ corresponds to complex valued functions on $\mathbb{C}$. The case $r = 0$ leads to antiholomorphic functions, the case $r = 1$ to holomorphic functions. Obviously, for $j \in \mathbb{N}_0$, a Gelfand-Tsetlin basis of the space $\mathcal{M}^0_{0,j}(\mathbb{C})$, resp. $\mathcal{M}^1_{j,0}(\mathbb{C})$, is formed by the unique polynomial $\bar{z}_1^j\, I$, resp. $z_1^j\, \mathfrak{f}_1^\dagger I$. Otherwise, the spaces $\mathcal{M}^r_{a,b}(\mathbb{C})$ are trivial.

(ii) Now assume that we know the Gelfand-Tsetlin bases in dimension $n-1$ for all spaces $\mathcal{M}^{r'}_{a',b'}(\mathbb{C}^{n-1})$, $r' = 0, \ldots, n-1$ and $a', b' \in \mathbb{N}_0$. Then we can construct the Gelfand-Tsetlin basis of the space $\mathcal{M}^r_{a,b}(\mathbb{C}^n)$ using the Cauchy-Kovalevskaya method as follows. First we use Theorem 16 to decompose the space of initial data, introduced in Theorem 15, into $U(n-1)$ irreducible components. Applying the Cauchy-Kovalevskaya extension map to this irreducible decomposition gives us the branching of the given space $\mathcal{M}^r_{a,b}(\mathbb{C}^n)$, that is, a decomposition of $\mathcal{M}^r_{a,b}(\mathbb{C}^n)$ into $U(n-1)$-irreducible pieces (see Theorem 15). Due to the induction assumption, we can use the explicit form of the Gelfand-Tsetlin bases in dimension $n-1$ to get an explicit basis of the space of initial data. Moreover, in [20], the Cauchy-Kovalevskaya extension map is described as a differential operator acting on initial data. In such a way, we can construct elements of the Gelfand-Tsetlin basis of the space $\mathcal{M}^r_{a,b}(\mathbb{C}^n)$ explicitly.

In [L10], it is shown that, in any complex dimension n, elements of the Gelfand-Tsetlin bases for Hermitian monogenics possess the Appell property with respect to the last variables z_n and $\overline{z}_n$. On the other hand, in complex dimension $n = 2$, the basis elements have the Appell property even with respect to all variables and, in this case, we know explicit formulas for the basis elements, see [L10] for details.

Chapter 4

Finely monogenic functions

In this chapter, we review results about finely monogenic functions obtained in a series of the papers [74, L2, L4, 75, L3, 76]. Finely monogenic functions are a generalization of B. Fuglede's finely holomorphic functions to higher dimensions in the context of Clifford analysis.

4.1 Finely holomorphic functions

In this section, we recall briefly the theory of finely holomorphic functions (see [56, 57, 61]). It generalizes the theory of holomorphic functions to plane domains open in a topology finer than the Euclidean topology, namely, in the fine topology from potential theory.

Recall that the fine topology $\mathcal{F}$ in $\mathbb{R}^{m+1}$, where $m \geq 1$, is the weakest topology making all subharmonic functions in $\mathbb{R}^{m+1}$ continuous, see e.g. [5, Chapter 7]. It is strictly finer than the Euclidean topology in $\mathbb{R}^{m+1}$. For example, if K is a dense countable subset of an open set $\Omega \subset \mathbb{R}^{m+1}$, then $U := \Omega \setminus K$ is a finely open set but it has no interior points in the usual sense. Let $U \subset \mathbb{R}^{m+1}$ be finely open and let $f : U \to \mathbb{R}^n$. Then we call the function f finely continuous on U if it is continuous from U endowed with the fine topology to $\mathbb{R}^n$ with the Euclidean topology. Denote by $\mathcal{F}_z$ the family of all finely open sets containing a point $z \in \mathbb{R}^{m+1}$. The fine limit of f at a point $\tilde{z} \in U$ can be understood as the usual limit along some fine neighourhood of $\tilde{z}$, that is, there is $V \in \mathcal{F}_{\tilde{z}}$ such that

$$\operatorname*{fine-lim}_{z \to \tilde{z}} f(z) = \lim_{z \to \tilde{z}, z \in V} f(z),$$

see [5, p. 207]. Moreover, we call a linear map $L : \mathbb{R}^{m+1} \to \mathbb{R}^n$ the fine differential of f at a point $\tilde{z} \in U$ if

$$\operatorname*{fine-lim}_{z \to \tilde{z}} \frac{f(z) - f(\tilde{z}) - L(z - \tilde{z})}{|z - \tilde{z}|} = 0.$$

We write $d_{\mathrm{f}} f(\tilde{z})$ for the fine differential L and, for $l = 0, \ldots, m$, we define the first order fine derivatives of f at the point $\tilde{z}$ by

$$\frac{\partial_{\mathrm{f}} f}{\partial x_l}(\tilde{z}) := d_{\mathrm{f}} f(\tilde{z})(e_l).$$

Here $(e_0, \ldots, e_m)$ is the standard basis of $\mathbb{R}^{m+1}$ and $z = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1}$.

Remark 6. In the case when $m > 1$, a definition of fine partial derivatives is not straightforward because

$$V := B(\tilde{z}, r) \setminus \bigcup_{l=0}^{m} \{\tilde{z} + te_l;\ t \in \mathbb{R}, t \neq 0\}$$

is a fine neighbourhood of a point $\tilde{z} \in \mathbb{R}^{m+1}$ for any $r > 0$. Here $B(\tilde{z}, r)$ is the ball in $\mathbb{R}^{m+1}$ with center $\tilde{z}$ and radius r.

Now we introduce some function spaces. Let us denote by fine-$\mathcal{C}^1(U)$ the set of all functions f finely differentiable everywhere on U whose fine differential $d_{\mathrm{f}}f$ is finely continuous on U. As usual we can define inductively the spaces fine-$\mathcal{C}^k(U)$ for all $k \in \mathbb{N}_0$. In particular, the space fine-$\mathcal{C}^0(U) =$ fine-$\mathcal{C}(U)$ is the set of finely continuous functions on U and the space fine-$\mathcal{C}^2(U)$ consists of functions $f \in$ fine-$\mathcal{C}^1(U)$ whose first fine derivatives belong to fine-$\mathcal{C}^1(U)$ as well. Finally, put

$$\text{fine-}\mathcal{C}^\infty(U) = \bigcap_{k=0}^{\infty} \text{fine-}\mathcal{C}^k(U).$$

See [62] for details. Moreover, for $k \in \mathbb{N}_0 \cup \{\infty\}$, we denote by $\mathcal{C}^k_{\text{f-loc}}(U)$ the set of all functions f on U such that, for each $z \in U$, there is $V \in \mathcal{F}_z$ and $F \in \mathcal{C}^k(\mathbb{R}^{m+1})$ with $F = f$ on V. It is easy to see that $\mathcal{C}^k_{\text{f-loc}}(U) \subset$ fine-$\mathcal{C}^k(U)$. A question whether these spaces coincide or not is discussed later on, see Section 4.3.

Finally, let us recall that the Sobolev space $W^{1,2}(\mathbb{R}^{m+1})$ consists of (Lebesgue) measurable functions F whose second power is integrable on $\mathbb{R}^{m+1}$ together with second powers of its first weak derivatives. Denote by $W^{1,2}_{\text{f-loc}}(U)$ the set of functions f on U satisfying that, for each $z \in U$, there exist $V \in \mathcal{F}_z$ and $F \in W^{1,2}(\mathbb{R}^{m+1})$ such that $F = f$ on V. For an account of the Sobolev spaces on fine domains, we refer to [73].

Finely holomorphic functions are closely related with finely harmonic ones (see [54]). For our purposes, let us recall one of their characterizations. A real-valued function f is finely harmonic on a finely open set $U \subset \mathbb{R}^{m+1}$ if and only if for every $z \in U$ there is $V \in \mathcal{F}_z$ such that $f|_V$, the restriction of f to V, is a uniform limit of functions f_n harmonic on open sets V_n containing V. Let us remark that f is harmonic on a usual open set $\Omega \subset \mathbb{R}^{m+1}$ if and only if f is finely harmonic and locally bounded (from above or below) on Ω. In case of $\mathbb{R}^2$ we need not assume local boundedness of f. Moreover, finely harmonic functions are finely continuous but, in general, have the first fine differential only almost everywhere (a.e.), see e.g. [58]. Next finely harmonic functions need not possess the unique continuation property. Indeed, by [81], there is a non-trivial finely harmonic function f in a fine domain U which vanish in some fine neighbourhood of a point of U.

Now we are ready to state some basic facts about finely holomorphic functions, see e.g. [56], [57] and [61]. Let $U \subset \mathbb{C}$ be finely open and let $f : U \to \mathbb{C}$. Then there are several equivalent definitions of finely holomorphic functions available. Indeed, a function f is finely holomorphic if one of the following (equivalent) conditions holds:

(FH1) f has a finely continuous fine derivative f' on U. Here

$$f'(\tilde{z}) = \text{fine-}\lim_{z\to\tilde{z}} \frac{f(z)-f(\tilde{z})}{z-\tilde{z}}, \ \tilde{z} \in U.$$

In other words, $f \in \text{fine-}\mathcal{C}^1(U)$ and $\bar{\partial}_{\mathrm{f}} f = 0$ on U. Here $z = x_0 + ix_1$ and

$$\bar{\partial}_{\mathrm{f}} f = \frac{1}{2}\left(\frac{\partial_{\mathrm{f}} f}{\partial x_0} + i\frac{\partial_{\mathrm{f}} f}{\partial x_1}\right).$$

(FH2) $f \in \mathcal{C}^1_{\text{f-loc}}(U)$ and $\bar{\partial}_{\mathrm{f}} f = 0$ on U.

(FH3) f is finely continuous on U, $f \in W^{1,2}_{\text{f-loc}}(U)$ and $\bar{\partial} f = 0$ on U,

(FH4) f is finely harmonic and $\bar{\partial}_{\mathrm{f}} f = 0$ a.e. on U.

(FH5) f and $zf(z)$ are finely harmonic (componentwise) on U.

(FH6) For each $z \in U$ there is $V \in \mathcal{F}_z$ such that the restriction $f|_V$ is a uniform limit of functions f_n holomorphic on open sets V_n containing V.

Now let us recall some remarkable properties of finely holomorphic functions. Obviously, by (FH5), a function f is holomorphic on a usual open set $\Omega \subset \mathbb{C}$ if and only if f is finely holomorphic on Ω because the same is true even for finely harmonic functions. It is a bit surprising that, in comparison with finely harmonic functions, they have much better properties. For example, if a function f is finely holomorphic, so is its fine derivative f'. Hence finely holomorphic functions are always infinitely fine differentiable. Moreover, finely holomorphic functions possess the unique continuation property. Namely, if f is finely holomorphic on a fine domain $U \subset \mathbb{C}$ and all its fine derivatives $f^{(k)}(\tilde{z})$, $k \in \mathbb{N}$, vanish at a point $\tilde{z} \in U$, then f is constant on U.

4.2 Finely monogenic functions

Before introducing finely monogenic functions we slightly modify in this part the definition of monogenic functions. As usual, let $\mathcal{C}\ell_m$ be the Clifford algebra $\mathbb{R}_{0,m}$ or $\mathbb{C}_m$ over $\mathbb{R}^m$, generated by the vectors $e_1, \ldots, e_m$. A vector $z = (x_0, \ldots, x_m)$ of $\mathbb{R}^{m+1}$ is now identified with the Clifford number $x_0 + x_1e_1 \cdots + x_me_m$ of $\mathcal{C}\ell_m$. Let $\mathcal{V}$ be an arbitrary $Pin(m)$-module, for example, $\mathcal{V} = \mathcal{C}\ell_m$. Then a function f defined and continuously differentiable in an open region Ω of $\mathbb{R}^{m+1}$ and taking values in the module $\mathcal{V}$ is called monogenic in Ω if it satisfies the equation $\bar{D}f = 0$ in Ω where the generalized Cauchy-Riemann operator $\bar{D}$ is defined as

$$\bar{D} = \partial_{x_0} + e_1\partial_{x_1} + \cdots + e_m\partial_{x_m}. \tag{4.1}$$

The advantage of this definition is the fact that, for $m = 1$ and $\mathcal{V} = \mathbb{R}_{0,1} \simeq \mathbb{C}$, monogenic functions obviously coincide with holomorphic ones without any other identifications. Put $D = \partial_{x_0} - e_1\partial_{x_1} - \cdots - e_m\partial_{x_m}$. As $\Delta = D\bar{D}$ monogenic functions are obviously harmonic. On the other hand, a function f is monogenic if and only if both f and $zf(z)$ are harmonic.

In what follows, we suppose that $\mathcal{V}$ is a $Pin(m)$-module, $U \subset \mathbb{R}^{m+1}$ is a finely open set and $f : U \to \mathcal{V}$ unless otherwise stated. As is obvious from the previous section, there are quite a few possible generalizations of finely holomorphic functions to higher dimensions, namely:

(FM1) $f \in$ fine-$\mathcal{C}^1(U)$ and $\bar{D}_{\mathrm{f}} f = 0$ on U. Here
$$\bar{D}_{\mathrm{f}} f = \frac{\partial_{\mathrm{f}} f}{\partial x_0} + e_1 \frac{\partial_{\mathrm{f}} f}{\partial x_1} + \cdots + e_m \frac{\partial_{\mathrm{f}} f}{\partial x_m}.$$

(FM2) $f \in \mathcal{C}^1_{\text{f-loc}}(U)$ and $\bar{D}_{\mathrm{f}} f = 0$ on U.

(FM3) f is finely continuous on U, $f \in W^{1,2}_{\text{f-loc}}(U)$ and $\bar{D} f = 0$ on U,

(FM4) f is finely harmonic and $\bar{D}_{\mathrm{f}} f = 0$ a.e. on U.

(FM5) f and $zf(z)$ are finely harmonic (componentwise) on U.

(FM6) For each $z \in U$ there is $V \in \mathcal{F}_z$ such that the restriction $f|_V$ is a uniform limit of functions f_n monogenic on open sets V_n containing V.

A natural question arises whether these conditions are equivalent to each other not only in dimension 2 but also in higher dimesions. In [L2], the following result is shown.

Theorem 17. *The conditions (FM3), (FM4) and (FM5) are equivalent to each other.*

In [L2], finely monogenic functions are defined as follows.

Definition 1. A function f is called finely monogenic if the function f and $zf(z)$ are both finely harmonic on U, that is, the condition (FM5) holds.

When $m = 1$ and $\mathcal{V} = \mathbb{R}_{0,1}$ finely monogenic functions obviously coincide with B. Fuglede's finely holomorphic functions. A function f is monogenic on a usual open set $\Omega \subset \mathbb{R}^{m+1}$ if and only if f is finely monogenic and locally bounded on Ω because the same is true even for finely harmonic functions. Moreover, when $m = 1$ we do not need to assume local boundedness of f. See [54, Theorem 10.16].

4.3 Finely differentiable monogenic functions

In this section, we discuss fine differentiability and finely differentiable monogenic functions. As for fine differentiability, we refer to [84, 85, 93, L3, L4, 62]. First we describe, in more detail, a relation between the function spaces fine-$\mathcal{C}^k(U)$ and $\mathcal{C}^k_{\text{f-loc}}(U)$ introduced in Section 4.1. Let us recall that the space fine-$\mathcal{C}^k(U)$ consists of finely continuously differentiable functions of order k on U and the space $\mathcal{C}^k_{\text{f-loc}}(U)$ is the set of functions on U which are finely locally extendable to usual $\mathcal{C}^k$ functions. As we mentioned before, we obviously have that $\mathcal{C}^k_{\text{f-loc}}(U) \subset$ fine-$\mathcal{C}^k(U)$. Moreover, the fact that $\mathcal{C}_{\text{f-loc}}(U) =$ fine-$\mathcal{C}(U)$ is known as the so-called Brelot property for finely continuous functions, see [55]. Now a quite natural question arises whether finely continuously differentiable functions have the corresponding Brelot property as well. In this connection, in [L3], it was proved that

Theorem 18. *If $U \subset \mathbb{R}^2$ is a finely open set, then*

$$\mathcal{C}^1_{\text{f-loc}}(U) = \text{fine-}\mathcal{C}^1(U).$$

In addition, if f is a function with $d_f f = 0$ on a fine domain U in $\mathbb{R}^2$, then the function f is constant on U.

Furthermore, in [L4], under a mild additional assumption, analogous results were obtained in higher dimensions as well. In particular, in [L4], it is shown that, for a given finely open set $U \subset \mathbb{R}^{m+1}$, we have that

$$\mathcal{C}^1_{\text{f-loc}}(U) = \text{fine-}\mathcal{C}^1(U) \cap W^{1,2}_{\text{f-loc}}(U).$$

Using this result and Theorem 17, it is not difficult to show the following characterization of finely differentiable monogenic functions (see [75]).

Theorem 19. *A function f is finely monogenic and $f \in \text{fine-}\mathcal{C}^1(U)$ if and only if $f \in \mathcal{C}^1_{\text{f-loc}}(U)$ and $\bar{D}_f f = 0$ on U.*

Finally, in [62], S. Gardiner obtained the following results.

Theorem 20. *If $U \subset \mathbb{R}^{m+1}$ is a finely open set and $k \in \mathbb{N} \cup \{\infty\}$, then*

$$\mathcal{C}^k_{\text{f-loc}}(U) = \text{fine-}\mathcal{C}^k(U).$$

In addition, if f is a function with $d_f f = 0$ on a fine domain U in $\mathbb{R}^{m+1}$, then the function f is constant on U.

Corollary 2. *Let $U \subset \mathbb{R}^{m+1}$ be a finely open set and let $f \in \text{fine-}\mathcal{C}^2(U)$.*

(i) Then f is finely harmonic if and only if $\Delta_f f = 0$ on U. Here the fine laplacian Δ_f is given by

$$\Delta_f = \frac{\partial_f^2}{\partial x_0^2} + \cdots + \frac{\partial_f^2}{\partial x_m^2}.$$

(ii) If, in addition, $m = 1$ and the function f is finely harmonic then $\partial_f f$ is finely holomorphic and $f \in \text{fine-}\mathcal{C}^\infty(U)$. Here

$$\partial_f f = \frac{1}{2}\left(\frac{\partial_f f}{\partial x_0} - i\frac{\partial_f f}{\partial x_1}\right).$$

Now we are ready to summarize what is known about finely monogenic functions.

Theorem 21. *The following statements about the conditions (FM1)-(FM6) of Section 4.2 hold.*

(i) The conditions (FM1) and (FM2) are equivalent to each other.

(ii) The conditions (FM3), (FM4) and (FM5) are equivalent to each other.

(iii) The condition (FM2) implies (FM3).

In addition, for $f \in \text{fine-}\mathcal{C}^1(U)$, the condition (FM3) implies (FM2) and the function f is finely monogenic if and only if $\bar{D}_f f = 0$ on U.

(iv) The condition (FM6) implies (FM5). The condition (FM2) implies (FM6).

(v) For $f \in \text{fine-}\mathcal{C}^1(U)$, the conditions (FM1)-(FM6) are all equivalent to each other.

In the proof of Theorem 21 given below, we use the Théodoresco integral transform T in $\mathbb{R}^{m+1}$. So let us recall briefly the definition and some properties of this transform, see [70, Sections 3.1 and 3.2] for details. Let β_{m+1} be the m-dimensional area of the unit sphere in $\mathbb{R}^{m+1}$ and define

$$E(z) = \frac{1}{\beta_{m+1}} \frac{\bar{z}}{|z|^{m+1}}, \;\; z \neq 0.$$

Then E is a fundamental solution of the differential operator $\bar{D}$, in particular, $\bar{D}E(z) = 0$ for $z \neq 0$. Of course, for $m = 1$, $E(z) = 1/(2\pi z)$ is the well-known Cauchy kernel from complex analysis. Denote by $\mathcal{C}_c^k(\mathbb{R}^{m+1})$ the set of functions of $\mathcal{C}^k(\mathbb{R}^{m+1})$ with a compact support in $\mathbb{R}^{m+1}$. Then, for $f \in \mathcal{C}_c(\mathbb{R}^{m+1})$, we define

$$T(f)(z) = -\int_{\mathbb{R}^{m+1}} E(z-y) f(y)\, dy, \;\; z \in \mathbb{R}^{m+1}.$$

Obviously, $T(f)$ is monogenic in an open subset Ω of $\mathbb{R}^{m+1}$ if $f = 0$ on Ω. Moreover, it is well-known that each function $f \in \mathcal{C}_c^1(\mathbb{R}^{m+1})$ can be expressed on $\mathbb{R}^{m+1}$ as $f = T(\bar{D}f)$ (see [70, Theorem 3.23]). In the proof, we also need the following routine estimate.

Lemma 2. *Let $V \subset \mathbb{R}^{m+1}$ be of a finite Lebesgue measure and let α_{m+1} be the Lebesgue measure of the unit ball in $\mathbb{R}^{m+1}$. Then we have that*

$$\int_V |E(z-y)|\, dy \leq (\lambda^{m+1}(V)/\alpha_{m+1})^{1/(m+1)}, \quad z \in \mathbb{R}^{m+1}.$$

Here λ^{m+1} is the Lebesgue measure in $\mathbb{R}^{m+1}$.

Proof. Let $z \in \mathbb{R}^{m+1}$ and take $r \geq 0$ such that $\lambda^{m+1}(V) = \lambda^{m+1}(B(z,r))$. Then we have that $r = (\lambda^{m+1}(V)/\alpha_{m+1})^{1/(m+1)}$ and

$$\int_V |E(z-y)|\, dy \leq \frac{1}{\beta_{m+1}} \int_{B(z,r)} \frac{dy}{|z-y|^m} = r,$$

which finishes the proof. □

Proof of Theorem 21. The statement (i) follows directly from Theorem 20. For the statement (ii), see Theorem 17. Finally, Theorems 19 and 20 give easily (iii). Moreover, obviously, the condition (FM6) implies (FM5).

It remains to show only that (FM2) implies (FM6). Let thus $f \in \mathcal{C}^1_{\text{f-loc}}(U)$, $\bar{D}_{\text{f}} f = 0$ on U and $\tilde{z} \in U$. Then there is a bounded set $V \in \mathcal{F}_{\tilde{z}}$ and a function $F \in \mathcal{C}_c^1(\mathbb{R}^{m+1})$ such that $F = f$ on V. Of course, we have that $\bar{D}F = 0$ on V. As we know, the function F can be expressed as $F = T(\bar{D}F)$. Furthermore, choose a sequence of open sets $V_n \subset \mathbb{R}^{m+1}$ containing V such that $\lambda^{m+1}(V_n \setminus V) \to 0$ as $n \to \infty$ and define $f_n = T(\bar{D}F(1 - \chi_{V_n}))$. Here χ_M is the characteristic function of a set M. Obviously, each function f_n is monogenic on V_n and, by Lemma 2, we have that

$$\sup_{z\in V} |f_n(z) - f(z)| = \sup_{z\in V} |T(\bar{D}F\chi_{V_n\setminus V})(z)| \leq$$
$$\leq (\lambda^{m+1}(V_n \setminus V)/\alpha_{m+1})^{1/(m+1)} \sup_{z\in\mathbb{R}^{m+1}} |\bar{D}F(z)|,$$

which tends to zero as $n \to \infty$. So the proof is finished. □

4.4 Open problems

As we have mentioned, in comparison with finely harmonic functions, finely holomorphic functions are infinitely fine differentiable everywhere and have the unique continuation property. It would be interesting to clear up to what extent these properties remain true for finely monogenic functions in higher dimensions. In particular, if finely monogenic functions were finely continuously differentiable then, by the statement (v) of Theorem 21, the conditions (FM1)-(FM6) of Section 4.2 would be all equivalent to each other.

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