Elementary Riemann surfaces

Riemann surfaces (RS) are natural domains of definition for multi-valued functions in Complex Analysis. At this moment, we treat RS informally and give a rigorous definition in the next lecture

LOGARITHM

We know that the exponential

$$\exp(z) := e^x(\cos y + i \sin y), \quad z = x+iy \in \mathbb{C}$$

has e.g. the following properties:

(a) $\exp: \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}\setminus\{0\}$ is holomorphic, i.e., complex differentiable

(b) exp is not one-to-one on $\mathbb{C}$ but it is $2\pi i$-periodic and even it holds that

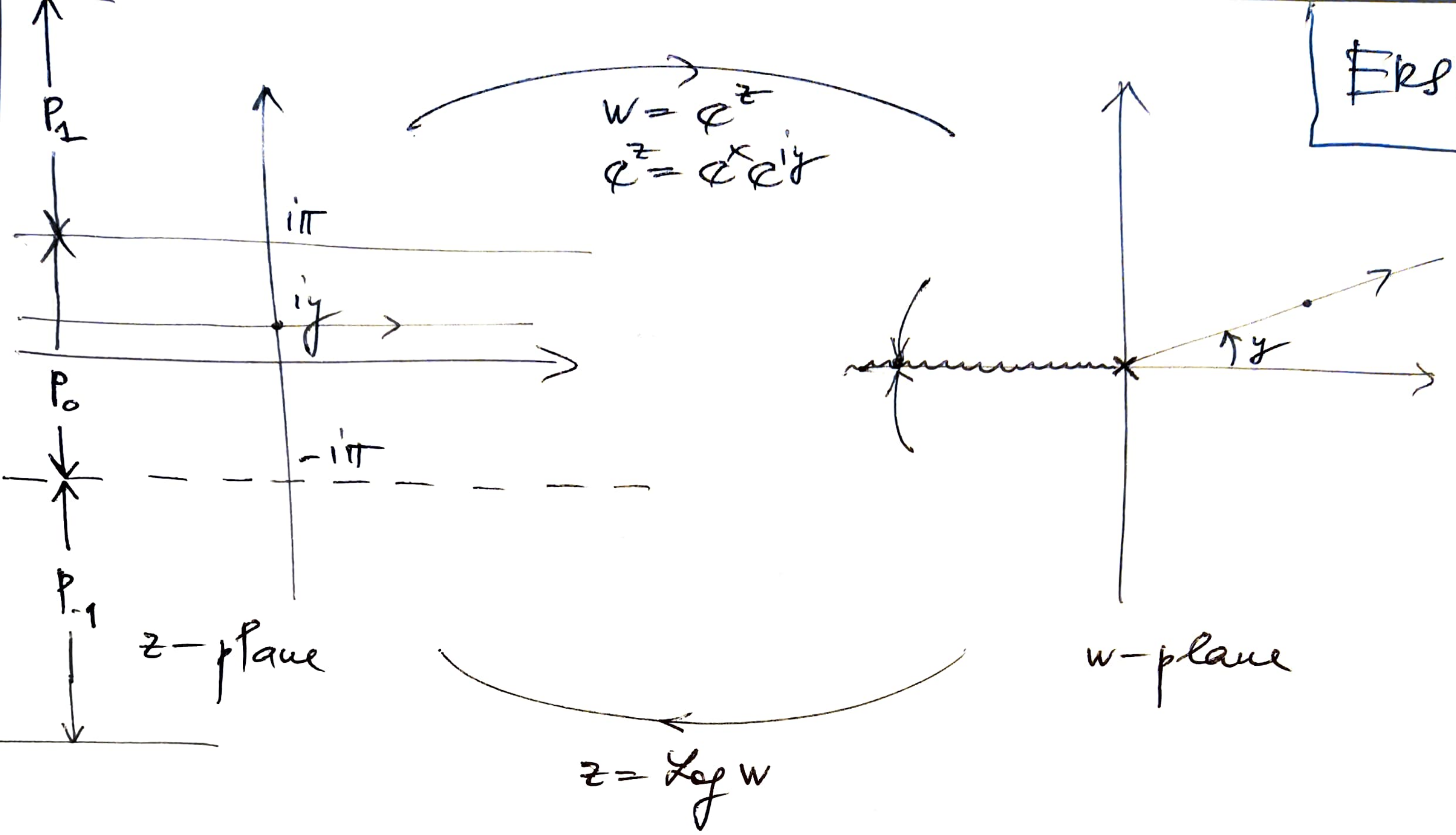
$$e^z = e^w \iff \exists k \in \mathbb{Z} : w = z + 2k\pi i$$

(c) Let $k \in \mathbb{Z}$ and

$$P_k := \{z \in \mathbb{C} \mid -\pi + 2k\pi < \operatorname{Im} z \leq \pi + 2k\pi\}$$

Then $\exp|_{P_k} : P_k \xrightarrow{\text{onto}} \mathbb{C}\setminus\{0\}$ is one-to-one.

Ers 2



$\boxed{\text{Problem}}$ What is an inverse of exp?

For $w \in \mathbb{C} \setminus \{0\}$, we get the multivalued function
$$\mathcal{L}og\, w \overset{\text{def.}}{:=} \{ z \in \mathbb{C} \mid w = e^z \}.$$
[OUR CHOICE!]

We have $0 \neq w = |w| e^{i \arg w}$ where $\arg w \in (-\pi, \pi]$ is the principal value of the argument of w.

Then $0 \neq w = e^{\log|w| + i \arg w} = e^z$ iff
$$z = \log|w| + i \cdot \arg w + 2k\pi i$$
for some $k \in \mathbb{Z}$. Then

(d) $\mathcal{L}og\, w = \{ \log w + 2k\pi i \mid k \in \mathbb{Z} \}$, $w \in \mathbb{C} \setminus \{0\}$

where $\log w := \log|w| + i \cdot \arg w$ is the principal value of the logarithm of w.

ERS3

(a) $\operatorname{Log} = \left(\exp|_{P_0}\right)^{-1}$ is holomorphic on $\mathbb{C}\setminus(-\infty,0]$ but <u>not</u> continuous at any $w\in(-\infty,0)$.

$\boxed{\text{QUESTION}}$ Is it possible to represent Log as a single-valued function? <u>YES!</u>

(1.) For each $k\in\mathbb{Z}$, consider the (continuous) holomorphic <u>branch</u> of Log

$$L_k(w) := \operatorname{Log} w + 2k\pi i, \quad w\in\mathbb{C}\setminus(-\infty,0] =: C_k$$

Of course, $L_k = \left(\exp|_{P_k^\circ}\right)^{-1}$. Here P_k° is the interior of the strip P_k.

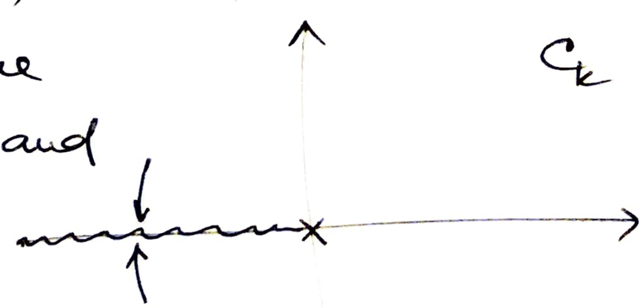
(2.) For $w\in(-\infty,0)$,

$$L_k(w+) = \operatorname{Log} w + 2k\pi i, \quad L_k(w-) = \operatorname{Log} w + 2(k-1)\pi i$$

where $L_k(w\pm) \overset{\text{def}}{:=} \lim\limits_{\substack{v\to w\\ \pm \operatorname{Im} v>0}} L_k(v)$.

In particular, $L_k(w+) = L_{k+1}(w-)$.

(3.) The cut $(-\infty,0)$ of the <u>sheet</u> C_k has the upper and the lower edge.



For each $k \in \mathbb{Z}$, glue together the upper edge of the cut of C_k and the lower " " C_{k+1}.

We obtain a surface R, which looks like a spiral stairway or a screw (infinite in the both directions).

By (2.), there is a unique <u>continuous single-valued</u> function L on R defined as L_k on any sheet C_k. This function L represents $\mathcal{L}og$. We call R the Riemann surface of $\mathcal{L}og$.

POWER FUNCTIONS

For $\alpha \in \mathbb{C}$ and $z \in \mathbb{C} \setminus \{0\}$, put

- $z^\alpha := \exp(\alpha \cdot \log z)$ ... principal value
- $M_\alpha(z) := \{\exp(\alpha \cdot w) \mid w \in \mathrm{Log}\, z\}$.

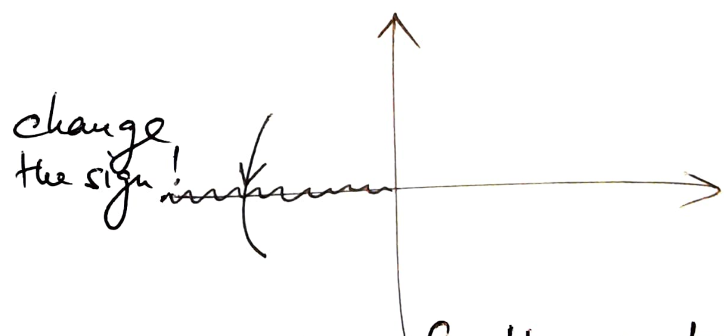
Then $M_\alpha(z) = \{z^\alpha e^{2\pi k \alpha i} \mid k \in \mathbb{Z}\}$.

Example Construct RS for the square root, i.e., for the multi-valued function

$$M_{1/2}(z) = \{w \in \mathbb{C} \mid w^2 = z\} = \{\pm \underbrace{\sqrt{z}}_{\text{principal value}}\}, \quad z \in \mathbb{C} \setminus \{0\}.$$

We have 2 branches

$$F_\pm(z) = \pm\sqrt{z}, \quad z \in \mathbb{C} \setminus (-\infty, 0] =: C_\pm$$



We glue together the upper edge of the cut of $C_\pm$
and the lower —"— $C_\mp$

Then we obtain the RS X of '$w = \sqrt{z}$'
$M_{1/2}$

and $M_{1/2}$ is represented by a continuous single-valued function on X.

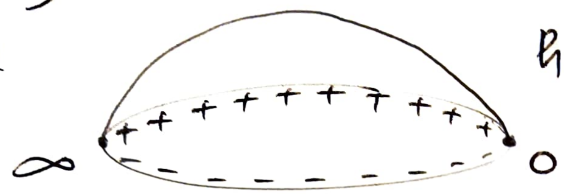
Moreover, it turns out that

$X \simeq S^2$ without 2 points corresponding $0, \infty$.

'looks like' $\equiv$ 'you can deform'

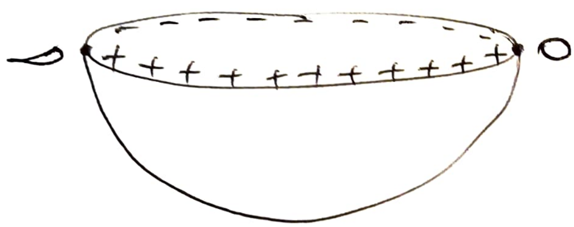
Indeed, we have

$C_+ \simeq$

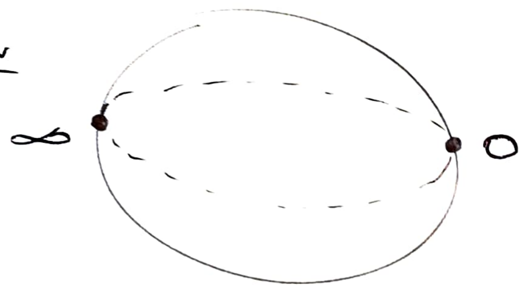


hemisphere

$C_- \simeq$



$X \simeq$



$\boxed{\text{EXERCISE}}$ Describe RS for '$w = \sqrt[n]{z}$', $n \in \mathbb{N}$.

$M_{1/n}$

$\boxed{\text{Ex.}}$ RS for "$w = \sqrt{(z^2-1)(z^2-4)}$", i.e.,

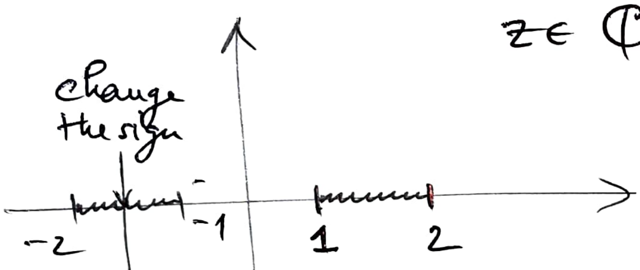
for $F(z) := \{ w \in \mathbb{C} \mid w^2 = (z^2-1)\cdot(z^2-4) \}$, $z \in \mathbb{C}$.

— x —

- $F(z)$ is double-valued for $z \neq \pm 1, \pm 2$;
  $F(z) = 0$ for $z = \pm 1, \pm 2$.

- Branches

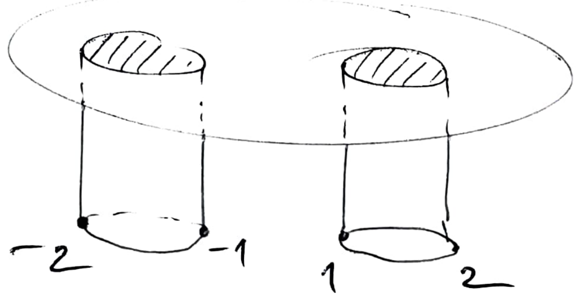
$$f_\pm(z) = \pm \sqrt{z-1}\cdot\sqrt{z-2}\cdot\sqrt{z+1}\cdot\sqrt{z+2},$$
$$z \in \mathbb{C} \setminus ([-2,-1] \cup [1,2]) =: C_\pm$$

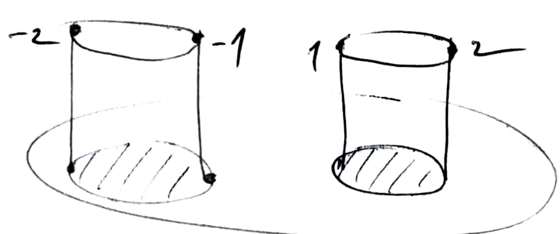


- We glue the upper edges of the cuts of $C_\pm$
  and lower —"— $C_\mp$.
  We obtain the RS X for F.

- Moreover, $X \simeq$ torus without 2 points
  (corresponding to ∞!)

Indeed, we have

$C_+ \simeq$ 

$C_- \simeq$ 

[two planes joined by two tubes]

$X \simeq$