Hyperelliptic RS

Let $w^2 = p(z)$ with a polynomial p of degree $k \geq 1$ having just <u>simple</u> roots.
Putting $P(z,w) := w^2 - p(z)$, consider the algebraic curve $X_P =: X$.

(1.) X is <u>regular</u>: Indeed, $\frac{\partial P}{\partial w} = 2w \neq 0$ if $w \neq 0$. If $w = 0$ and $p(z) = 0$, then

$$\frac{\partial P}{\partial z}(z,0) = -p'(z) \neq 0$$

because z is a <u>simple</u> root of p.

(2.) Let $\pi : X \longrightarrow \mathbb{C}$ be the z-projection, i.e.,

$$\pi(z,w) := z, \quad (z,w) \in X.$$

If z is not a root of p, then

$$\pi^{-1}(z) = \{(z, \pm p(z)^{1/2})\}.$$

Every root z of p is a <u>critical value</u> (<u>branching point</u>) of π. Thus we have

$$\boxed{\deg \pi = 2} \quad \text{and} \quad \boxed{b_{\pi} = k}$$

(3.) <u>Points at ∞</u>: Let $p(z) = a_0 z^k + \dots + a_k$, $a_0 \neq 0$.
(a) Let $\boxed{k = 2m}$. Then there is $R > 0$ big enough such that the multivalued function $w^2 = p(z)$

RH13

has two <u>different</u> holomorphic branches

$$w_\pm(z) := \pm\sqrt{a_0}\, z^m \cdot \left(1+\frac{a_1}{a_0 z}+\dots+\frac{a_k}{a_0 z^k}\right)^{1/2},$$

$|z|>R$.

Of course, both $w_\pm$ have a pole at ∞ of degree m.
Hence $\boxed{N=2}$ and $g = 1+m-\frac{1}{2}\cdot 2-\frac{1}{2}\cdot 2$

$\boxed{g=m-1}$.

$\boxed{\text{Conclusion}}$ $X \simeq \Sigma_{m-1}$ with 2 points at ∞ removed

(b) Let $\boxed{k=2m+1}$. Then there is $R>0$ such that $w^2=p(z)$ has branches

$$w_\pm(z) := \pm\sqrt{a_0}\, z^m \cdot z^{1/2} \left(1+\frac{a_1}{a_0 z}+\dots+\frac{a_k}{a_0 z^k}\right)^{1/2},$$

$|z|>R$ a $z\notin(-\infty,0]$.

As we know (see Lecture 1) we can 'glue together' these branches across the cut to get one 'double' point at ∞.
Hence $\boxed{N=1}$ and $g = 1+m+\frac{1}{2}-\frac{1}{2}\cdot 2-\frac{1}{2} \Rightarrow$

$\boxed{g=m}$.

$\boxed{\text{Conclusion}}$ $X \simeq \Sigma_m$ with 1 point at ∞ removed

RH14

As for 3. (a), put $\overline{X} := X \cup \{\infty_\pm\}$

and $U_\pm := \{(z, w_\pm(z)) \mid |z| > R_1\} \cup \{\infty_\pm\}$,

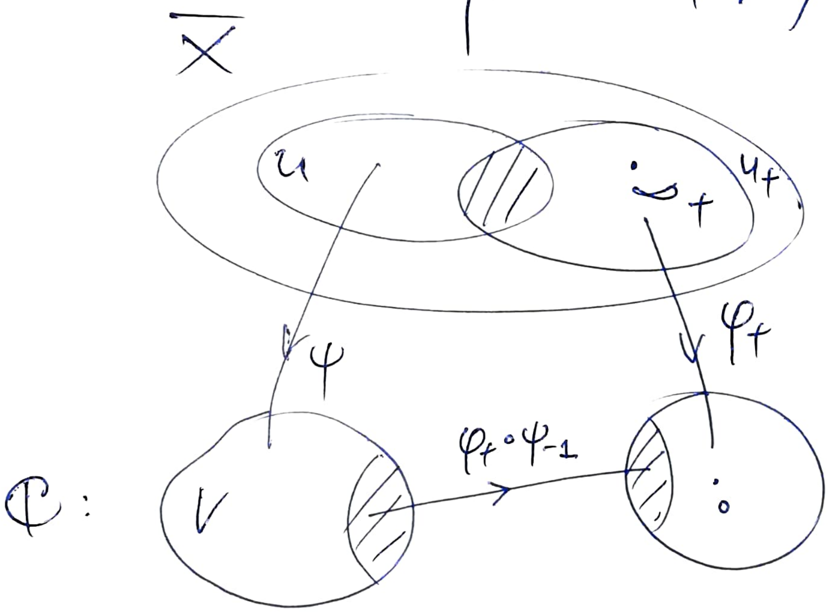
$$\varphi_\pm(z,w) := \frac{1}{z}^{*)}, \quad (z,w) \in U_\pm \setminus \{\infty_\pm\};$$

$$:= 0, \quad (z,w) = \infty_\pm.$$

(i) Let $\mathcal{A}$ be a conformal atlas on X.

Then $\overline{\mathcal{A}} := \mathcal{A} \cup \{(U_\pm, U(0, 1/R_1), \varphi_\pm)\}$ is a conformal atlas on $\overline{X}$.

Indeed, let (U, V, φ) be a local chart on X such that, say, $U \cap U_+ \neq \emptyset$. Then the transition function



$$\varphi_+ \circ \varphi^{-1} = \frac{1}{\pi \circ \varphi^{-1}}$$

is holomorphic on $\varphi(U \cap U_+)$ because $\pi : X \longrightarrow \mathbb{C}$ is holomorphic.

(ii) $\overline{X}$ is <u>connected</u> because X is connected

(iii) $\overline{X}$ is <u>compact</u>: The sets $U(\infty_\pm, \varepsilon) := U_\pm \cap \{(z,w) \in \mathbb{C}^2 \mid |z| > 1/\varepsilon\} \cup \{\infty_\pm\}$ for $0 < \varepsilon \leq 1/R_1$ form a basis of nbh's of $\infty_\pm$

*) $\varphi_\pm = \frac{1}{\pi}$

Let $\{U_\alpha \mid \alpha \in A\}$ be an open cover of $\overline{X}$. Then there are $\alpha_\pm \in A$ s.t. $U_{\alpha_\pm} \ni \infty_\pm$ **RH15**

Then there is $\varepsilon \in (0, 1/R]$ s.t.

$$U_{\alpha_\pm} \supset U(\infty_\pm, \varepsilon).$$

Since π is proper the set $\pi^{-1}(\overline{U(0, 1/\varepsilon)})$ is compact and thus there is a finite set K in A s.t. $\{U_\alpha \mid \alpha \in K\}$ is a cover of $\pi^{-1}(\overline{U(0, 1/\varepsilon)})$. Then $\{U_\alpha \mid \alpha \in K \cup \{\alpha_\pm\}\}$ is a finite subcover of $\overline{X}$.

Pr: $w^2 = z^{2m+1} \iff w = \pm z^{1/2} z^m \iff w = \xi^{2m+1}$, $\xi = \pm z^{1/2}$, $\xi^2 = z$

singular at $(0,0)$

We have $X := \{(z, w) \in \mathbb{C}^2 \mid w^2 = z^{2m+1}\}$

$$= \{(\xi^2, \xi^{2m+1}) \mid \xi \in \mathbb{C}\}.$$

Moreover, ξ is uniquely determined by ξ^2 and ξ^{2m+1}.

As for (3.) (b),

put $P(\infty, \varepsilon) := \{ z \in \mathbb{C} \mid |z| > 1/\varepsilon \}$ and

$U(\infty, \varepsilon) := P(\infty, \varepsilon) \cup \{\infty\}$ for $\varepsilon > 0$.

Then $w^2 = h(z)^{2m+1}$ where

$$h(z) := \sqrt[k]{a_0}\, z \cdot \left(1 + \frac{a_1}{a_0 z} + \cdots + \frac{a_k}{a_0 z^k}\right)^{1/k}, \quad z \in P(\infty, \varepsilon)$$

Since h has a simple pole at ∞ we can assume that $\varepsilon > 0$ is so small that h is a conformal map on $U(\infty, \varepsilon)$. For $z \in P(\infty, \varepsilon)$, $(z, w) \in X$ iff

$w = \pm h(z)^{1/2} (h(z))^m$ iff $w = \xi^{2m+1}$ for $\xi = \pm h(z)^{1/2}$

$\xi^2 = h(z)$

$z = h^{-1}(\xi^2)$

Remark: ξ is uniquely determined by ξ^2 and ξ^{2m+1}.

Put $\overline{X} := X \cup \{\infty_0\}$ and

$U_0 := \{ (h^{-1}(\xi^2), \xi^{2m+1}) \mid \xi \in P(\infty, \delta) \} \cup \{\infty_0\}$,

$\varphi_0 (h^{-1}(\xi^2), \xi^{2m+1}) := \frac{1}{\xi}$, $\xi \in P(\infty, \delta)$;

$\varphi_0(\infty_0) := 0$. Here $\delta > 0$ is small enough.

(i) Let $\mathcal{A}$ be a conformal atlas on X.
Then $\overline{\mathcal{A}} := \mathcal{A} \cup \{ (U_0, U(0, \delta), \varphi_0) \}$ be a conformal atlas on $\overline{X}$.

(ii) $\overline{X}$ is connected and compact.

$\boxed{\text{Exercise}}$

EXERCISE Show that RH17

1. $X := \{ (z,w) \in \mathbb{C}^2 \mid w^3 = z^3 - z \} \simeq \Sigma_1$ without 3 points at ∞;

2. Affine Fermat curve: For $n \in \mathbb{N}$,

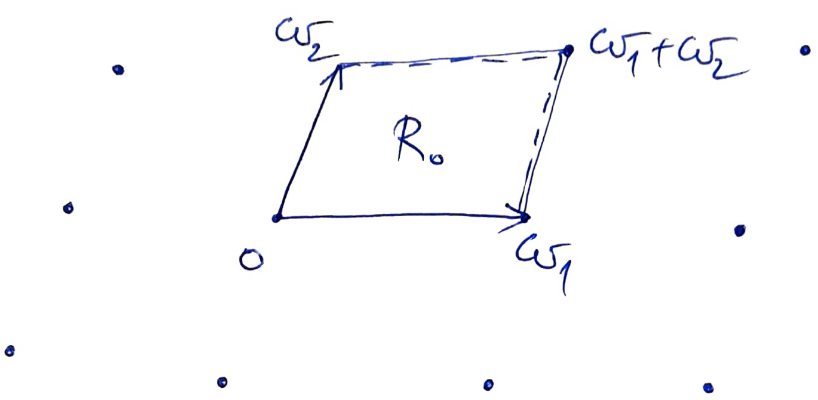
$X_n := \{ (z,w) \in \mathbb{C}^2 \mid w^n + z^n = 1 \} \simeq \Sigma_g$ without n points at ∞ where $g = \frac{1}{2}(n-1)\cdot(n-2)$

Elliptic functions

Let $\omega_1, \omega_2 \in \mathbb{C}\setminus\{0\}$, $\omega_2/\omega_1 \notin \mathbb{R}$ and $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$.
Then

$$R_0 := \{t_1\omega_1 + t_2\omega_2 \mid t_1, t_2 \in [0,1)\}$$

is the <u>fundamental parallelogram</u> for the lattice L.



Recall that <u>elliptic functions</u> f (i.e., $f \in M(\mathbb{C}/L)$) are the functions meromorphic and doubly periodic on $\mathbb{C}$, i.e.,

$$f(z+\omega_1) = f(z) = f(z+\omega_2) \quad \forall z \in \mathbb{C}.$$

$\boxed{\text{Exercise}}$ Let X be a <u>connected</u> RS. Then $M(X)$ is a field, the so-called <u>field of fractions for X</u>.

We know $M(\hat{\mathbb{C}}) = \{\text{rational functions}\}$.

$\boxed{\text{Problem}}$ Describe $M(\mathbb{C}/L)$.

Theorem: Let $f \not\equiv 0$ be an elliptic function. | EL2

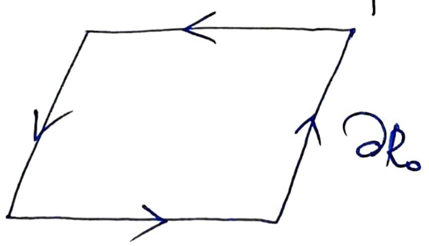
Then (1.) in R_0 the number of zero points of f is equal to the number of poles of f if multiplicities are counted, i.e.,

$$\sum_{\substack{z \in R_0 \\ f(z)=0}} m_f(z) = \sum_{\substack{z \in R_0 \\ f(z)=\infty}} m_f(z);$$

(2.) $$\sum_{\substack{z \in R_0 \\ f(z)=\infty}} \operatorname{res}_z f = 0;$$

(3.) $$\sum_{\substack{z \in R_0 \\ f(z)=0}} m_f(z) \cdot z \sim \sum_{\substack{z \in R_0 \\ f(z)=\infty}} m_f(z) z$$ where $z_1 \sim z_2$ means that $z_2 - z_1 \in L$.

Pf: We can view ∂R_0 as a positively oriented closed curve



WLOG, we can assume that on ∂R_0 there are no zero points nor poles of f. Otherwise, we move ∂R_0 a bit!

EL3

(1.) We have

$$0 \overset{(b)}{=} \int_{\partial R_0} \frac{f'}{f} \overset{RT}{=} 2\pi i \sum_{z_0 \in R_0} \operatorname{res}_{z_0} \frac{f'}{f} \overset{(a)}{=} \sum_{\substack{z_0 \in R_0 \\ f(z_0)=0}} n_f(z_0) - \sum_{\substack{z_0 \in R_0 \\ f(z_0)=\infty}} n_f(z_0).$$

the residue theorem

Indeed, we have

(a) Let $f(z) = (z-z_0)^n g(z)$ on a ngh of z_0 for some $n \in \mathbb{Z}$ and $g(z_0) \neq 0$, g holomorphic at z_0. Then on a ngh of z_0

$$\frac{f'(z)}{f(z)} = \frac{n(z-z_0)^{n-1} g(z) + (z-z_0)^n g'(z)}{f(z)} = \frac{n}{z-z_0} + \frac{g'(z)}{g(z)}$$

holomorphic at z_0

Hence $\operatorname{res}_{z_0} \frac{f'}{f} = n$.

(b) Since f is doubly periodic we have

$$\int_0^{\omega_1} f = \int_{\omega_2}^{\omega_1+\omega_2} f \quad \text{and} \quad \int_0^{\omega_2} f = \int_{\omega_1}^{\omega_1+\omega_2} f, \text{ so } \int_{\partial R_0} f = 0.$$

Moreover, f'/f is also doubly periodic.

(2.) We have

$$0 \overset{(b)}{=} \int_{\partial R_0} f \overset{RT}{=} 2\pi i \sum_{\substack{z \in R \\ f(z)=\infty}} \operatorname{res}_z f.$$

EL4

(3.) We have

$$L \overset{(a)}{\Rightarrow} \frac{1}{2\pi i} \int_{\partial R_0} z \frac{f'(z)}{f(z)}\, dz \overset{RT}{=} \sum_{z_0 \in R_0} \operatorname{res}_{z_0}\left(z \frac{f'(z)}{f(z)} \right) =$$

$\overset{(c)}{=}$ the difference of the sums in (3.)

Indeed, we have

(c) $$z \frac{f'(z)}{f(z)} = z_0 \frac{f'(z)}{f(z)} + (z - z_0) \frac{f'(z)}{f(z)}$$ on a ngh of z_0

Then $\operatorname{res}_{z_0}\left(z \frac{f'(z)}{f(z)} \right) = n z_0$ if f is as in (a).

(d) We have

$$\int_0^{\omega_1} z \frac{f'(z)}{f(z)}\, dz - \int_{\omega_2}^{\omega_1 + \omega_2} z \frac{f'(z)}{f(z)}\, dz =$$

$$-\|- \quad - \int_0^{\omega_1} (z + \omega_2) \frac{f'(z)}{f(z)}\, dz = - \omega_2 \int_0^{\omega_1} \frac{f'}{f} \in 2\pi i \omega_2 \mathbb{Z}$$

Indeed, for the closed curve $\varphi(t) := f(t\omega_1)$, $t \in [0,1]$,

$$\int_0^{\omega_1} \frac{f'}{f} = \int_0^1 \frac{\varphi'(t)}{\varphi(t)}\, dt = \int_\varphi \frac{dz}{z} = 2\pi i \underbrace{\operatorname{ind}_\varphi 0}_{\in \mathbb{Z}}.$$

∎

Remark (i) The condition (1.) of Theorem is valid for $M(X)$ where X is a compact and connected RS (see Theorem on the degree of a holomorphic map).

ELF

(ii) It is possible to prove

Unique presentation by principal parts: An elliptic function is determined uniquely, up to an additive constant, by its principal parts at all poles in R_0. Such an elliptic function exists iff the prescribed principal parts satisfy (2.) of Theorem.

Unique presentation by zeroes and poles: An elliptic function $f \not\equiv 0$ is determined uniquely, up to a multiple, by its zeroes and poles in R_0 with multiplicities. Such an elliptic function exists iff the prescribed zeroes and poles satisfy (1.) and (3.) of Theorem.

Exercise

[A proof of uniqueness is very easy, existence is more difficult.]