The Weierstrass $\wp$-function

Let $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$, $\omega_2/\omega_1 \notin \mathbb{R}$ and $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$.
We know that an elliptic function cannot have a single simple pole in the period parallelogram R_0.
Try a double pole at 0.

DEF. Put

$$(*) \quad \wp(u) := \frac{1}{u^2} + \sum_{\omega \in L^*} \left(\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2} \right),$$

where $L^* := L \setminus \{0\}$.

Properties of $\wp$:

(1.) $\wp$ is a meromorphic function on $\mathbb{C}$. In particular, the series in $(*)$ converges uniformly on every compact $K \subset \mathbb{C}$ provided that we remove finitely many terms with poles in K.

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$\boxed{\text{LEMMA}}$ For $n \in \mathbb{N}$, put $P_n := \{ z \in \mathbb{C} \mid n-1 \le |z| < n \}$.
Then there is $C \in (0, +\infty)$ such that

$$\# P_n \cap L \le C \cdot n, \quad n \in \mathbb{N}.$$

Here $\# A$ is the number of elements of a set A.

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<u>Proof</u>: Let us consider the partition of the plane $\mathbb{C}$ into the parallelograms with vertices at points of the given lattice L. For $n \in \mathbb{N}$ denote by k_n the number of closed parallelograms of the partition which intersect P_n and by $d>0$ the diameter of R_0.

Then

$$k_n \, |R_0| \leq \left| \{ z \in \mathbb{C} \mid n-d-1 \leq |z| \leq n+d \} \right| =$$

the area of R_0

$$= \pi \cdot \left((n+d)^2 - (n-d-1)^2 \right) = \pi \cdot (2n-1) \cdot (2d+1)$$

$$\leq C_1 \cdot n, \quad n \in \mathbb{N},$$

for some $C_1 \in (0, +\infty)$. Hence, for each $n \in \mathbb{N}$,

$$\# P_n \cap L \leq k_n \leq \left(C_1 / |R_0| \right) n. \quad \blacksquare$$

<u>Proof of 1.</u>: Let $|u| < R < +\infty$ and $|\omega| > 2R$.

Then

$$\left| \frac{1}{(u-\omega)^2} - \frac{1}{\omega^2} \right| \leq \frac{|u|^2 + 2|u||\omega|}{|u-\omega|^2 |\omega|^2} \leq \frac{4R^2}{|\omega|^4} + \frac{8R}{|\omega|^3}$$

because $|u-\omega| = |\omega| \cdot \underbrace{\left| 1 - \frac{u}{\omega} \right|}_{|\cdots| < \frac{1}{2}} > \frac{1}{2} |\omega|$.

By LEMMA, $\sum_{\omega \in L^*} \frac{1}{|\omega|^\lambda} < +\infty$ if $\lambda > 2$.

Indeed, we have

$$\sum_{\substack{\omega \in L \\ |\omega| \geq 1}} \frac{1}{|\omega|^\lambda} = \sum_{n=2}^{+\infty} \sum_{\omega \in P_n \cap L} \frac{1}{|\omega|^\lambda} \leq \sum_{n=2}^{+\infty} \underbrace{\frac{c \cdot n}{(n-1)^\lambda}}_{\frac{1}{n^{\lambda-1}}} < +\infty$$

for $\lambda > 2$. ▨

(2.) We have $\wp'(u) = -2 \sum_{\omega \in L} \frac{1}{(u-\omega)^3}$

Obviously, $\wp'$ is an elliptic function with a triple pole at 0 and $\wp'$ is odd (i.e., $\wp'(-u) = -\wp'(u)$)

(3.) $\wp$ is an elliptic function with a double pole at 0 and $\wp$ is even (i.e., $\wp(-u) = \wp(u)$).

By $(*)$, $\wp$ is even.

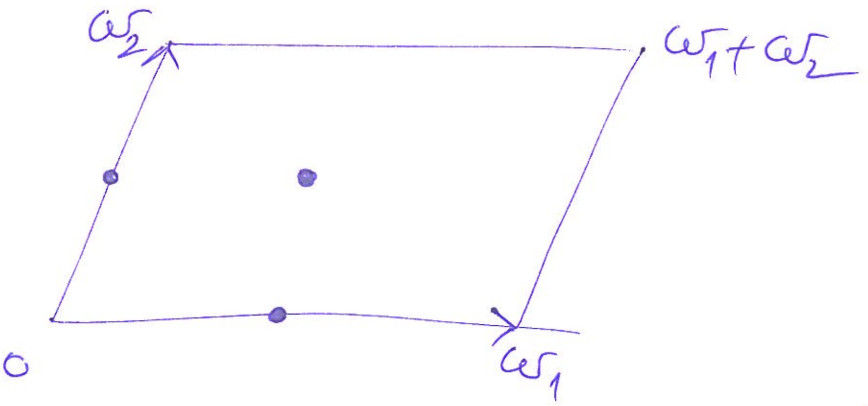
For $i = 1,2$, $\frac{d}{du}\left(\wp(u+\omega_i) - \wp(u)\right) = \wp'(u+\omega_i) - \wp'(u)$
$= 0$,

thus $\wp(u+\omega_i) - \wp(u)$ is constant. Moreover, for $u = -\omega_i/2$, this function is zero because $\wp$ is even.

(4.) $\wp$ and $\wp'$ is a holomorphic map of $\mathbb{C}/L$ onto $\mathbb{S}$ of degree 2 and 3, respectively

⌜because $\wp$ and $\wp'$ have a single pole at 0 of degree 2 and 3 modulo L.⌟

(5.) The only zeroes of $\wp'$ modulo L are $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$ and $\frac{\omega_1+\omega_2}{2}$, and they are all simple zeroes.



⌜Indeed, $\wp'\left(\frac{\omega_1}{2}\right) = \wp'\left(\frac{\omega_1}{2}-\omega_1\right) = -\wp'\left(\frac{\omega_1}{2}\right) = 0$. Similarly, $\wp'\left(\frac{\omega_2}{2}\right) = 0 = \wp'\left(\frac{\omega_1+\omega_2}{2}\right)$. By (4.) $\wp'$ has no other zeroes and all of these zeroes are simple.⌟

(6.) The critical values of $\wp$ are just e_1, e_2, e_3 and ∞ with $e_1 := \wp(\omega_1/2)$, $e_2 := \wp(\omega_2/2)$ and $e_3 := \wp((\omega_1+\omega_2)/2)$.

Moreover, $e_1, e_2, e_3 \in \mathbb{C}$ are all different.

Theorem Every elliptic function is a rational function of $\wp$ and $\wp'$. Every elliptic function f can be uniquely expressed as

$$(\Delta) \quad f(u) = R(\wp(u)) + \wp'(u)\, Q(\wp(u))$$

for some rational functions R and Q.

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Pf: $\boxed{\Leftarrow}$ obvious; $\boxed{\Rightarrow}$ Let f be an elliptic function. We can decompose f uniquely into the even and odd part as

$$f(u) = \frac{f(u)+f(-u)}{2} + \frac{f(u)-f(-u)}{2}.$$

WLOG, we can assume that f is, in addition, even. Indeed, for odd f, the function $f(u)/\wp'(u)$ is even. We show that there is a unique $R \in M(\mathbb{S})$ such that $f = R(\wp)$.

Put $R(s) := f(u)$ whenever $s = \wp(u)$ and $u \in \mathbb{C}$.

(i) the definition of R is correct because $\wp(\mathbb{C}) = \mathbb{S}$ and $\wp(u) = \wp(v)$ iff $v = \pm u$ modulo L.

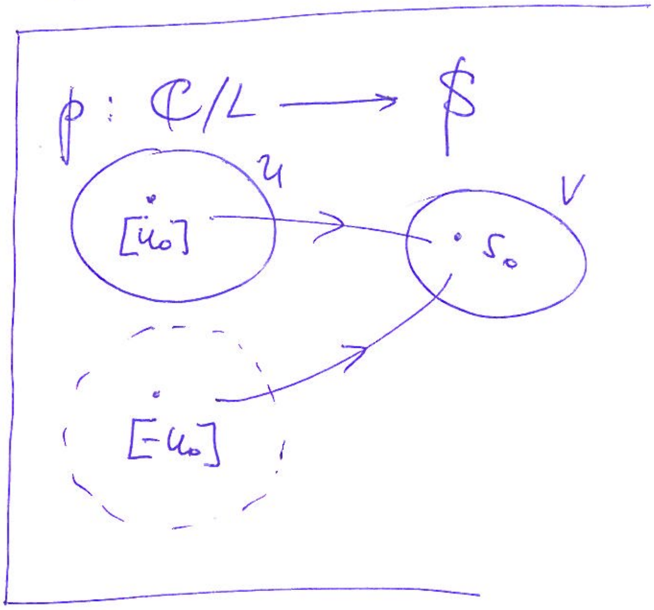
(ii) the map $R : \mathbb{S} \to \mathbb{S}$ is continuous.

$$\begin{tikzcd} \mathbb{S} & \mathbb{S} \arrow[l, "R"'] & \mathbb{C} \arrow[l, "\wp"'] \arrow[ll, bend left, "f"] \end{tikzcd}$$

Indeed, for an open $\mathcal{U}$ in $\$$, it is easy to see that $R^{-1}(\mathcal{U}) = \underbrace{\wp(\underbrace{f^{-1}(\mathcal{U})}_{\text{open}})}_{\text{open}}$ is open in $\$$.

(iii) R is holomorphic on $\overset{\text{open}}{\$ \setminus \{e_1, e_2, e_3, \infty\}}$, and by (ii) the points e_1, e_2, e_3, ∞ are removable singularities of R.

Indeed, let $s_0 \in \$ \setminus \{e_1, e_2, e_3, \infty\}$, i.e., s is not a <u>critical value</u> of $\wp$. Let $u_0 \in \mathbb{C}$ such that $\wp(u_0) = s$. Then $[u_0] \neq [-u_0]$ where $[u_0] := u_0 + L \in \mathbb{C}/L$. Since $n_\wp([u_0]) = 1$ there is a ngh $\mathcal{U}$ of $[u_0]$ in $\mathbb{C}/L$ and a ngh V of s_0 in $\$$ such that $\wp|_{\mathcal{U}} : \mathcal{U} \xrightarrow{\text{onto}} V$ is a conformal map.



Then $R = f \circ (\wp|_{\mathcal{U}})^{-1}$ is holomorphic on V. ∎

$\boxed{\text{Remark}}$ By Theorem, we can express $(\wp'(u))^2$ using $\wp(u)$.