Quotients – continuation

The easy part of the uniformisation theorem tells us that if X is a connected RS, then $X \simeq \tilde{X}/\Gamma$ where $\tilde{X}$ is a simply connected RS and a subgroup Γ of $\mathrm{Aut}(\tilde{X})$ acts on $\tilde{X}$ properly discontinuously. The hard part says that $\tilde{X} = \mathbb{S}, \mathbb{C}$ or $\mathbb{H}$. We give more details on the easy part.

Fundamental groups

Let X be a topological space, $u : [\alpha, \beta] \to X$ and $v : [\gamma, \delta] \to X$ be continuous curves. As usual we define

$$(\dot{-}u)(t) := u(\beta - t), \quad t \in [0, \beta - \alpha] \text{ and}$$
$$(u \dot{+} v)(t) := u(t), \quad t \in [\alpha, \beta]$$
$$:= v(t - \beta + \gamma), \quad t \in [\beta, \beta - \gamma + \delta]$$

whenever $u(\beta) = v(\gamma)$.

WLOG, we can assume that every curve in X is defined ~~on~~ on $[0,1]$. Otherwise, we can make a linear reparametrization.

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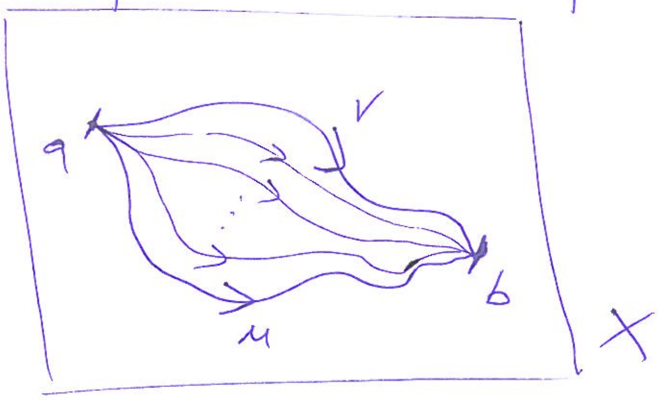
Let $a,b \in X$ and $u,v : [0,1] \to X$ be (continuous) curves from the point a to the point b.



Then the curves u,v are called <u>homotopic</u> (in X) if there is a continuous map $A : [0,1] \times [0,1] \to X$ such that, for $u_s(t) := A(s,t)$, we have

(i) $u_0 = u$

(ii) $u_1 = v$

(iii) $u_s(0) = a$, $u_s(1) = b$ $\forall s \in [0,1]$

<u>Remark</u>: The curves u,v are homotopic in X iff u can be continuously deformed onto v in X. We write $u \sim v$. This relation is obviously an equivalence.

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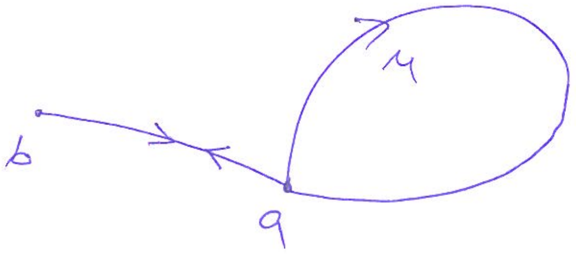
We call $u:[0,1]\to X$ a <u>loop based at</u> $a\in X$ if u is a closed continuous curve with $u(0)=a=u(1)$.



Denote by $\pi_1(X,a)$ the set of all homotopic classes of loops in X based at a. It is easy to see that $\pi_1(X,a)$ forms a group with the operations

$$[u]\dot{+}[v] := [u\dot{+}v], \quad \dot{-}[u] := [\dot{-}u]$$

and the neutral element given by the homotopic class of the constant loop at a.

We call $\pi_1(X,a)$ the <u>fundamental group</u> of X at a. If X is <u>path-connected</u> (i.e., every two points in X can be connected by a curve), then the groups $\pi_1(X,a)$ and $\pi_1(X,b)$ are isomorphic for any $a,b\in X$. Then we write simply $\pi_1(X)$.

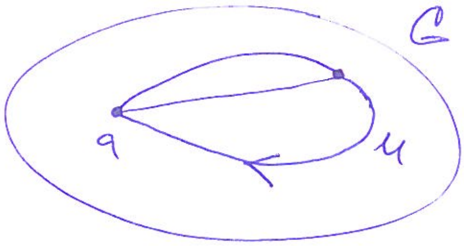
DEFINITION A path-connected topological space X is called <u>simply connected</u> if

$$\pi_1(X) = 0,$$

i.e., every loop in X is homotopic to a constant one.

Example (1.) <u>Simply connected spaces</u>: $\mathbb{S}, \mathbb{C}, \mathbb{D}$

Let G be a <u>convex domain</u> in $\mathbb{C}$, $a \in G$ and let u be a loop in G based at a. Then u is homotopic in G to the constant loop at a.

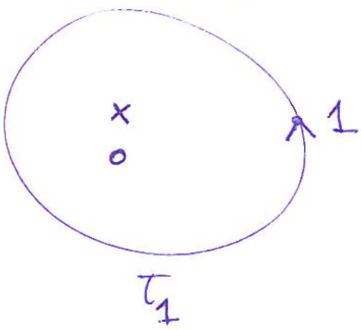


Indeed, use $A(s,t) := (1-s)\,u(t) + s a$, $s,t \in [0,1]$.

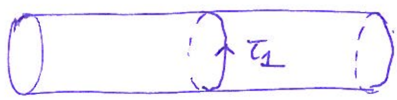
(2.) $\underset{\text{cylinder}}{\pi_1(\mathbb{C} \setminus \{0\}, 1)} = \{n[\tau_1] \mid n \in \mathbb{Z}\} \simeq \mathbb{Z}$ where

$$\tau_1(t) := e^{2\pi i t}, \quad t \in [0,1]$$

$$n[\tau_1] = \underbrace{[\tau_1] + \cdots + [\tau_1]}_{n \text{ times}}$$

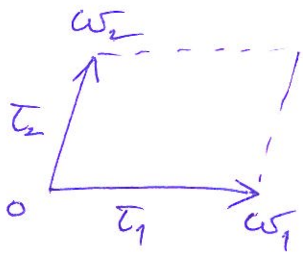


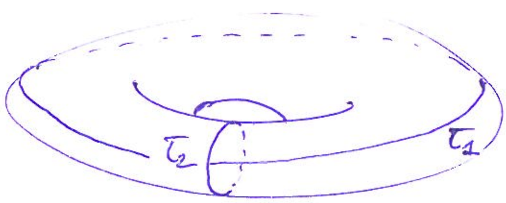
Here $n = \operatorname{ind}_0 u$!



(3.) $\underset{\text{torus}}{\pi_1(\mathbb{C}/L, 0)} = \{n[\tau_1] + m[\tau_2] \mid n, m \in \mathbb{Z}\} \simeq \mathbb{Z} \times \mathbb{Z}$

where $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$





Universal cover

DEFINITION: Let X, Y be topological spaces. Then a mapping $p: Y \to X$ is called a covering map if each $x \in X$ has a ngh. U such that

$$(*) \qquad p^{-1}(U) = \bigcup_{\alpha \in I} V_\alpha$$

where V_α, $\alpha \in I$, are pairwise disjoint open sets and, for each $\alpha \in I$, $p|V_\alpha$ is a homeomorphism of V_α onto U.

Remark: A covering map is a local homeomorphism but the converse is not true in general. On the other hand, a proper local homeomorphism is a covering map. Exercise

Examples

1. For $k \in \mathbb{N}$, $p_k(z) := z^k$ is a covering map of $\mathbb{C} \setminus \{0\}$ onto $\mathbb{C} \setminus \{0\}$.
2. $\exp: \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is a covering map
3. For RS X, the canonical projection

$$p: X \longrightarrow X/\Gamma$$

is a covering map. [This can be generalized for a topological space X.]

4. The mapping $p: \mathbb{D} \to \mathbb{C}$, $p(z) := z$, is a local homeomorphism but not a covering map.

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In what follows, we assume that any space $X, Y, Z \ldots$ is a topological surface (or manifold) unless stated otherwise.

$\boxed{\text{Theorem 1}}$ Let $p: Y \to X$ be a covering map.

1. Lifting of curves: Let $u: [0,1] \to X$ be a continuous curve, $u(0) = x$ and $y \in p^{-1}(x)$. Then there is a unique continuous curve $\tilde{u}: [0,1] \to Y$ such that

$$p \circ \tilde{u} = u \quad \text{and} \quad \tilde{u}(0) = y,$$

the so-called lift of u.

$$\begin{tikzcd} Y \arrow[d, "p"] & y \arrow[r, bend left, "\tilde{u}"] & {} \\ X & x \arrow[r, bend left, "u"] & {} \end{tikzcd}$$

2. Lifting of continuous maps: Let Z be simply connected, $f: Z \to X$ be continuous, $z_0 \in Z$ and $y_0 \in p^{-1}(f(z_0))$. Then there is a unique continuous $\tilde{f}: Z \to Y$ such that

$$p \circ \tilde{f} = f \quad \text{and} \quad \tilde{f}(z_0) = y_0,$$

the so-called lift of f.

$$\begin{tikzcd} Y \arrow[d, "p"'] & \\ X & Z \arrow[l, "f"] \arrow[ul, dashed, "\tilde{f}"'] \end{tikzcd}$$

$y_0 \in Y$, $f(z_0)$, Z s.con. $\ni z_0$

Remark For $Z = [0,1]$ we get 1.

3. If X is simply connected and Y is connected, then $p: Y \xrightarrow{\text{onto}} X$ is a homeomorphism. The only cover of a simply connected space is the space itself.

Proof (sketch)

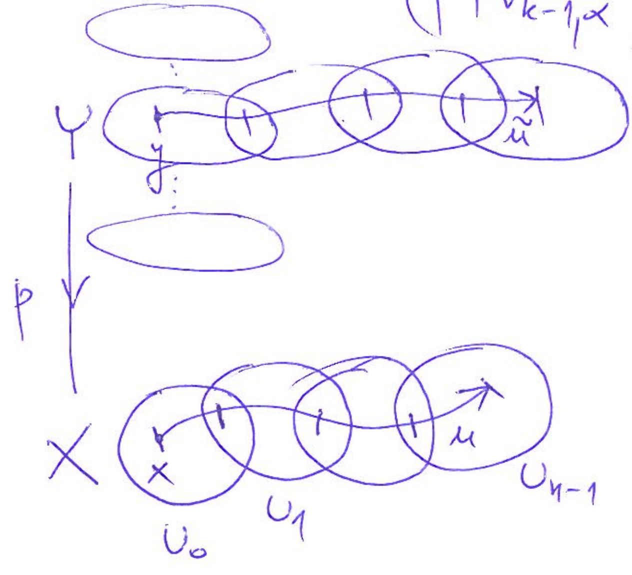
(1.) By compactness of $[0,1]$, there is a division $0=t_0<t_1<\dots<t_n=1$ and open sets U_k in X such that (i) $u([t_k,t_{k+1}])\subset U_k$,

(ii) $p^{-1}(U_k)=\bigcup_{\alpha\in I_k} V_{k,\alpha}$ as in $(*)$.

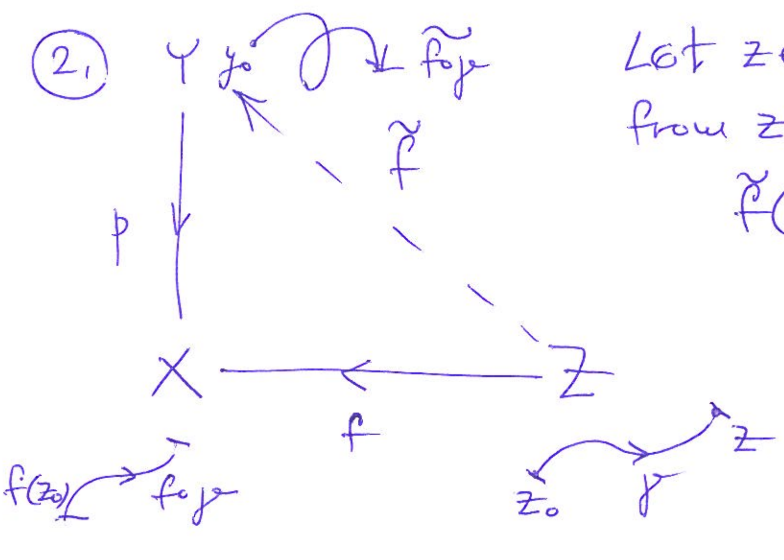
By induction, we construct the lift $\tilde{u}$ as follows. Assume that we have constructed $\tilde{u}$ on $[0,t_{k-1}]$ and put $y_{k-1}:=\tilde{u}(t_{k-1})$.

Since $p(y_{k-1})=u(t_{k-1})\in U_{k-1}$ there is a unique $\alpha\in I_{k-1}$ such that $y_{k-1}\in V_{k-1,\alpha}$.

Put $\tilde{u}(t):=\left(p|_{V_{k-1,\alpha}}\right)^{-1}(u(t))$, $t\in[t_{k-1},t_k]$.



(2.)



Let $z\in Z$ and γ be a curve in Z from z_0 to z. Put

$$\tilde{f}(z):=(\widetilde{f\circ\gamma})(1)$$

(3.) follows from (2.).

$\boxed{\text{Example}}$ Let X be a simply connected RS and $f: X \to \mathbb{C}\setminus\{0\}$ be a holomorphic map. Then the lift F of f for the covering map

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"'] \\ \mathbb{C}\setminus\{0\} & X \arrow[l, "f"] \arrow[u, dashed, "F"'] \end{tikzcd}$$

X s. con.

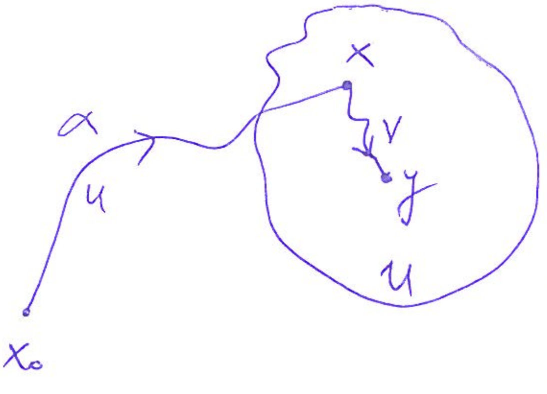
$\exp: \mathbb{C} \to \mathbb{C}\setminus\{0\}$ is a holomorphic branch of logarithm of f, i.e., a holomorphic $F: X \to \mathbb{C}$ such that $\exp(F) = f$.

In an analogous way, we get branches of the k-th root of f for $k \in \mathbb{N}$.

$\boxed{\text{Theorem 2}}$ Let X be a connected surface. Then there are uniquely determined (up to an isomorphism) simply connected surface $\tilde{X}$ and a covering map $p: \tilde{X} \to X$. We call $\tilde{X}$ the universal cover of X.

Construction of $\tilde{X}$: Let $x_0 \in X$. For every $x \in X$ denote by $\pi(x_0, x)$ the set of all homotopic classes of curves in X from x_0 to x. Put

$\tilde{X} := \{(x, \alpha) \mid x \in X, \alpha \in \pi(x_0, x)\}$ and $p: \tilde{X} \to X$, $p(x, \alpha) := x$.



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Topology on $\tilde{X}$: Let $(x,\alpha) \in \tilde{X}$.

Let $U \subset X$ be an open, connected and simply connected set and $x \in U$. Then define the set $[U,\alpha]$ in $\tilde{X}$ as follows: $(y,\beta) \in [U,\alpha]$ iff $y \in U$ and $\beta = [u+v]$ for curves u,v in X such that $\alpha = [u]$ and v is a curve in U connecting x and y.

Uniqueness of $\tilde{X}$ follows from Theorem 1, 3.

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For RSs, we have: Let X be a connected RS. Let $p: \tilde{X} \to X$ be the universal covering map. Then $\tilde{X}$ is simply connected RS and $X \simeq \tilde{X}/\Gamma$ where

$$\Gamma := \{ g \in Aut(\tilde{X}) \mid p \circ g = p \}.$$

"g commutes with p"

Moreover, we have $\Gamma \simeq \pi_1(X)$.

[Actually, an analogue of this result is true for more general topological spaces.]

QU021

Application

Little Picard Theorem: If $f \in \mathrm{Hol}(\mathbb{C})$ does not attain at least two values then f is constant.

$\boxed{\text{Example}}$ $\exp(\mathbb{C}) = \mathbb{C} \setminus \{0\}$

Proof: WLOG, we can assume that $f : \mathbb{C} \to \mathbb{C} \setminus \{0,1\}$ is holomorphic. We know that the universal cover of $\mathbb{C} \setminus \{0,1\}$ is $\mathbb{D}$. Then its lift $\tilde{f} : \mathbb{C} \to \mathbb{D}$ is constant by Liouville's theorem, and so is f.

$$
\begin{tikzcd}
\mathbb{D} \arrow[d, "p"'] & \\
\mathbb{C} \setminus \{0,1\} & \mathbb{C} \arrow[l, "f"'] \arrow[ul, dashed, "\tilde{f}"']
\end{tikzcd}
$$

$\blacksquare$